ON STRICT AND SIMPLE TYPE EXTENSIONS

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ABSTRACT. Let \((Y, \tau)\) be an extension of a space \((X, \tau')\). Let \(\mathcal{O}_p = \{W \cap X: W \in \tau, p \in W\}\). For \(U \in \tau'\), let \(\mathcal{o}(U) = \{p \in Y: U \in \mathcal{O}_p\}\). In 1964, Banaschewski introduced the strict extension \(Y^s\), and the simple extension \(Y^+\) of \(X\) (induced by \((Y, \tau)\)) having base \(\{\mathcal{o}(U): U \in \tau'\}\) and \(\{U \cup \{p\}: p \in Y, \text{ and } U \in \mathcal{O}_p\}\), respectively. The extensions \(Y^s\) and \(Y^+\) have been extensively used since then. In this paper, the open filters \(\mathcal{L}^p = \{W \in \tau': W \supseteq \text{int}_X \text{cl}_X(U)\}\) for some \(U \in \mathcal{O}_p\), and \(\mathcal{U}^p = \{W \in \tau': \text{int}_X \text{cl}_X(W) \in \mathcal{O}_p\}\) are used to define some new topologies on \(Y\). Some of these topologies produce nice extensions of \((X, \tau')\). We study some interrelationships of these extensions with \(Y^s\), and \(Y^+\) respectively.

KEY WORDS AND PHRASES: Extension, simple extension, strict extension, H-closed, s-closed, almost realcompact, near compact.

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1. INTRODUCTION

A topological space \(Y\) is an extension of a space \(X\) if \(X\) is a dense subspace of \(Y\). If \(Y_1\) and \(Y_2\) are two extensions of a space \(X\), then \(Y_2\) is said to be projectively larger than \(Y_1\), written \(Y_2 \geq Y_1\) for \(Y_1 \leq Y_2\), provided that there exists a continuous map \(f: Y_1 \to Y_2\) such that \(f|_X = i_X\), the identity map on \(X\). Two extensions \(Y_1\) and \(Y_2\) of \(X\) are called equivalent if \(Y_1 \leq Y_2\) and \(Y_2 \leq Y_1\). We shall identify two equivalent extensions of \(X\). With this convention, the class \(E(X)\) of all the Hausdorff extensions of a Hausdorff space \(X\) is a set. Let \((Y, \tau) \in E(X)\) and set \(p \in Y\). If \(N_p\) is the open neighborhood filter of \(p\) in \(Y\), the set \(\mathcal{O}_p = \{N \cap X: N \in N_p\}\) (called the trace of \(N_p\) on \(X\)) is an open filter on \(X\). If \(U\) is open in \(X\), denote \(\mathcal{o}_U = \{p \in Y: U \in \mathcal{O}_p\}\).
In 1964 Banaschewski [1] introduced the extensions \( Y^* \) (resp. \( Y^+ \)) the strict extension (resp. the simple extension) of \( X \) induced by \( Y \) satisfying \( Y^* \leq Y \leq Y^+ \). The topology \( \tau^* \) on \( Y^* \) (resp. \( \tau^+ \) on \( Y^+ \)) has for an open base the collection \( \{ \alpha, (U): U \text{ open in } X \} \) (resp., the collection \( \{ U \cup \{ p \}: p \in Y, \text{ and } U \in \Omega_f \} \)). The extensions \( Y^*, \) and \( Y^+ \) have been studied extensively and have proved extremely useful regarding some properties weaker than compactness, such as nearly compact, almost realcompact, feebly compact, \( H \)-closed, \( s \)-closed, etc. In this paper we introduce new extensions \( Y', Y^*, Y'^*, \) and \( Y'^{**} \), study some of their properties, and compare them with \( Y, Y^*, \) and \( Y^+ \). All spaces under consideration are Hausdorff.

2. THE EXTENSIONS \( Y' \) AND \( Y'^* \).

In this section, we introduce several topologies on \( Y \), and compare them with \( \tau \). Some of these topologies yield interesting extensions of \( (X, \tau') \).

**DEFINITION 2.1.** Let \( (Y, \tau) \) be an extension of a space \( (X, \tau') \). For \( p \in Y \) define

\[
U^p = \{ W: W \in \tau', \text{int}_X cl_X W \in \Omega_f \},
\]

\[
\mathcal{L}^p = \{ W: W \in \tau', W \supseteq \text{int}_X cl_X U \text{ for some } U \in \Omega_f \}.
\]

**LEMMA 2.1.**

(a) Both \( U^p \) and \( \mathcal{L}^p \) are open filters on \( X \) such that \( \mathcal{L}^p \subseteq \Omega_f \subseteq U^p \).

