ANGULAR ESTIMATIONS OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. The object of the present paper is to derive some argument properties of certain integral operators. Our results contain some interesting corollaries as the special cases.

KEY WORDS AND PHRASES: Argument, integral operators, starlike functions, Bazilević functions.

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1. INTRODUCTION

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists a Schwarz function $w(z)$ in $U$ such that $f(z) = g(w(z))$.

A function $f \in A$ is said to be in the class $S^*[E, F]$ if

$$\frac{zf'(z)}{f(z)} < \frac{1+Ez}{1+Fz} \quad (z \in U, -1 < F < E \leq 1).$$

The class $S^*[E, F]$ was studied in [1,2]. In particular, $S^*[1-2\alpha, -1] \equiv S^*(\alpha)(0 \leq \alpha < 1)$ is the well known class of starlike functions of order $\alpha$. We observe [2] that a function $f$ is in $S^*[E, F]$ if and only if

$$\left| \frac{zf'(z)}{f(z)} - \frac{1-EF}{1-F^2} \right| < \frac{E-F}{1-F^2} \quad (z \in U, F \neq -1) \quad (1.2)$$

and

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \frac{1-E}{2} \quad (z \in U, F = -1). \quad (1.3)$$

A function $f \in A$ is said to be in the class $B(\mu, \alpha, \beta)$ if it satisfies

$$\Re \left\{ \frac{zf'(z)g^{\mu-1}}{g^\alpha(z)} \right\} > \beta(z \in U)$$

for some $\mu(\mu > 0), \beta(0 \leq \beta < 1)$ and $g \in S^*(\alpha)$. Furthermore, we denote $B_1(\mu, \alpha, \beta)$ by the subclass of $B(\mu, \alpha, \beta)$ for $g(z) \equiv z \in S^*(\alpha)$. The classes $B(\mu, \alpha, \beta)$ and $B_1(\mu, \alpha, \beta)$ are the subclasses of Bazilević functions in $U$ [3]. We also note that $B(1, \alpha, \beta) \equiv C(\alpha, \beta)$ is an important subclass of close-to-convex functions [4].

For a positive real number $\mu > 0$ and a function $f \in A$, we define the integral operator $J_{c,\mu}$ by
Kumar and Shukla [5] showed that the integral operator \( J_{c,\mu}(f) \) defined by (1.4) belongs to the class \( S^*[E,F] \) for \( c \geq \frac{\mu(E-1)}{1-F} \), whenever \( f \in S^*[E,F] \). The operator \( J_{1,1} \), when \( c \in \mathbb{N} = \{1,2,3,\ldots\} \), was introduced by Bernardi [6]. Further, the operator \( J_{1,1} \) was studied earlier by Libera [7] and Livingston [8].

In the present paper, we give some argument properties of the integral operator defined by (1.4). We also generalize the previous results of Libera [7], Owa and Srivastava [9] and Owa and Obradović [10].

2. MAIN RESULTS

In proving our main results, we shall need the following lemmas.

**LEMMA 1** ([11]). Let \( M(z) \) and \( N(z) \) be regular in \( U \) with \( M(0) = N(0) = 0 \), and let \( \beta \) be real. If \( N(z) \) maps \( U \) onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

\[ \Re N'(z) > (z \in U) = \Re M(z) \]

and

\[ \Re N''(z) < (z \in U) = \Re N'(z) < (z \in U). \]

**LEMMA 2** ([12]). Let \( p(z) \) be analytic in \( U \), \( p(0) = 1 \), \( p(z) \neq 0 \) in \( U \) and suppose that there exists a point \( z_0 \in U \) such that

\[ |\arg p(z)| < \frac{\pi \beta}{2} \quad \text{for} \quad |z| < |z_0| \]

and

\[ |\arg p(z_0)| = \frac{\pi \beta}{2}, \]

where \( \beta > 0 \). Then we have

\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \]

where

\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi \beta}{2} \]

and

\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi \beta}{2} \]

where

\[ p(z_0)^{\frac{1}{a}} = \pm ia (a > 0). \]

With the help of Lemma 1 and Lemma 2, we now derive

**THEOREM 1.** Let \( c \) and \( \mu \) be real numbers with \( c \geq 0, \mu > 0 \) and \( -1 \leq F < E \leq 1 \) and let \( f \in A \). If

\[ J_{c,\mu}(f) = \left( \frac{c + \mu}{z^c} \int_0^z t^{c-1} f^{\mu}(t) dt \right)^{\frac{1}{\mu}} (c > -\mu). \]
for some $g \in S^*[E,F]$, then

$$\left| \arg \left( \frac{zJ_{c,\mu}(f)}{J_{c,\mu}(g)} - \beta \right) \right| < \frac{\pi \delta}{2} \quad (0 \leq \beta < 1, 0 < \delta \leq 1)$$

where $J_{c,\mu}$ is the integral operator defined by (1.4) and $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \eta + \frac{2}{\pi} \tan^{-1} \left( \frac{\eta \sin \frac{\pi}{2} (1 - t_c(E,F))}{c + \frac{1}{1 + \beta} + \eta \cos \frac{\pi}{2} (1 - t_c(E,F))} \right) & \text{for } F \neq -1, \\ \eta & \text{for } F = -1, \end{cases} \quad (2.1)$$

when

$$t_c(E,F) = \frac{2}{\pi} \sin^{-1} \left( \frac{E - F}{c(1 - F^2) + 1 - EF} \right). \quad (2.2)$$

