\section{Introduction}

\Gamma - Banach algebras and \( \alpha \) - derivations are generalisations of ordinary Banach algebras and derivations respectively. The set of all \( m \times n \) rectangular matrices and the set of all bounded linear transformations from an infinite dimensional normed linear space \( X \) into a Banach space \( Y \) are nice examples of \( \Gamma \)-Banach algebras which are not general Banach algebras. Similarly a derivation can't be defined on these spaces as there appears to be no natural way of introducing an algebraic multiplication into these. So, a new concept of derivation known as \( \alpha \)-derivation is introduced on a \( \Gamma \)-Banach algebra. Bhattacharya and Maity have defined a \( \Gamma \)-Banach algebra in their paper [1] and have discussed in their another paper [2] various tensor products of \( \Gamma \)-Banach algebras over fields which are isomorphic to another field with a real valued valuation by using semilinear transformations, [3]. Derivations and tensor products of general Banach algebras are discussed in many papers. [4, 5, 6, 7, 8]. Now there are some natural questions: Does every pair of derivations \( D_1 \) and \( D_2 \) on Gamma Banach algebras \( (V, \Gamma) \) and \( (V', \Gamma') \) respectively give rise to a derivation \( D \) on their projective tensor product? If yes, then can one estimate the norm of \( D \) with the help of norms of \( D_1 \) and \( D_2 \)? Can one evaluate the spectrum of \( D \) with the help of \( \sigma(\Gamma) \) and \( \sigma(\Gamma') \)?
of those of \( D_1 \) and \( D_2 \)? Are the converses of the above problems true? We give affirmative answers to some of these questions. The useful terminologies are forwarded below:

**DEFINITION 1.1.** Let \( X (F) \) and \( Y (F) \) be given normed linear spaces over fields \( F \) and \( F' \), which are isomorphic to a field \( F \) with a real valued valuation. (Refer to Backman's book [9]). If \( u = \sum (x_i \otimes y_i) \) is an element of the algebraic tensor product \( X \otimes Y \), then the projective norm \( p \) is defined by

\[
p(u) = \inf \left\{ \sum \| x_i \| \| y_i \| : x_i \in X, y_i \in Y \right\},
\]

where the infimum is taken over all finite representations of \( u \). Further the weak norm \( w \) on \( u \) is defined by

\[
w(u) = \sup \left\{ \| f(x_i) \| \| g(y_i) \| : f \in X^*, g \in Y^*, \| f \| \leq 1, \| g \| \leq 1 \right\}.
\]

[Here \( X^* \) and \( Y^* \) are respective dual spaces of \( X \) and \( Y \); and \( F_1, F_2 \) are isomorphic to \( F \) under isomorphisms \( \xi_1 \) and \( \xi_2 \). The projective tensor product \( X \otimes_p Y \) and the weak tensor product \( X \otimes_w Y \) are the completions of \( X \otimes Y \) with their respective norms. For details, see Bonsall and Duncan's book [10].]

**DEFINITION 1.2.** Let \( (V, F) \) be a \( F \)-Banach algebra and \( a, a' \) be fixed elements of \( F \). Then \( a \)-identity, \( l_a \), is an element of \( V \) satisfying the conditions \( xa l_a = x \) and \( l_a ax = x \) for every \( x \) in \( V \).

**DEFINITION 1.3.** A linear operator \( D \) of \( (V, F) \) into itself is called an \( \alpha \)-derivation if

\[
D(xa) = (Dx) a + xa(Dy), \quad x, y \in V.
\]

Every \( x \in V \) gives rise to an \( \alpha \)-derivation \( D_x \) defined by \( D_x(y) = xay - yax \). Such a derivation is called an \( \alpha \)-inner derivation. Further, if \( (V, \Gamma) \) is an involutive \( \Gamma \)-Banach algebra with an involution \( * \), then an \( \alpha \)-derivation \( D \) is called an \( \alpha \)-star-derivation if \( D(x^* y) = (Dx)^* y + x ay \) being the adjoint of \( x \). Again, we define an operation \( o \) by \( xoy = xay + yax, \) \( x, y \in V \). A linear map \( D \) on \( (V, \Gamma) \) is called an \( \alpha \)-Jordan derivation if \( D(xoy) = (Dx) o y + x o (Dy) \) for all \( x \) and \( y \) in \( V \).