(b) \( U^p = \{ W: W \in \tau', \text{int}_X cl_X W \in \mathcal{L}^p \} \)

\[= \cap \{ U: U \text{ is an open ultrafilter on } X, \Omega_f \subseteq U \} \]

**PROOF.** We prove (b). Let \( U = \{ W: W \in \tau', \text{int}_X cl_X W \in \mathcal{L}^p \} \). If \( W \in U \), then \( W \in \tau' \) and \( \text{int}_X cl_X W \supseteq \text{int}_X cl_X U \) for some \( U \in \Omega_f \). Therefore, \( \text{int}_X cl_X W \in \Omega_f \), whence \( W \in U^p \). Thus, \( U \subseteq U^p \). To prove the reverse inequality, let \( W \in U^p \). Then \( \text{int}_X cl_X W \in \Omega_f \). Since \( \text{int}_X cl_X W \supseteq \text{int}_X cl_X (\text{int}_X cl_X W) \) it follows that \( \text{int}_X cl_X W \in \mathcal{L}^p \). Hence \( W \in U \). This proves the first equality in (b). The second equality follows from [9], completing the proof of the lemma.

**REMARK 2.1.** Since \( \Omega_f = \Omega_f^* = \Omega_f^+ \), it follows that each one of \( Y, Y' \) and \( Y'^* \) yield the same \( \mathcal{L}^p \) (resp., \( U^p \)) for all \( p \in Y \). Moreover, if \( Z \in E(X) \) has the same underlying set as \( Y \), and is such that \( Y^* \leq Z \leq Y^+ \), then \( Y \) and \( Z \) induce the same \( \mathcal{L}^p \) (resp., \( U^p \)) for all \( p \in Y \). Also, if \( p \neq q \) are distinct elements of \( Y \) then \( \mathcal{L}^p \neq \mathcal{L}^q \) and \( U^p \neq U^q \). Obviously, if \( U \in \Omega_f \), then \( \text{int}_X cl_X (U) \in \mathcal{L}^p \). Moreover, \( U \in U^p \) if and only if \( \text{int}_X cl_X (U) \in U^p \).

**DEFINITION 2.2.** Let \( (Y, \tau) \) be an extension of \( (X, \tau') \). For \( G \in \tau' \), define

\[
\alpha(G) = G \cup \{ p: p \in Y \setminus X, G \in \mathcal{L}^p \}
\]
The proof of the Propositions 2.1, and 2.2 is straightforward.

**PROPOSITION 2.1.** Let \((Y,\tau)\) be an extension of \((X,\tau')\). Then for all \(U, V \in \tau'\),

(a) \(o_1(\varnothing) = \varnothing, o_1(X) = Y\),
(b) \(o_1(U) \cap X = U\),
(c) \(o_1(U \cap V) = o_1(U) \cap o_1(V)\),
(d) The family \(\{o_1(G) : G \in \tau'\}\) is an open base for a Hausdorff topology \(\tau_1\) on \(Y\) and \((Y, \tau_1)\) is an extension of \(X\).

**PROPOSITION 2.2.** Let \((Y,\tau)\) be an extension of \((X,\tau')\). Then for all \(U, V \in \tau'\),

(a) \(o_\alpha(\varnothing) = \varnothing, o_\alpha(X) = Y\),
(b) \(o_\alpha(U) \cap X = U\),
(c) \(o_\alpha(U \cap V) = o_\alpha(U) \cap o_\alpha(V)\),
(d) The family \(\{o_\alpha(G) : G \in \tau'\}\) is an open base for a Hausdorff topology \(\tau_\alpha\) on \(Y\) and \((Y, \tau_\alpha)\) is an extension of \(X\).

**PROPOSITION 2.3.** Let \((Y,\tau)\) be an extension of \((X,\tau')\). Then for all \(U, V \in \tau'\),

(a) \(a_1(\varnothing) = \varnothing, a_1(X) = Y\),
(b) \(a_1(U) \cap X \subseteq U\),
(c) \(a_1(U \cap V) = a_1(U) \cap a_1(V)\),
(d) \(a_1(U) = \cup \{W : W \in \tau \text{ and } \text{int}_X \text{cl}_X(W \cap X) \subseteq U\}\)
(e) The family \(\{a_1(G) : G \in \tau'\}\) is an open base for a coarser Hausdorff topology \(\tau_{\alpha_1}\) on \(Y\), \(X\) is dense in \((Y, \tau_{\alpha_1})\), but \((Y, \tau_{\alpha_1})\) may not be an extension of \(X\).

