FUNCTIONAL EVOLUTION EQUATIONS WITH NONCONVEX LOWER SEMICONTINUOUS MULTIVALUED PERTURBATIONS

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ABSTRACT. In this paper we prove some existence theorems concerning the solutions and integral solution for functional (delay) evolution equations with nonconvex lower semicontinuous multivalued perturbations.

KEY WORDS AND PHRASES: Functional evolution equations, m-accretive operators, integral solutions

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1. INTRODUCTION

Let $E$ be a Banach space, $r, T \in \mathbb{R}^+$ and $I = [a, b]$. Let us denote $C_E([-r, T])$ the vector space of all continuous functions from $[-r, T]$ to $E$ endowed with the uniform topology.

For all $t \geq 0$, $s_t : C_E([-r, t]) \to C_E([-r, 0])$,

$$(s_t f)(\theta) = f(t + \theta), \quad \forall \theta \in [-r, 0].$$

$A : I \times E \to 2^E$ such that $A(t, \cdot)$ is an m-accretive multivalued operator

$P_{wc}(E)$ the family of nonempty weakly compact subsets of $E$

In this paper we are concerned with the following problems

1) Existence of solutions of the perturbated evolution equation with delay

$$(P) \begin{cases} u'(t) \in -A(t, u(t)) + F(t, s_t u) & \text{a.e. on } I, \\ u \equiv \psi & \text{on } [-r, 0] \end{cases}$$

where $F : I \times C_E([-r, 0]) \to P_{wc}(E)$ is a multivalued function such that $F(t, \cdot)$ is lower semicontinuous and $\psi \in C_E([-r, 0])$ is arbitrary but fixed.

2) Existence of solutions of the perturbated evolution equation with delay

$$(Q) \begin{cases} u'(t) \in -N_{\Gamma(t)}(u(t)) + F(t, s_t u) & \text{a.e. on } I, \\ u \equiv \psi & \text{on } [-r, 0] \end{cases}$$

where $N_{\Gamma(t)}(x)$ is the normal cone of the convex set $\Gamma(t)$ at the point $x \in E, t \in I$. It should be noticed that the problem $(Q)$ is not a special case of the problem $(P)$.

3) Existence of integral solutions of $(P)$, when the operator $A$ is independent of $t$, under conditions that are weaker than those imposed in $(P)$

The results obtained in the present paper generalized the following interesting known cases

Problem $(P)$ for which the dual of $E$ is uniformly convex, $A(t, \cdot)$ is an $m$-accretive single-valued operator and $F$ is a Lipschitz single-valued function cf Kartsatos and Parrott [1].
Problem (P) for which \( E \) is reflexive, \( A(t, \cdot) \) is an \( m \)-accretive multivalued operator and \( F \) is a Lipschitz single-valued function cf Tanaka [2]

Problems (P) and (Q) without delay cf Cichon [3], [4], Ibrahim [5] and the references therein

2. NOTATIONS AND DEFINITIONS

Let \( E^* \) be the dual of \( E \), \( E_0 \) the Banach space \( E \) endowed with the weak topology \( \sigma(E, E^*) \). If \( B \) is a multivalued operator from \( E \) to \( 2^E \) then \( B \) is said to be accretive if for each \( \lambda > 0 \), \( x_1, x_2 \in D(B) \) (the domain of \( B \)), \( y_1 \in B(x_1) \) and \( y_2 \in B(x_2) \) we have

\[
\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|.
\]

We say that \( B \) is \( m \)-accretive if \( B \) is accretive and if there exists \( \lambda > 0 \) such that \( R(I + \lambda B) = E \), where \( I \) is the identity map. It is known that if \( B \) is \( m \)-accretive, then for every \( \lambda > 0 \) the resolvent \( J_\lambda B = (I + \lambda B)^{-1} \) and the Yosida approximation of \( B \), \( B_\lambda = (I - J_\lambda B) / \lambda \), are defined everywhere. The generalized domain of \( B \) is defined by

\[
D'(B) = \left\{ x \in E : |B(x)| = \lim_{\lambda \to \infty} \|B_\lambda x\| < \infty \right\}.
\]

For the properties of \( m \)-accretive multivalued operators refer to [6] and [7]

Now we recall some concepts concerning multivalued functions. Let \( Y \) be a locally convex space and let \( G : 2^Y \to \mathcal{P} \) We say that \( G \) is lower semicontinuous (resp. upper semicontinuous) if for every open \( V \) in \( Y \) the set \( \{ x \in E : G(x) \cap V \neq \emptyset \} \) (resp. \( \{ x \in E : G(x) \subseteq V \} \) is open in \( E \). We say that \( G \) is lower semicontinuous (resp. upper semicontinuous) in the Kuratowski sense if for all \( v_n \to v \) in \( E \), \( G(v) \subseteq \lim_{n \to \infty} \inf_{n} G(v_n) \) (resp. \( \lim_{n \to \infty} \sup_{n} G(v_n) \subseteq G(v) \)), where

\[
\lim_{n \to \infty} \inf_{n} G(v_n) = \left\{ z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall n \geq 1 \right\},
\]

\[
\lim_{n \to \infty} \sup_{n} G(v_n) = \left\{ z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall k \geq 1 \right\}.
\]