2. **THE MAIN RESULTS**

Throughout our discussion, unless stated otherwise, \( (V, \Gamma) \) and \( (V', \Gamma') \) are Gamma-Banach algebras over \( F_1 \) and \( F_2 \), isomorphic to \( F \) which possesses a real valued valuation; \( \alpha \) and \( \alpha' \) are fixed elements of \( \Gamma \) and \( \Gamma' \); and \( l_{\alpha}, l_{\alpha}' \) are \( \alpha \)-identity and \( \alpha' \)-identity of \( V \) and \( V' \) respectively. Moreover, suppose that \( \| l_{\alpha} \| = k_1 \neq 0 \) and \( \| l_{\alpha}' \| = k_2 \neq 0 \).

The following proposition is fundamental for our purpose, and we refer to Bhattacharya and Maity [2] for its proof.

**PROPOSITION 2.1.** The projective tensor product \( (V, \Gamma) \otimes_p (V', \Gamma') \) with the projective norm is a \( \Gamma \otimes \Gamma' \)-Banach algebra over the field \( F \), where multiplication is defined by the formula

\[
(x \otimes y)(\gamma \otimes \delta) = (x \gamma x')(\delta^{-1} y'), \quad \text{where } x, y, \gamma, \delta \in V; \quad x', y', \gamma', \delta' \in V'; \quad \gamma \in \Gamma; \quad \delta \in \Gamma'.
\]

**THEOREM 2.1.** Let \( D_1 \) and \( D_2 \) be bounded \( \alpha \)-derivation and \( \alpha' \)-derivation on \( (V, \Gamma) \) and \( (V', \Gamma') \) respectively. Then

(i) there exists a bounded \( \alpha \otimes \alpha' \)-derivation \( D \) on the projective tensor product \( (V, \Gamma) \otimes_p (V', \Gamma') \) defined

...
by the relation

$$D(u) = \sum_i \left[ D_i x_i \otimes y_i + x_i \otimes (D_i y_i) \right],$$

for each vector $u = \sum x_i \otimes y_i (V, \Gamma) \otimes_p (V', \Gamma')$.

(ii) If $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- inner derivations implemented by the elements $r, s \in V$ and $s_1 \in V'$ respectively then $D$ is an $\alpha \otimes \alpha'$- inner derivation implemented by $r \otimes s' + s \otimes s'$.

(iii) If $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- Jordan derivations, then $D$ is an $\alpha \otimes \alpha'$- Jordan derivation.

(iv) If $(V, \Gamma)$ and $(V', \Gamma')$ are involutive Gamma-Banach algebras, and if $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- star derivations, then $D$ is $\alpha \otimes \alpha'$- star derivation.

**Proof.** (i) We define a map $D : (V, \Gamma) \otimes_p (V', \Gamma') \rightarrow (V, \Gamma) \otimes_p (V', \Gamma')$ by the rule

$$D(u) = \sum_i \left[ D_i x_i \otimes y_i + x_i \otimes (D_i y_i) \right].$$

Clearly, $D$ is well defined. Before establishing the linearity of $D$, we first aim at proving the boundedness of $D$. For any arbitrary element $u \in (V, \Gamma) \otimes_p (V', \Gamma')$ and $\epsilon > 0$, the definition of the projective norm provides a finite representation $\sum x_i \otimes y_i$, such that $\|u\|p + \epsilon \geq \sum \|x_i\| \|y_i\|$. Therefore, for this representation of $u$, we obtain

$$\|D(u)\|p \leq \sum_i \left( \|D_i x_i\| \|y_i\| + \|x_i\| \|D_i y_i\| \right).$$

Since a projective norm is a cross norm, we have

$$\leq \sum_i \left( \|D_i\| \|x_i\| \|y_i\| + \|x_i\| \|D_i\| \|y_i\| \right) = \sum_i \left( \|D_i\| \|x_i\| \|y_i\| + \|x_i\| \|D_i\| \|y_i\| \right).$$

Thus, $\|D(u)\|p \leq K (\|u\|p + \epsilon)$, where $K = \|D_1\| \|D_2\|$.

Therefore, we have

$$\|D(u)\|p \leq K (\|u\|p + \epsilon).$$

Since the left hand side is independent of $\epsilon$, and $\epsilon$ was arbitrary, it follows that $\|D(u)\|p \leq K \|u\|p$ for every $u \in (V, \Gamma) \otimes_p (V', \Gamma')$. Consequently, $D$ is bounded.