**PROOF.** We prove (d). The rest is straightforward. Let \(p \in a_1(U)\). Then \(U \in \mathcal{L}_\alpha\). Therefore, \(U \supseteq \text{int}_X \text{cl}_X V\) for some \(V \in \mathcal{O}_\alpha\). Therefore, there exists \(W \in \tau\) such that \(p \in W\) and \(W \cap X = V\). It follows that \(\text{int}_X \text{cl}_X(W \cap X) \subseteq U\). Conversely, if \(W \in \tau\) is such that \(\text{int}_X \text{cl}_X(W \cap X) \subseteq U\) and \(p \in W\), then \(W \cap X \in \mathcal{O}_\alpha\). So, \(\text{int}_X \text{cl}_X(W \cap X) \in \mathcal{L}_\alpha\). This implies that \(U \in \mathcal{L}_\alpha\) and hence \(p \in a_1(U)\). The proof of the proposition is now complete.

**PROPOSITION 2.4.** Let \((Y,\tau)\) be an extension of \((X,\tau')\). Then for all \(U, V \in \tau'\),

(a) \(a_\alpha(\varnothing) = \varnothing, a_\alpha(X) = Y\),
(b) \( a_x(U) \cap X = \text{int}_x \text{cl}_x(U) \),
(c) \( a_x(U \cap V) = a_x(U) \cap a_x(V) \),
(d) \( a_x(U) = \cup \{W : W \in \tau \land W \cap X \subseteq \text{int}_x \text{cl}_x(U)\} \)
(e) The family \( \{a_x(G) : G \in \tau'\} \) is an open base for a coarser Hausdorff topology \( \tau_m \) on \( Y \). \( X \) is dense in \( (Y, \tau_m) \), but \( (Y, \tau_m) \) may not be an extension of \( X \).

**PROOF.** We prove (d). The rest is straightforward. Let \( p \in a_x(U) \). Then \( U \in \tau^p \). Therefore, \( \text{int}_x \text{cl}_x(U) \subseteq \tau^p \). It follows that there exists \( W \in \tau \) such that \( p \in W \land W \cap X \subseteq \text{int}_x \text{cl}_x(U) \).
Conversely, if \( W \in \tau \) is such that \( W \cap X \subseteq \text{int}_x \text{cl}_x(U) \) and \( p \in W \), then \( W \cap X \subseteq \tau^p \). So, \( \text{int}_x \text{cl}_x(U) \subseteq \tau^p \). Therefore, \( U \in \tau^p \) and \( p \in a_x(U) \).

**DEFINITION 2.3.** The spaces \((Y, \tau')\), \((Y, \tau_w)\), \((Y, \tau_m)\), and \((Y, \tau_m)\) described in propositions 2.1-2.4 will, henceforth, be denoted by \( Y'\), \( Y''\), \( Y'\) and \( Y''\) respectively. If \( A \subseteq Y \), then \( \text{int}_r(A) \) (resp. \( \text{cl}_r(A) \)) will be denoted by \( \text{int}_r(A) \) (resp. \( \text{cl}_r(A) \)). Likewise, \( \text{int}_r(A), \text{cl}_r(A), \text{int}_m(A), \text{cl}_m(A), \text{int}_\omega(A) \), and \( \text{cl}_\omega(A) \) are defined in an analogous manner.

**LEMMA 2.2.** If \( U \in \tau' \), then
(a) \( a_x(U) \subseteq o_x(U) \subseteq o_r(U) \subseteq o_x(\text{int}_x \text{cl}_x(U)) = a_x(\text{int}_x \text{cl}_x(U)) \),
(b) \( a_x(U) \setminus X = o_x(U) \setminus X \) and \( a_x(U) \setminus X = o_r(U) \setminus X \)
(c) \( o_x(\text{int}_x \text{cl}_x(U)) \setminus X = a_x(U) \setminus X \), and
(d) if \( U \) is regular open (i.e. \( U = \text{int}_x \text{cl}_x(U) \)), then \( a_x(U) = a_r(U) \), and the equality holds in (a).