If \( E \) is metrizable then lower semicontinuity and lower semicontinuity in the Kuratowski sense are equivalent (cf [8], [9])

The following known result will be used in the sequel

**Lemma 2.1** [6]. For every \( t \in I \), let \( A(t, \cdot) \) be an \( m \)-accretive multivalued operator from \( E \) to \( 2^E \) satisfying the following condition:

\( (C_1) \) There exist \( \lambda_0 > 0 \), a continuous function \( h : [0, \infty) \to [0, \infty) \) and a nondecreasing continuous function \( L : [0, \infty) \to [0, \infty) \) such that for all \( \lambda \in (0, \lambda_0) \) and for almost \( t, s \in I \),

\[
\|A_\lambda(t, x) - A_\lambda(s, x)\| \leq \|h(s) - h(t)\|L(\|x\|), \quad \forall x \in E.
\]

Then \( D^*(A(t, \cdot)) \) and \( \overline{D}(A(t, \cdot)) \) are independent of \( t \).

So if \( A \) is as in Lemma 2.1 we may write \( D^*(A) := D^*(A(t, \cdot)) \) and \( \overline{D}(A) := \overline{D}(A(t, \cdot)); t \in I \) respectively

**Lemma 2.2** [10]. Let \( E \) be a Banach space and \( M \) a compact metric space. If \( T \) is a lower semicontinuous multivalued function on \( M \) and with nonempty closed decomposable values in \( L^\infty_c(I) \), then \( T \) has a continuous selection.

3. EXISTENCE OF SOLUTIONS FOR THE PROBLEMS (P) AND (Q)

To prove our results we need the following lemmas

**Lemma 3.1.** Let \( \psi \) be an element of \( CE([-\tau, 0]) \) and \( \beta \) be a positive real number. The set
\[ \chi = \left\{ u \in C_E([-r, 0]) : u \equiv \psi \text{ on } [-r, 0] \text{ and } u(t) = \psi(0) + \int_0^t f(s) \, ds; f \in K_\beta \right\}, \]

is nonempty and convex, where \( K_\beta = \{ f \in L_E^1(I) : |f(t)| \leq \beta \text{ a.e. on } I \} \). If \( E \) is reflexive then \( \chi \) is compact subset of \( C_{E_\omega}([-r, T]) \) If, in addition, \( E \) is separable then \( \chi \) is metrizable.

**Proof.** It is obvious that \( \chi \) is nonempty, convex and equicontinuous and that the set \( \{ u(t) : u \in \chi \}; t \in I, \) is bounded. So, if \( E \) is reflexive then, \( \chi \) is relatively compact in \( C_{E_\omega}([-r, T]) \) by Ascoli's theorem. Let us verify that \( \chi \) is closed in \( C_{E_\omega}([-r, T]) \). Let \( (u_n) \) be a sequence in \( \chi \) converging to \( u \in C_{E_\omega}([-r, T]) \). Then \( u \equiv \psi \text{ on } [-r, 0] \text{ and for each } n \geq 1 \text{ there exists } f_n \in K_\beta \text{ such that } u_n(t) = \psi(0) + \int_0^t f_n(s) \, ds; t \in I \). Since \( E \) is reflexive, \( K_\beta \) is weakly compact in \( L_E^1(I) \). Hence, the sequence \( (f_n) \) has a subsequence, denoted again by \( (f_n) \), converging weakly to \( f \in K_\beta \). Then \( u(t) = \psi(0) + \int_0^t f(s) \, ds; t \in I \). This proves that \( \chi \) is closed in \( C_{E_\omega}([-r, T]) \). Now if \( E \) is separable then so is \( L_E^1(I) \). Consequently, \( K_\beta \) is metrizable. Since \( \chi \) is isomorphic to \( \{ \psi(0) \} \times K_\beta \), then \( \chi \) is metrizable.