Next to establish the linearity, let $u = \sum x_i \otimes y_i$ and $v = \sum r_j \otimes s_j$ be any two elements of $(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u + v = \sum x_i \otimes y_i + \sum r_j \otimes s_j$, where $x_{ij} = r_j$ and $y_{ij} = s_j$, $i = 1, 2, \ldots, m$.

Now, $D(u + v) = D(\sum x_i \otimes y_i)$

$$= \sum_i \left[ D_i x_i \otimes y_i + x_i \otimes (D_i y_i) \right]$$

$$= \sum_i \left[ D_i x_i \otimes y_i + x_i \otimes (D_i y_i) \right] + \sum_j \left[ D_i r_j \otimes s_j + r_j \otimes (D_i s_j) \right] = D(u) + D(v).$$

The boundedness of $D$ implies that the result, $D(u + v) = D(u) + D(v)$, is also true for any infinite
representations of $u$ and $v$. Similarly it can be shown easily that $D(au) = aD(u)$ for any scalar $a$. Consequently $D$ is a bounded linear map.

To show that $D$ is an $\alpha \otimes \alpha'$-derivation, we suppose that $u = x \otimes y$ and $v = r \otimes s$ are any two elementary tensors of $(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u \alpha \otimes \alpha' v = x \alpha r \otimes y \alpha' s$. Now

$$D(u \alpha \otimes \alpha' v) = (D_1 x \alpha r) \otimes y \alpha' s + x \alpha r \otimes (D_2 y \alpha' s)$$

$$= \left[ \left( D_1 x \right) \alpha r + x \alpha (D_1 r) \right] \otimes y \alpha' s + x \alpha r \otimes \left[ \left( D_2 y \right) \alpha' s + y \alpha' (D_2 s) \right]$$

$$= \left[ \left( D_1 x \right) \alpha r \otimes y \alpha' s + x \alpha r \otimes (D_2 y) \alpha' s \right] + \left[ x \alpha (D_1 r) \otimes y \alpha' s + x \alpha r \otimes y \alpha' (D_2 s) \right]$$

$$= (Du) \alpha \otimes \alpha' v + u \alpha \otimes \alpha' (Dv).$$

Similarly, if $u = \sum x_i \otimes y_i$ and $v = \sum r_j \otimes s_j$ be any two elements of $(V, \Gamma) \otimes_p (V', \Gamma')$, then summing over $i$ and $j$ we can prove easily that $D(u \alpha \otimes \alpha' v) = (Du) \alpha \otimes \alpha' v + u \alpha \otimes \alpha' (Dv)$. So $D$ is an $\alpha \otimes \alpha'$-derivation.

(ii) Let $D_1$ and $D_2$ be $\alpha$- and $\alpha'$-inner derivations implemented by the vectors $r_\alpha$ and $s_\alpha$, respectively.

So, $D_1(x) = r_\alpha x - x \alpha r_\alpha, \forall x \in V$ and $D_2(y) = s_\alpha y - y \alpha s_\alpha, \forall y \in V'$.

Now, $D(u) = \sum_i \left[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \right]$

$$= \sum_i \left[ (r_\alpha x_i - x_i \alpha r_\alpha) \otimes y_i + x_i \otimes (s_\alpha y_i - y_i \alpha s_\alpha) \right]$$

$$= \sum_i \left[ x_i \alpha r_\alpha \otimes y_i - x_i \alpha r_\alpha \otimes y_i + x_i \alpha y_i - x_i \otimes y_i \alpha s_\alpha \right]$$

$$= \sum_i \left[ (r_\alpha \otimes 1_{x_i}) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (r_\alpha \otimes 1_{x_i}) \right]$$

$$+ \left[ (1_{x_i} \otimes s_\alpha) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (1_{x_i} \otimes s_\alpha) \right]$$

$$= \sum_i \left[ (r_\alpha \otimes 1_{x_i} + 1_{x_i} \otimes s_\alpha) (\alpha \otimes \alpha') (x_i \otimes y_i) - (x_i \otimes y_i) (\alpha \otimes \alpha') (r_\alpha \otimes 1_{x_i} + 1_{x_i} \otimes s_\alpha) \right]$$

$$= D_0(u), \text{ where } t_\alpha = r_\alpha \otimes 1_{x_i} + 1_{x_i} \otimes s_\alpha.$$

Consequently, $D$ is an $\alpha \otimes \alpha'$-inner derivation implemented by $t_\alpha$.