**PROOF.** Part (a): We show that \( o_x(\text{int}_x \text{cl}_x(U)) = a_x(U) \), the rest being straightforward. Certainly, \( o_x(\text{int}_x \text{cl}_x(U)) \cap X = \text{int}_x \text{cl}_x(U) = a_x(U) \cap X \). Let \( p \in o_x(\text{int}_x \text{cl}_x(U)) \setminus X \). Then \( \text{int}_x \text{cl}_x(U) \not\subseteq X \). Therefore, \( U \not\in \tau^p \), and \( p \in a_x(U) \setminus X \). Conversely, let \( p \in a_x(U) \setminus X \). Then, \( U \in \tau^p \). So, \( p \in o_x(\text{int}_x \text{cl}_x(U)) \setminus X \). The above arguments prove (a).

To prove (c), let \( q \in o_x(\text{int}_x \text{cl}_x(U)) \setminus X \). Then, \( \text{int}_x \text{cl}_x(U) \subseteq a_x(U) \setminus X \). Therefore, \( q \in o_x(G) \setminus X \). Thus, \( o_x(\text{int}_x \text{cl}_x(U)) \setminus X \subseteq o_x(U) \setminus X \). To prove the reverse inequality, let \( q \in o_x(G) \setminus X \). Then, \( G \in \tau^p \), whence \( \text{int}_x \text{cl}_x(G) \subseteq a_x(U) \setminus X \). Therefore, \( q \in o_x(\text{int}_x \text{cl}_x(U)) \setminus X \) and \( o_x(G) \setminus X \subseteq o_x(\text{int}_x \text{cl}_x(U)) \setminus X \). Hence, \( o_x(\text{int}_x \text{cl}_x(U)) \setminus X = o_x(G) \setminus X \). The rest of the lemma is straightforward.

Given a space \((X, \tau')\), the family \( \{\text{int}_x \text{cl}_x(U) : U \in \tau'\} \) forms an open base for a coarser Hausdorff topology \( \tau'_m \) on \( X \). The space \( X' = (X, \tau'_m) \) is called the semiregularization of \( X \). A space \((X, \tau')\) is called semiregular if \((X, \tau') = X\).

**THEOREM 2.1.** If \( X \) is semiregular, and \((Y, \tau)\) (not necessarily semiregular) is an extension of \( X \), then \( Y' \) is an extension of \( X \) such that \( Y' \leq Y \).

**PROOF.** If \( X \) is semiregular, then \( o_x(U) = a_x(U) \) for all \( U \in \tau' \). Hence, \( Y' \) is an extension of \( X \) such that \( Y' = Y'' \leq Y \).
THEOREM 2.2. The spaces $\mathcal{Y}'$ and $\mathcal{Y}''$ are homeomorphic.

PROOF. For all $U \in \tau'$, $\sigma_{\tau}(\delta_{\tau}(U)) = \eta_{\tau}(\delta_{\tau}(U)) = a_{\tau}(U)$ implies that $\tau' \subseteq \tau''$. Also, if $G \in \tau' \subseteq \tau''$, and if $q \in a_{\tau}(U)$, then $\delta_{\tau}(U) \subseteq \mathcal{P}$, and $q \in a_{\tau}(G)$. Therefore, $p \in a_{\tau}(U) \subseteq a_{\tau}(G)$. Hence, $\tau' \subseteq \tau''$. This proves the theorem.

LEMMA 2.3. Let $(Y, \delta)$ be an extension of $(X, \delta')$. Then, for all $G \in \tau'$ the following are true.

(a) $\delta_{\tau}(G) \subseteq \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))$
(b) $\delta_{\tau}(G) = \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$
(c) $\delta_{\tau}(G) = \delta_{\tau}(G)$
(d) $\delta_{\tau}(G) = \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$
(e) $\delta_{\tau}(\delta_{\tau}(G)) = \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$

PROOF. Part (a): Let $p \in \delta_{\tau}(G)$, and let $o_{\tau}(U)$ be a basic open neighborhood of $p$ in $Y$. If $p \in o_{\tau}(U) \cap X$, then $p \in U \subseteq \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))).$ Therefore, $\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))$ is an open neighborhood of $p$ in $\tau'$. Consequently, $\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))) \neq \emptyset$. Hence $U \cap G \neq \emptyset$. This in turn implies that $o_{\tau}(U) \cap G \neq \emptyset$, and $p \in \delta_{\tau}(G)$. If $p \in o_{\tau}(U) \cap X$, then $U \in \mathcal{P}$. Now, $\delta_{\tau}(G) \subseteq \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))$, whence $p \in \delta_{\tau}(G)$.