**PROOF.** Let us put

$$p(z) = \frac{M(z)}{N(z)}$$

where

$$M(z) = \frac{1}{1 - \beta} \left\{ z^\alpha f^{\mu}(z) - c \int_0^z t^{\alpha-1} f^{\mu}(t) dt - \beta \mu \int_0^z t^{\alpha-1} g^{\mu}(t) dt \right\}$$

and

$$N(z) = \mu \int_0^z t^{\alpha-1} g^{\mu}(t) dt.$$

Then $p(z)$ is analytic in $U$ with $p(0) = 1$. By a simple calculation, we have

$$\frac{M'(z)}{N'(z)} = p(z) \left( 1 + \frac{N(z)}{zN'(z)} \frac{zp'(z)}{p(z)} \right) = \frac{1}{1 - \beta} \left( \frac{z^\alpha f^{\mu-1}(z)}{g^{\mu}(z)} - \beta \right).$$

Since $g \in S^*[E,F]$, $J_{c,\mu}(g) \in S^*[E,F]$ [5] and hence $N(z)$ is (possibly many-sheeted) starlike function with respect to the origin. Therefore, from our assumption and Lemma 1, $p(z) \neq 0$ in $U$.

If there exists a point $z_0 \in U$ such that

$$\left| \arg p(z) \right| < \frac{\pi \eta}{2} \quad \text{for } |z| < |z_0|$$

and

$$\left| \arg p(z_0) \right| = \frac{\pi \eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i \kappa \eta,$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = \frac{\pi \eta}{2}.$$
and
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when} \quad \arg p(z_0) = -\frac{\pi \eta}{2} \]
where
\[ p(z_0)^{\frac{1}{2}} = \pm ia(a > 0). \]

Since \( J_{c,\mu}(g) \in S^*[E, F] \), from (1.2) and (1.3), we have
\[ \frac{zN'(z)}{N(z)} = \frac{z(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c = \rho e^{\frac{i\pi}{2}}, \]
where
\[
\begin{cases}
  c + \frac{1 - E}{1 - F} < \rho < c + \frac{1 + E}{1 + F}, \\
  -t_c(E, F) < \phi < t_c(E, F) \quad \text{for } F \neq -1,
\end{cases}
\]
when \( t_c(E, F) \) is given by (2.2), and
\[
\begin{cases}
  c + \frac{1 - E}{2} < \rho < \infty, \\
  -1 < \phi < 1 \quad \text{for } F = -1.
\end{cases}
\]

At first, suppose that \( p(z_0)^{\frac{1}{2}} = ia(a > 0) \). For the case \( F \neq -1 \), we obtain
\[ \arg \left( \frac{z_0f'(z_0)f^{\mu-1}(z_0)}{g^\mu(z_0)} - \beta \right) = \arg \left( \frac{(1 - \beta)M'(z_0)}{N'(z_0)} \right) \]
\[ = \arg p(z_0) + \arg \left( 1 + \frac{1}{\rho \frac{(J_{c,\mu}(g))'}{J_{c,\mu}(g)} + c} \frac{z_0p'(z_0)}{p(z_0)} \right) \]
\[ = \frac{\pi \eta}{2} + \arg \left( 1 + \left( \rho e^{\frac{i\pi}{2}} \right)^{-1} \right) i\eta k \]
\[ = \frac{\pi \eta}{2} + \tan^{-1} \left( \frac{\eta k \sin \frac{\pi}{2} (1 - \phi)}{\rho + \eta k \cos \frac{\pi}{2} (1 - \phi)} \right) \]
\[ \geq \frac{\pi \eta}{2} + \tan^{-1} \left( \frac{\eta \sin \frac{\pi}{2} (1 - t_c(E, F))}{c + \frac{1 + F}{1 - F} + \eta \cos \frac{\pi}{2} (1 - t_c(E, F))} \right) \]
\[ = \frac{\pi \delta}{2}, \]
where \( t_c(E, F) \) and \( \delta \) are given by (2.2) and (2.1), respectively. Similarly, for the case \( F = -1 \), we have
\[ \arg \left( \frac{z_0f'(z_0)f^{\mu-1}(z_0)}{g^\mu(z_0)} - \beta \right) \geq \frac{\pi \eta}{2}. \]

These are a contradiction to the assumption of our theorem.