(iii) The proof is routine.

(iv) Let $D_1$ and $D_2$ be star derivations. If $u = \sum x_i \otimes y_i$ is an element of $(V, \Gamma) \otimes_p (V', \Gamma')$, then the adjoint of $u$ is given by $u^* = \sum x_i^* \otimes y_i^*$. Now,

$$Du^* = D(\sum x_i^* \otimes y_i^*)$$

$$= \sum_i \left[ D_1 x_i^* \otimes y_i^* + x_i^* \otimes D_2 y_i^* \right]$$

$$= \sum_i \left[ - (D_1 x_i) x_i^* \otimes y_i^* + x_i^* \otimes (-D_2 y_i^*) \right], \text{ because } D_1 \text{ and } D_2 \text{ are star derivation.}$$
α-DERIVATIONS AND THEIR NORMS IN TENSOR PRODUCTS

\[ \sum \left[ (D_x x', y') \otimes y + x' \otimes (D_y y') \right] = -(Du) \]

So, D is a star-derivation. Q.E.D.

**REMARK 2.1.** (i) The above theorem can be extended to the projective tensor product of n number of G- Banach algebras.

(ii) If \( u = x \otimes 1 \), \( \epsilon (V, G) \otimes_p (V', G') \), then from the definition of D, we get

\[
Du = D_1 x \otimes 1_x \quad \text{because } D_1 1_x = 0 \quad \ldots \quad (2.1)
\]

From this result, we can ascertain that for each derivation D on \( (V, G) \otimes_p (V', G') \), there may not exist derivations \( D_1 \) and \( D_2 \) on \( (V, G) \) and \( (V', G') \) respectively such that \( D_1 \) and \( D_2 \) are connected by the relation given in Theorem 2.1. For example, let \( D' \) be an \( \alpha \otimes \alpha' \)- inner derivation implemented by an element \( r \otimes s \), where \( s \) is not a scalar multiple of the identity element \( 1_x \). Then

\[
D' u = (r \otimes s) (\alpha \otimes \alpha') (x \otimes 1) = (\alpha \otimes \alpha') (r \otimes s) \quad \text{for every } u \in (V, G) \otimes_p (V', G'). \]

Now if \( u = x \otimes 1 \), then

\[
D' u = (r \otimes s) (\alpha \otimes \alpha') (x \otimes 1) - (x \otimes 1) (\alpha \otimes \alpha') (r \otimes s)
\]

\[
= r \alpha x \otimes s \alpha' 1_x - x \alpha \otimes 1_x \alpha' s = (r \alpha x - x \alpha r) \otimes s
\]

\[
= (D_1 x \otimes s) \quad \text{where } D_1 \text{ is a derivation on } (V, G) \text{ implemented by } r \quad \ldots \quad (2.2)
\]

From the results (2.1) and (2.2) we can conclude that unless \( s \) is a scalar multiple of the identity element \( 1_x \), \( D' (x \otimes 1_x) \) may not be of the form \( x \otimes 1_x \), where \( x \in V \), \([x \text{ may be different from } x] \). This implies that \( D' \) may not equal D in general. However, we have a converse of Theorem 2.1 as follows. Recall that an element \( x \in V \) is called an \( \alpha \)-idempotent element if \( \alpha \alpha x = x \).

**THEOREM 2.2.** The following results are true:

(i) If \( D \) is a derivation on \( (V, G) \otimes_p (V', G') \) such that \( D (\sum x_i \otimes y_i) = \sum x_i \otimes y_i + x, \epsilon V \) and \( y_i \)'s are \( \alpha' \)-idempotent elements of \( V' \), then there exists an \( \alpha' \)-derivation \( D_1 \) on \( V \) defined by the rule \( D_1 x \otimes y = D (x \otimes y) \) for all \( x \in V \) and for every \( \alpha' \)-idempotent element \( y \in V' \);

(ii) If \( D \) is bounded, so is \( D_1 \);

(iii) If \( D \) is an \( \alpha \otimes \alpha' \)-inner derivation implemented by an element \( w = \sum x_i \otimes y_i \), where \( y_i \)'s are \( \alpha' \)-idempotent elements, then \( D_1 \) is also an \( \alpha \)-inner derivation implemented by the element \( \sum x_i \);