Part (b): Let $p \in \delta_{\tau}(G)$, and let $o_{\tau}(U)$ be a basic open neighborhood of $p$ in $Y$. Since $o_{\tau}(U) \subseteq \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))$, and $\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))$ is an open neighborhood of $p$ in $\tau'$. Therefore, $\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))) \subseteq \delta_{\tau}(G)$. Consequently, $\delta_{\tau}(G) \subseteq \delta_{\tau}(G)$. Therefore, $\delta_{\tau}(G) \subseteq \delta_{\tau}(G)$. Hence, $\delta_{\tau}(G) \subseteq \delta_{\tau}(G)$. The other half of (b) is straightforward.

The proof of (c) is straightforward.

Part (d): Let $p \in \delta_{\tau}(G)$, and let $W$ be an open neighborhood of $p$ in $Y$. Then, $W \cap X \circ \delta_{\tau}(G) \subseteq \mathcal{P}$ shows that $\sigma_{\tau}(W \cap X)$ is an open neighborhood of $p$ in $\tau''$. Therefore, $\sigma_{\tau}(W \cap X) \neq \emptyset$. This shows that $W \cap G \neq \emptyset$, whence $p \in \delta_{\tau}(G)$. Conversely, let $p \in \delta_{\tau}(G)$, and let $p \in \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$. Then, $U \in \mathcal{P}$. So, $\sigma_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$ is an open neighborhood of $p$ in $\tau''$ such that $\sigma_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))) \cap G \neq \emptyset$. This implies that $\sigma_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G)))) \cap G \neq \emptyset$. Hence, $p \in \delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(\delta_{\tau}(G))))$. The rest follows from (c).

THEOREM 2.3. The spaces $Y' \setminus X, Y'' \setminus X$, and $Y'' \setminus X$ are pairwise homeomorphic.
PROOF. To prove the continuity of the identity map \( i:Y^w \setminus X \to Y^t \setminus X \), let \( o_i(G) \setminus X \) be a basic open neighborhood of \( p \) in \( Y^t \setminus X \). Then, \( G \in \mathcal{P} \). Hence \( G \supseteq \text{int}_x \text{cl}_x U \) for some \( U \in \mathcal{O}_p \subseteq U^p \). Therefore, \( o_i(U) \setminus X \) is an open neighborhood of \( p \) in \( Y^w \) such that \( o_i(U) \setminus X \subseteq o_i(G) \setminus X \). To prove that the identity map \( i:Y^t \setminus X \to Y^w \setminus X \) is continuous, let \( o_i(G) \setminus X \) be a basic open neighborhood of \( p \) in \( Y^w \setminus X \). Then \( o_i(\text{int}_x \text{cl}_x G) \setminus X \) is an open neighborhood of \( p \) in \( Y^t \setminus X \) such that \( o_i(\text{int}_x \text{cl}_x G) \setminus X = o_i(G) \setminus X \). Hence, the spaces \( Y^t \setminus X \) and \( Y^w \setminus X \) are homeomorphic. The rest of the theorem follows directly from Lemma 2.2.

Let \( Z_1 \) and \( Z_2 \) be spaces. A map \( f:Z_1 \to Z_2 \) is called \( \theta \)-continuous [3] if for every \( p \in Z_1 \) and for every open neighborhood \( V \) of \( f(p) \) in \( Z_2 \), there exists an open neighborhood \( U \) of \( p \) in \( Z_1 \) such that \( f(\text{cl}_Z U) \subseteq \text{cl}_Z (V) \). \( f \) is called perfect if \( f \) is a closed map (not necessarily continuous) such that \( f^{-1}(z) \) is compact in \( Z_1 \) for every \( z \in Z_2 \). Also, \( f \) is called irreducible if \( f \) is closed and there is no proper closed subset \( K \) of \( Z_1 \) for which \( f(K) = Z_2 \). Two extensions \( Z_1 \) and \( Z_2 \) of a space \( X \) are called \( \theta \)-equivalent if there exists a \( \theta \)-homeomorphism \( f \) from \( Z_1 \) onto \( Z_2 \) such that \( f|_X = i_X \), the identity map on \( X \).