Next, suppose that \( p(z_0)^{\frac{1}{2}} = -ia(a > 0) \). For the case \( F \neq -1 \), applying the same method as the above, we have
\[ \arg \left( \frac{z_0f'(z_0)f^{\mu-1}(z_0)}{g^\mu(z_0)} - \beta \right) = \leq -\frac{\pi \eta}{2} - \tan^{-1} \left( \frac{n \sin \frac{\pi}{2} (1 - t_c(E, F))}{c + \frac{1 + F}{1 - F} + n \cos \frac{\pi}{2} (1 - t_c(E, F))} \right) \]
where \( t_c(E, F) \) and \( \delta \) are given by (2.2) and (2.1), respectively and for the case \( F = -1 \), we have
which are contradictions to the assumption. Therefore we complete the proof of our theorem.

Taking $E = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $F = -1$ in Theorem 1, we have

**COROLLARY 1.** Let $c \geq 0$, $\mu > 0$ and $f \in A$. If

$$\left| \arg \left( \frac{zf'(z)f_{\mu-1}^-(z)}{g^\mu(z)} - \beta \right) \right| < \frac{\pi \delta}{2} (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*(\alpha)$, then

$$\left| \arg \left( \frac{z(J_{e,\mu}(f))'J_{e,\mu}^-(f)}{J_{e,\mu}^\mu(g)} - \beta \right) \right| < \frac{\pi \delta}{2},$$

where $J_{e,\mu}$ is the integral operator defined by (1.4).

**REMARK 1.** For $\delta = 1$, Corollary 1 is the result obtained by Owa and Obradović [10].

Setting $E = 1$, $F = -1$, $\mu = 1$, $\delta = 1$ and $g(z) = z$ in Theorem 1, we have

**COROLLARY 2.** Let $c \geq 0$ and $f \in A$. If

$$\text{Re} \ f'(z) > \beta (0 \leq \beta < 1),$$

then

$$\text{Re} \ (J_{e,1}(f))' > \beta,$$

where $J_{e,1}$ is the integral operator defined by (1.4).

Letting $\mu = 1$ in Theorem 1, we have

**COROLLARY 3.** Let $c \geq 0$ and $-1 \leq F < E \leq 1$ and let $f \in A$. If

$$\left| \arg \left( \frac{zf'(z)}{g(z)} - \beta \right) \right| < \frac{\pi \delta}{2} (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in S^*[E, F]$, then

$$\left| \arg \left( \frac{z(J_{e,1}(f))'}{J_{e,1}(g)} - \beta \right) \right| < \frac{\pi \eta}{2},$$

where $J_{e,1}$ is the integral operator defined by (1.4) and $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.1).

Taking $E = 1 - 2\alpha$ ($0 \leq \alpha < 1$) and $F = -1$ in Corollary 3, we have

**COROLLARY 4.** Let $c \geq 0$ and $f \in A$. If

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi \delta}{2} (0 \leq \alpha < 1, 0 < \delta \leq 1),$$

then

$$\left| \arg \left( \frac{z(J_{e,1}(f))'}{J_{e,1}(f)} - \alpha \right) \right| < \frac{\pi \delta}{2},$$

where $J_{e,1}$ is the integral operator defined by (1.4).

Putting $E = 1 - 2\alpha$ ($0 \leq \alpha < 1$), $F = -1$ and $\delta = 1$ in Corollary 3 and Corollary 4, we obtain the following result of Owa and Srivastava [9].

**COROLLARY 5.** If the function $f$ defined by (1.1) is in the class $C(\alpha, \beta)$, then the integral operator $J_{e,1}(f)(c \geq 0)$ defined by (1.4) is also in the class $c(\alpha, \beta)$.

**REMARK 2.** Taking $\alpha = \beta = 0$ and $c = 1$ in Corollary 5, we obtain the result given earlier by Libera [7].
By using the same technique as in proving Theorem 1, we have

**Theorem 2.** Let \( c \) and \( \mu \) be real numbers with \( c \geq 0, \mu > 0 \) and \( -1 \leq F < E \leq 1 \) and let \( f \in A \). If

\[
\left| \arg \left( \beta - \frac{zf'(z)f^{n-1}(z)}{g^{n}(z)} \right) \right| < \frac{\pi \delta}{2} \quad (\beta > 1, 0 < \delta \leq 1)
\]

for some \( g \in S^*[E, F] \), then

\[
\left| \arg \left( \beta - \frac{z(J_{c,\mu}(f))^n J_{c,\mu}^{n-1}(f)}{J_{c,\mu}^n(g)} \right) \right| < \frac{\pi \eta}{2},
\]

where \( J_{c,\mu} \) is the integral operator defined by (1.4) and \( \eta(0 < \eta \leq 1) \) is the solution of the equation (2.1)

Putting \( E = 1 - 2\alpha(0 \leq \alpha < 1), F = -1, \mu = 1 \) and \( \delta = 1 \) in Theorem 2, we have the following result by Owa and Srivastava [9].

**Corollary 6.** Let \( c \geq 0 \) and \( f \in A \). If

\[
\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} < \beta (\beta > 1)
\]

for some \( g \in S^*(\alpha) \), then

\[
\text{Re} \left\{ \frac{z(J_{c,1}(f))^n}{J_{c,1}(g)} \right\} < \beta,
\]

where \( J_{c,1} \) is the integral operator defined by (1.4).

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**References**