(iv) If \( (V, G) \) and \( (V', G') \) are involutive Gamma-Banach algebras, and \( D \) is a star derivative, then so is \( D_1 \);

(v) If \( D \) is an \( \alpha \otimes \alpha' \)-Jordan derivation then \( D_1 \) is an \( \alpha \)-Jordan derivation;

(vi) If \( D \) is an \( \alpha \otimes \alpha' \)- derivation on \( (V, G) \otimes_p (V', G') \) such that \( D (\sum x_i \otimes y_i) = \sum x_i \otimes s_i \) for \( \alpha' \)-idempotent elements \( x_i \)'s in \( V \), and \( s_i \in V' \), then there exists an \( \alpha' \)- derivation \( D_2 \) on \( (V', G') \) given by the relation \( x \otimes D_2 y = D (x \otimes y) \) for every \( \alpha \)-idempotent element \( x \in V \) and for all elements \( y \in V' \). The above results (ii), (iii), (iv) and (v) are also true for \( D_2 \).

**PROOF.** (i) We define a map \( D_1 : V \rightarrow V \) by

\[ D_1 x \otimes y = D (x \otimes y), \quad \text{for all } x \in V \text{ and for every } \alpha' \text{-idempotent element } y \in V'. \]

Clearly, \( D_1 \) is well-defined. In particular, we have \( D_1 x \otimes 1_x = D (x \otimes 1_x), \forall x \in V \). We first establish the linearity of \( D_1 \). Let \( x_1, x_2 \in V \).
Then
\[ D_1 (x_1 + x_2) \otimes l_{a'} = D((x_1 + x_2) \otimes l_{a'}) \]
\[ = D (x_1 \otimes l_{a'} + x_2 \otimes l_{a'}) \]
\[ = D (x_1 \otimes l_{a'}) + D (x_2 \otimes l_{a'}) \]
\[ = (D_1 x_1 \otimes l_{a'}) + D_1 x_2 \otimes l_{a'} \]
\[ = (D_1 x_1 + D_1 x_2) \otimes l_{a'} \]

So, \((D_1 (x_1 + x_2) \otimes l_{a'}) (f, g) = ((D_1 x_1 + D_1 x_2) \otimes l_{a'}) (f, g)) \quad \forall f \in V', \forall g \in V".\]

This gives, \(f(D_1 (x_1 + x_2))g(\lambda) = f(D_1 x_1 + D_1 x_2)g(\lambda) \quad \forall f \in V', \forall g \in V".\]

The Hahn-Banach theorem provides a functional \(g_0 \in V"\) in such a way that \(g_0(1_\omega) = \|1_\omega\| = k_\omega.\)

Then, \(f(D_1 (x_1 + x_2)) = f(D_1 x_1 + D_1 x_2) \forall f \in V'\). This yields, \(D_1 (x_1 + x_2) = D_1 x_1 + D_1 x_2.\)

By appealing to the same mechanism, we can show that \(D_1 (ax) = aD_1 (x)\) for any scalar \(a\). So \(D_1\) is linear.

Next, to show that \(D_1\) is an \(\alpha\)-derivation.
\[ D_1 (x_1, \alpha x_2) \otimes l_{a'} = D (x_1, \alpha x_2 \otimes l_{a'}) \quad (x_1, x_2 \in V) \]
\[ = D \left[ (x_1 \otimes l_{a'}) (\alpha \otimes \alpha') (x_2 \otimes l_{a'}) \right] \]
\[ = (D (x_1 \otimes l_{a'})) (\alpha \otimes \alpha') (x_2 \otimes l_{a'}) + (x_1 \otimes l_{a'}) (\alpha \otimes \alpha') D (x_2 \otimes l_{a'}) \]
\[ = (D_1 x_1 \otimes l_{a'}) (\alpha \otimes \alpha') (x_2 \otimes l_{a'}) + (x_1 \otimes l_{a'}) (\alpha \otimes \alpha') (D_1 x_2 \otimes l_{a'}) \]
\[ = (D_1 x_1 \alpha x_2 \otimes l_{a'}) + (x_1 \alpha (D_1 x_2)) \otimes l_{a'} = \left[ (D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2) \right] \otimes l_{a'} \]

So, \(D_1 (x_1, \alpha x_2) = (D_1 x_1) \alpha x_2 + x_1 \alpha (D_1 x_2)\). Therefore, \(D_1\) is an \(\alpha\)-derivation. The rest of the results are routine.