The next theorem depicts some of the several interrelationships between the spaces \( Y, Y^w, Y^t, Y^e, \) and \( Y^{ad} \).

THEOREM 2.4. Let \( (Y, \tau) \) be an extension of a space \( (X, \tau') \). The following statements are true.
(a) The identity map \( i:Y^w \to Y \) is perfect, irreducible and \( \theta \)-continuous.
(b) The identity map \( i:Y^e \to Y^w \) is perfect, irreducible and \( \theta \)-continuous.
(c) The identity map \( i:Y^e \to Y^* \) is \( \theta \)-continuous.
(d) The identity map \( i:Y^* \to Y^t \) is \( \theta \)-continuous.
(e) The identity map \( i:Y^* \to Y^w \) is \( \theta \)-continuous.
(f) The identity map \( i:Y^t \to Y^* \) is \( \theta \)-continuous.
(g) The identity map \( i:Y^t \to Y^e \) is \( \theta \)-continuous.
(h) The identity map \( i:Y^{ad} \to Y^* \) is \( \theta \)-continuous.
(i) The identity map \( i:Y^{ad} \to Y^t \) is \( \theta \)-continuous.
(j) The identity map \( i:Y^{ad} \to Y^w \) is \( \theta \)-continuous.
(k) The identity map \( i:Y^{ad} \to Y^e \) is \( \theta \)-continuous.
(l) The identity map \( i:Y^{ad} \to Y^{ad} \) is \( \theta \)-continuous.

PROOF. Below, we outline the proofs of some parts of the theorem. The rest of the proofs are analogous.

Part (a) Since \( \tau_{ad} \subseteq \tau \), \( i:Y \to Y^{ad} \) is continuous. Hence, \( i:Y \to Y^{ad} \) is irreducible and perfect. To prove the \( \theta \)-continuity of \( i:Y^{ad} \to Y \), let \( V \) be an open neighborhood of \( p \) in \( Y \). Then \( V \cap X \in \mathcal{O}_p \) and \( \text{int}_x \text{cl}_x (V \cap X) \in \mathcal{P} \). Therefore, \( o_i(\text{int}_x \text{cl}_x (V \cap X)) \) is an open neighborhood of \( p \) in \( Y^{ad} \) such that
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\[ \text{cl}_x(a_i(int_x \text{ cl}_x(V \cap X)) = \text{cl}_x(a_i(int_x \text{ cl}_x(V \cap X)) \cap X) = \text{cl}_x(a_i(int_x \text{ cl}_x(V \cap X))) \subseteq \text{cl}_x(V). \] Hence \( i: Y^w \to Y \) is \( \theta \)-continuous.

Part (b): For all \( G \in \tau' \), \( o_x(G) = o_x(int_x \text{ cl}_x G) \in \tau_x \) shows that \( i: Y^w \to Y^{ow} \) is continuous. Therefore, \( i: Y^{ow} \to Y \) is irreducible and perfect. Let \( o_x(G) \) be a basic open neighborhood of \( p \) in \( Y^w \). Since \( o_x(G) \subseteq a_x(G), a_x(G) \) is an open neighborhood of \( p \) in \( Y^{ow} \) such that \( \text{cl}_w(a_x(G)) = \text{cl}_x(o_x(G)) \), establishing the \( \theta \)-continuity of \( i: Y^{ow} \to Y^w \).

Part (c): To prove the \( \theta \)-continuity of \( i: Y^{ow} \to Y^w \), let \( p \in Y \) and let \( o_x(G), G \in \tau' \) be a basic open neighborhood of \( p \) in \( Y^w \). Then, \( G \in \sigma_x \subseteq \tau' \) implies that \( a_i(int_x \text{ cl}_x G) \) is an open neighborhood of \( p \) in \( Y^{ow} \) such that \( \text{cl}_w(a_i(int_x \text{ cl}_x G)) \subseteq \text{cl}_x(o_x(G)) \).

Part (d): Let \( o_x(G) \) be a basic open neighborhood of \( p \) in \( Y' \). Then, \( o_x(G) \) is an open neighborhood of \( p \) in \( Y^w \) such that \( \text{cl}_x(o_x(G)) \subseteq \text{cl}_x(a_x(G)) \), establishing the \( \theta \)-continuity of \( i: Y^w \to Y' \).