3. THE NORM OF D

We now shift our attention to study the possibility of the result \(\|D\| = \|D_1\| + \|D_2\|\). when \(D\).

\(D_1 \) and \(D_2\) are related as in Theorem 2.1.

THEOREM 3.1. If \(D, D_1\) and \(D_2\) are related as in Theorem 2.1, then
\[ \|D\| \leq \|D_1\| + \|D_2\| \leq 2 \|D\|. \]

PROOF. For each \(u \in (V, \Gamma') \otimes_p (V', \Gamma')\) with \(\|u\|_{p'} = 1\) and for each \(\epsilon > 0\), \(\exists\) a (finite) representation
\[ u = \sum x_i \otimes y_i \text{ such that } \|u\|_{p'} + \epsilon \geq \sum \|x_i\| \|y_i\|. \]

Now, \(\|D\| = \sup_u \left\{ \|Du\|_{p'} : \|u\|_{p'} = 1 \right\}\).
\[\begin{align*}
\sup_u \left\{ \Sigma \left[ \| D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \|_p : \| u \|_p = 1 \right] \right\} \\
\leq \sup_u \left\{ \Sigma \left[ \| D_1 x_i \otimes y_i \|_p + \| x_i \otimes D_2 y_i \|_p : \| u \|_p = 1 \right] \right\} \\
= \sup_u \left\{ \Sigma \left[ \| D_1 x_i \|_1 \| y_i \|_1 + \| x_i \|_1 \| D_2 y_i \|_1 : \| u \|_p = 1 \right] \right\} \\
\leq \left( \| D_1 \| + \| D_2 \| \right) \sup_u \left\{ 1 + \varepsilon : \| u \|_p = 1 \right\} \\
= \left( \| D_1 \| + \| D_2 \| \right) (1 + \varepsilon)
\end{align*}\]

Since \(\varepsilon\) was arbitrary, it follows that \(\| D \| \leq \| D_1 \| + \| D_2 \|\) \hspace{1cm} (3.1)

Next, let \(x \in V\) be such that \(\| x \| = 1\). Then \(\| x/k_2 \otimes 1_{\alpha} \| = \| x/k_2 \| \| 1_{\alpha} \| = 1\).

Now, \(\| D \| = \sup_u \left\{ \| Du \|_p : \| u \|_p = 1 \right\}\)

\[\| D \| \geq \| D \left( x/k_2 \otimes 1_{\alpha} \right) \|_p = \| D_1 \left( x/k_2 \right) \otimes 1_{\alpha} \|_p \] (Since \(D_2 \left( 1_{\alpha} \right) = 0\) = \(\| D_1 \| \| x \|\)).

Thus, \(\| D_1 \| \leq \| D \|\) for every \(x \in V\) with \(\| x \| = 1\). This gives \(\| D_1 \| \leq \| D \|\). Similarly, we can prove that \(\| D_2 \| \leq \| D \|\). Hence, we have \(\| D_1 \| + \| D_2 \| \leq 2 \| D \|\) \hspace{1cm} (3.2)

The inequalities (3.1) and (3.2) together imply \(\| D \| \leq \| D_1 \| + \| D_2 \| \leq 2 \| D \|\). Q.E.D.

Our next question is - can one improve the above result - ? We illustrate the possibility with the help of examples:

Let \(V\) be the set of \(2 \times 3\) rectangular matrices and \(\Gamma\) be the set of all \(3 \times 2\) rectangular matrices with real (or complex) entries. Then \(V\) and \(\Gamma\) are Banach spaces under usual matrix addition, scalar multiplication, and the norm defined by \(\| A \|_w = \max \| a_{ij} \|\), where \(A = (a_{ij})\). Then \((V, \Gamma)\) is a \(\Gamma\) - Banach algebra.

Now the following result is true:

**THEOREM 3.2.** For a fixed \(\alpha \in \Gamma\), each \(\alpha\) - derivation on \(V\) is inner.

Since \(\alpha\) -derivations on a finite dimensional \(\Gamma\)-Banach algebra are all inner, the result follows immediately, see [10].

We show below with an example in the \(\Gamma\) - Banach algebra of \(2 \times 3\) rectangular matrices that the equality \(\| D \| = \| D_1 \| + \| D_2 \|\) holds.

**AN EXAMPLE 3.1.**

Let \(\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}\) be a fixed element in \(\Gamma\), and let \(D_{1a}\) and \(D_{2a}\) be two \(\alpha\) - derivations on \(V\) implemented by \(A_0\) and \(B_0\) respectively, where \(A_0 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}\) and \(B_0 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}\).

Now \(\| A_0 \| = 2\) and \(\| B_0 \| = 3\), and \(D_{1a}(A) = A_0 \alpha A = A_0 A_0\), \(\forall A \in V\).

Then \(\| D_{1a} A \| \leq 2 \| A_0 \| \| \alpha \| \| A \| = 2 \| A_0 \| \| A \|\), because \(\| \alpha \| = 1\).

Hence, \(\| D_{1a} \| \leq 2 \| A_0 \| = 2.2 = 4\). Next, suppose that \(X_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 \end{pmatrix}\). Then \(\| X_0 \| = 1\).
Also \( \| A_0 aX_0 - X_0 aA_0 \| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| = 4 \). Hence \( \| D_{1a} \| = 4 \).

Similarly we can show that \( \| D_{2a} \| = 6 \). So \( \| D_{1a} \| + \| D_{2a} \| = 4 + 6 = 10 \).

If \( D \) is the derivation defined by the relation as in Theorem 3.1, then we always have

\[
\| D \| \leq \| D_{1a} \| + \| D_{2a} \| = 10 . \tag{3.1}
\]

Next, consider the element \( u_o = e_i \otimes e_j \), where \( e_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Then \( \| u_o \|_p = 1 \).

Now, \( \| D \| \geq \| Du_o \|_p \)

\[
= \| D_{1a} e_i \otimes e_j + e_i \otimes D_{2a} e_j \|_p 
\geq \| D_{1a} e_i \otimes e_j + e_i \otimes D_{2a} e_j \|_w
\]

(because the projective norm is always greater than or equal to the weak norm)

\[
= \sup \left\{ \| f(D_{1a} e_i)g(e_j) + f(e_i)g(D_{2a} e_j) \| : f, g \in V^*, \| f \| = \| g \| = 1 \right\} . \tag{3.2}
\]

Again \( D_{1a} e_i = A_0 a e_i - e_i a A_0 \)

\[
= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\( D_{2a} e_i = B_0 a e_i - e_i a B_0 \)

\[
= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 & 0 \end{pmatrix}
\]

We know that if we define \( f_i(e_j) = 1 \) if \( i = j \) and \( 0 \) if \( i \neq j \), then \( \{ f_1, f_2, f_3, f_4, f_5, f_6 \} \) is a basis for \( V^* \).

In (3.2) put \( f = g = f_i \). Then we find that \( \| D \| \geq 10. \tag{3.3} \)

The inequalities (3.1) and (3.3) combinedly give \( \| D \| = 10 \). Hence \( \| D \| = \| D_{1a} \| + \| D_{2a} \| \)

ANOTHER EXAMPLE 3.2.

Next we wish to illustrate that the result in Theorem 3.1 cannot be improved in general. If we assume \( V \) and \( \Gamma \) represent the same set of all \( 2 \times 2 \) real matrices, then \( ( V, \Gamma ) \) is a particular \( \Gamma \) - Banach algebra with the usual operations. The ordinary identity matrix \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is the identity of \( ( V, \Gamma ) \) under multiplication.
If \( e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \) then \( \beta = \{ e_1, e_2, e_3, e_4 \} \) is the standard basis for \(( V, \Gamma )\). For a simple example, let \( D_1 \) and \( D_2 \) be derivations on \(( V, \Gamma )\) implemented by the matrices \( A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 4 & -7 \\ 0 & 2 \end{pmatrix} \) respectively. Then the matrix representations of \( D_1 \) and \( D_2 \) with respect to the basis \( \beta \) are respectively

\[
[D_1]_{\beta} = \begin{pmatrix} 0 & 0 & 3 & 0 \\ -3 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}
\]

and

\[
[D_2]_{\beta} = \begin{pmatrix} 0 & 0 & -7 & 0 \\ 7 & 2 & 0 & -7 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}
\]

So, \( \| D_1 \| = 3 \) and \( \| D_2 \| = 7 \). Again, \( \gamma = \{ e_i \otimes e_j : i, j = 1, 2, 3, 4 \} \) is a basis for \(( V, \Gamma ) \otimes_{\phi} ( V, \Gamma )\) and the matrix representation of \( D \) with respect to the basis \( \gamma \) is

\[
[D]_{\gamma} = \begin{pmatrix}
0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 2 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 7 & 3 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 2 & 0 & -7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Hence \( \| D \| = 7 \). Thus the strict inequality \( \| D \| < \| D_1 \| + \| D_2 \| < 2 \| D \| \) holds.