Part (h): Let \( o_x(G) \) be a basic open neighborhood of \( p \) in \( Y^w \). Then, \( o_i(int_x \text{ cl}_x G) \) is an open neighborhood of \( p \) in \( Y^{ow} \) satisfying \( \text{cl}_w(o_i(int_x \text{ cl}_x G)) \subseteq \text{cl}_x(o_x(G)) \).

Part (l): Let \( p \in Y \) and let \( a_i(G), G \in \tau' \) be a basic open neighborhood of \( p \) in \( Y^{ow} \). Then, \( G \supseteq int_x \text{ cl}_x U \) for some \( U \in \sigma_x \). So, \( p \in o_x(U) \). Now, \( \text{cl}_x(o_x(U)) = \text{cl}_x(o_x(U) \cap X) = \text{cl}_x(U) \subseteq \text{cl}_w(U) \subseteq \text{cl}_w(a_i(G)) \).

We now summarize the results proved above in the following theorem.

**Theorem 2.5.** The spaces \( Y, Y^w, Y', Y^* \), and \( Y'' \), are pairwise \( \theta \)-homeomorphic. The spaces \( Y', Y^w, \) and \( Y^* \) are \( \theta \)-equivalent extensions of \( X \) with homeomorphic remainders.

It is well known that spaces \( Y \) and \( Z \) are \( \theta \)-homeomorphic if and only if their semiregularizations are homeomorphic. [11] Hence, we have the following corollary.

**Corollary 2.1.** Let \( (Y, \tau) \) be an extension of a space \( (X, \tau) \). Then, the spaces \( Y', Y^w, Y', Y^* \), and \( Y'' \) are pairwise homeomorphic. Moreover, \( Y', Y^w, \) and \( Y^* \) are equivalent extensions of \( X \).

**3. The Extensions \( Y'^w \) and \( Y'^* \).**

In this section, we define extensions \( Y'^w \) and \( Y'^* \), analogous to the simple extension \( Y' \) of \( (X, \tau') \) induced by an extension \( (Y, \tau) \) of \( X \). The spaces \( Y'^w, Y'^w, Y^w, \) and \( Y^{ow} \) all have the same underlying set as the set \( Y \). An open base for the topology \( \tau'^w \) on \( Y'^w \) (respectively, \( \tau'^w \) on \( Y'^* \)) is the family \( \tau' \cup \{ G \cup \{ p \} : p \in Y \setminus X, G \in \sigma_y \} \) (respectively, \( \tau' \cup \{ G \cup \{ p \} : G \in \sigma_y \} \)). An open base for the topology \( \tau'^* \) on \( Y'^* \) (respectively, \( \tau'^* \) on \( Y^{ow} \)) is the family \( \tau' \cup \{ G \cup \{ p \} : p \in Y \setminus X, G \in \tau_x \} \) (respectively, \( \tau' \cup \{ G \cup \{ p \} : G \in \tau_x \} \)). For any \( A \subseteq Y, \text{cl}_x(A) \) will denote the closure of \( A \) in \( Y' \), with analogous notations in other cases. The proofs of the following statements are straightforward, and we omit the details. Obviously, the spaces \( Y'^w \setminus X, Y'^w \setminus X, Y'^w \setminus X, \) and \( Y^{ow} \setminus X \) are all discrete.

**Theorem 3.1.** The spaces \( Y'^w \) and \( Y'^w \) are extensions of \( (X, \tau') \) such that \( Y' \geq Y'^w \geq Y'^w \). The set \( X \) is dense in the spaces \( Y'^w \) and \( Y'^w \). But, \( Y'^w \) and \( Y'^w \) may not be extensions of \( X \).
Lemma 3.1. For each \( G \in \tau' \), \( cl_{\mu}(o(G)) = cl_{\lambda}(o(G)) \), and \( cl_{\lambda}(o(G)) = cl_{\mu}(o(G)) \).

Theorem 3.2. Each one of the identity maps \( i: Y \to Y'' \), and \( i: Y'' \to Y^* \) is \( \theta \) - continuous.

Theorem 3.3. The spaces \( Y' \), \( Y^* \), \( Y'' \), \( Y''' \), and \( Y^{'''} \) are \( \theta \) - homeomorphic. Moreover, \( Y' \), \( Y^* \), and \( Y'' \) are \( \theta \) - equivalent extensions of \( X \) with homeomorphic remainders.