4. **THE SPECTRUM OF D**

We next devote to studying the validity of the result \( \text{sp} ( D ) = \text{sp} ( D_1 ) + \text{sp} ( D_2 ) \). Recall that \( \text{sp} ( D ) \) consists of all scalars \( \lambda_i \) such that \( D - \lambda_i I_1 \) is singular. Analogous definitions apply to \( \text{sp} ( D_1 ) \) and \( \text{sp} ( D_2 ) \). Further, for the singularity and invertibility of a rectangular matrix, see Joshi [11].

**THEOREM 4.1.** The derivations \( D, D_1 \) and \( D_2 \) are defined as in Theorem 2.1. Then

\[
\text{sp} ( D_1 ) + \text{sp} ( D_2 ) \subseteq \text{sp} ( D )
\]

**PROOF.** Let \( \lambda_i \in \text{sp} ( D_1 ) \) and \( \lambda_j \in \text{sp} ( D_2 ) \).

\[
\Rightarrow D_1 - \lambda_i I_1 \text{ and } D_2 - \lambda_j I_2 \text{ are singular}
\]

\[
\Rightarrow \exists \text{ nonzero vectors } x_o \in V \text{ and } y_o \in V' \text{ such that } (D_1 - \lambda_i I_1) x_o = 0 \text{ and } (D_2 - \lambda_j I_2) y_o = 0
\]

Now, \( x_o \otimes y_o \) is a non-zero element in \(( V, \Gamma ) \otimes_{\phi} ( V, \Gamma )\).
Again, $[D - (\lambda_1 + \lambda_2) I](x_0 \otimes y_0) = D(x_0 \otimes y_0) - (\lambda_1 + \lambda_2)(x_0 \otimes y_0)\]

$$= D_1 x_0 \otimes y_0 + x_0 \otimes D_2 y_0 - (\lambda_1 + \lambda_2)x_0 \otimes y_0$$

$$= (D_1 - \lambda_1 I_1)x_0 \otimes y_0 + x_0 \otimes (D_2 - \lambda_2 I_2)y_0 = 0$$

So, $D - (\lambda_1 + \lambda_2) I$ is singular and hence $\lambda_1 + \lambda_2 \notin \text{sp } D$. Thus, we obtain $\text{sp } (D_1) \cup \text{sp } (D_2) \subseteq \text{sp } (D)$. Q.E.D

**REMARK 4.1.** (i) We conjecture that the above result cannot be improved in general.

(ii) However, the equality holds in finite dimensional $\Gamma$-Banach algebras. For, if dim $\langle V, \Gamma' \rangle = m$, dim $\langle V', \Gamma' \rangle = n$, then dim $\langle (V, \Gamma) \otimes (V', \Gamma') \rangle = mn$. So, $\text{sp } (D_1)$, $\text{sp } (D_2)$ and $\text{sp } (D)$ have $m, n$ and $mn$ eigenvalues respectively. Again, $\text{sp } (D_1) \cup \text{sp } (D_2)$ gives mn values which are precisely the eigenvalues of D.

Further, we have the following illuminating result.

**THEOREM 4.2.** As usual, let $D_1$, $D_2$ and $D$ be derivations connected by the relation as in Theorem 2.1(i). If $(V, \Gamma)$ and $(V', \Gamma')$ are finite dimensional Gamma-Banach algebras, $D_1$ and $D_2$ are implemented by $r \in V$ and $s \in V'$ respectively, then

$$\text{sp } (D_1) = \{ a = \lambda - \mu | \lambda, \mu \in \text{sp } (r) \}$$

$$\text{sp } (D_2) = \{ b = \lambda' - \mu' | \lambda', \mu' \in \text{sp } (s) \}$$

and $\text{sp } (D) = \{ a + b | a \in \text{sp } (D_1), b \in \text{sp } (D_2) \}$.

**PROOF.** The first two results will follow from Proposition 9, §18, Ch2 in [10], and the last result will follow from Remark 4.1 (ii). Q.E.D

**REFERENCES**