Corollary 3.1. If \( (Y, \tau) \) is an extension of a space \( (X, \tau') \), then the spaces \( Y', Y^*, Y'' \), \( Y''' \), and \( Y^{'''} \) are homeomorphic in pairs. Moreover, the spaces \( Y' \), \( Y^* \), and \( Y'' \) are equivalent extensions of \( X \).

Remarks 3.1. (a) If \( P \) is any property of topological spaces which is preserved under \( \theta \) - continuous surjections, and if \( (Y, \tau) \) is a \( P \)-extension of \( (X, \tau') \), then \( Y', Y^*, Y'' \), and \( Y^{'''} \) are also \( P \)-extensions of \( X \).

(b) The extensions \( Y', Y^*, Y'' \), and \( Y^{'''} \) introduced above are, in general, all distinct from \( Y, Y^* \), and \( Y'' \). It would be interesting to find a characterization of spaces \( Y \) for which \( Y'' = Y' \). A space \( Z \) is called \( H \)-closed if it is closed in every Hausdorff space in which it is embedded [see 11 for more details]. The Katetov (respectively, Fomin) extension of a space \( (X, \tau') \) is the space \( \kappa X \) (respectively, \( \sigma X \)) whose underlying set is the set \( X \cup \{ p : p \) is a free open ultrafilter on \( X \} \), and whose topology has for an open base the family \( \tau' \cup \{ U \cup \{ p \} : U \in p, p \in \kappa X \setminus X \} \) (respectively, the family \( \{ o_{x}(U) : U \in \tau' \} \)). The spaces \( \kappa X \), and \( \sigma X \) are \( H \)-closed extensions of \( X \) such that \( (\sigma X)' = \kappa X \), and \( (\kappa X)^* = \sigma X \) [3, 6, 11]. In general, \( (\sigma X)' \neq \sigma X, (\kappa X)^* \neq \kappa X \), and \( (\kappa X)^* \neq \sigma X \). Analogous remarks apply to the Banaschewski-Fomin-Shanin extension \( \mu X \) [13] of a Hausdorff space \( X \).

(c) A space \( Z \) is called compact like, or nearly compact if every regular open cover of \( Z \) is reducible to a finite subcover. A space \( X \) has a compactlike extension if and only if \( X' \) is Tychonoff [14]. Compactlike extensions (=near compactifications) of Hausdorff almost completely regular spaces \( X \) (whence, \( X' \) is Tychonoff) have been constructed in [2] via EF-Proximities. For a Hausdorff space \( X \) whose semiregularization \( X' \) is Tychonoff, a maximal compactlike extension \( BX \) of \( X \), satisfying \( BX = \beta X' \), is constructed in [14]. If \( (X, \tau') \) is any Hausdorff almost completely regular space, and if \( (Y, \tau) \) is any near compactification of \( (X, \tau') \), then so are \( Y', Y^*, Y'' \), and \( Y^{'''} \).

(d) A space \( Z \) is called almost real compact if every open ultrafilter on \( Z \) with countable closed intersection property in \( Z \) converges in \( Z \). A space \( Z \) is almost realcompact if and only if \( Z' \) is almost realcompact [12]. Almost realcompactifications of a Hausdorff space have been constructed (among others) in [7], and [12]. If \( (X, \tau') \) is any Hausdorff space, and if \( (Y, \tau) \) is any almost realcompactification of \( (X, \tau') \), then so are \( Y', Y^*, Y'' \), and \( Y^{'''} \).

(e) A Hausdorff space \( Z \) is called extremally disconnected if for each open subset \( U \) of \( Z \), \( cl_{\mu}(U) \) is open. A space \( Z \) is extremally disconnected if and only if each dense subspace of \( Z \) [respectively, if and only if \( Z' \)] is extremally disconnected [see 11 for more details]. A Hausdorff space \( Z \) is called \( s \)-closed if it is \( H \)-closed and extremally disconnected [8]. A Hausdorff space \( Z \) is \( s \)-closed if and only if \( Z' \) is \( s \)-closed. It is shown in [8] that every extremally disconnected space \( X \) admits an \( s \)-closed extension, viz.
moreover, an extension $Y$ of $X$ is $s$-closed if and only if $X$ is $C^*$-embedded in $Y$. If $(X, \tau')$ is any extremally disconnected Hausdorff space, and if $(Y, \tau)$ is any $s$-closed extension of $(X, \tau')$, then so are $Y', Y^*, Y''$, and $Y'''$.

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