**Lemma 3.2.** Let \( G \) be a multivalued function from \( E_\omega \) to the nonempty closed subsets of \( E \) such that \( G \) is lower semicontinuous in the Kuratowski sense. If \( (x_n) \) is a sequence converging to \( x \) in \( E_\omega \), then for every \( z \in E \),

\[ \lim_{n \to \infty} \sup_{x \in G(x_n)} d(z, G(x_n)) \leq d(z, \lim_{n \to \infty} \inf_{x \in G(x_n)} z) \leq d(z, G(x)). \]

**Proof.** Let \( y \in \lim_{n \to \infty} \inf_{x \in G(x_n)} G(x_n) \). Then there exists a sequence \( (y_n) \) such that \( y_n \in G(x_n); n \geq 1 \) and \( y_n \to y \) as \( n \to \infty \). For any \( z \in E \) we have

\[ \lim_{n \to \infty} \sup_{x \in G(x_n)} d(z, G(x_n)) \leq \lim_{n \to \infty} \sup_{x \in G(x_n)} \| x_n - y_n \| = \| z - y \|, \]

which proves the first inequality. The second inequality follows from the lower semicontinuity of \( G \).

**Theorem 3.1.** Let \( E \) be a reflexive separable Banach space. Let \( A(t,.) \); \( t \in I \) be an m-accretive multivalued operator from \( E \) to \( 2^E - \{ \phi \} \) satisfying condition (C1) together with the following conditions

\[(C_2) \text{ There exist } \mu > 0 \text{ such that for all } x \in E, \text{ the function } \omega_x : t \to (I + \mu A(t, .))^{-1} \text{ belongs to } L_E^1(I). \]

\[(C_3) \text{ For all } r > 0 \text{ there exists } \delta(r) > 0 \text{ such that for all } \lambda > 0 \text{ and all } x \in \bar{D}(A), \text{ with } \| x \| < r, \]

\[ \| J_x A(0, x) - x \| \leq \lambda \delta(r). \]

Let \( F \) be a measurable multivalued function from \( I \times C_E([-r, 0]) \) to \( P_{ac}(E) \) satisfying the following conditions

\[(F_1) \text{ There exists } \alpha > 0 \text{ such that } \]

\[ \sup \{ \| y \| : y \in F(t, u) \} \leq \alpha, \quad \forall (t, u) \in I \times C_E([-r, 0]). \]

\[(F_2) \text{ For all } t \in I, F(t, .) \text{ is lower semicontinuous in the sense of Kuratowski from } C_E([-r, 0]) \text{ to } E. \]

\[(F_3) \text{ For all } u \in C_E([-r, 0]) \text{ the multivalued function } t \to F(t, s_t u) \text{ admits a measurable selection.} \]

Then for every \( \psi \in C_E([-r, 0]) \) with \( \psi(0) \in D^*(A) \), the problem \((P)\) has a solution.

**Proof.** We split the proof into the following three steps

1. Let \( f \in K_\alpha = \{ g \in L_E^1(I) : \| g(t) \| \leq \alpha \text{ a.e. on } I \} \). Since \( A \) satisfies conditions (C1), (C2) and (C3), then by Theorem 4 of [5], there exists a unique absolutely continuous function \( u_f : I \to E \) such that

\[(i) \quad u_f(t) \in -A(t, u_f(t)) + f(t) \text{ a.e. on } I, \quad u_f(0) = \psi(0), \]

\[(ii) \quad \| u_f(t) \| \leq \beta_1 = (\alpha + 1)T + L(r) \sup_{t \in I} \| h(t) \| + \delta(r), \quad \forall t \in I, \text{ where } \]

\[ r = \alpha(1 + L(\| \psi(0) \|)) + |A(0, x_0)|, \]
(iii) the function $f \to u_f$ is continuous from $K_\alpha$ to $C_{E_\alpha}(I)$

(2) Set $\chi_1 = \left\{ u \in C_E([-r, T]) : u \equiv \psi$ on $[-r, 0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)\,ds, f \in K_\alpha \right\}$. By Lemma 3.1, $\chi_1$ is a compact subset of $C_0([-r, T])$ and is metrizable. Define a multivalued function $T_1$ on $\chi_1$ by $T_1(u) = \{ f \in K_\alpha : f(t) \in F(t, s_iu) \text{ a.e. on } I \}$ In this step we prove that $T_1$ has a continuous selection $V_1 : \chi_1 \to K_\alpha$. For this purpose, we show that $T_1$ satisfies the conditions of Lemma 2.2 Condition (F3) assures that the values of $T_1$ are nonempty Moreover, if $D$ is a measurable subset of $I$ and $g_1, g_2 \in T_1(u)$ for some $u \in \chi_1$, then the function $g = N_D g_1 + N_{I-D} g_2$ belongs to $T_1(u)$, where $N$ is the characteristic function. Then the values of $T_1$ are decomposable. It remains to prove that $T_1$ is lower semicontinuous. Since $\chi_1$ is compact metrizable in $C_{E_\alpha}([-r, T])$, it suffices to show that $T_1$ is lower semicontinuous in the Kuratowski sense. So, let $(u_n)$ be a sequence in $\chi_1$ converging to $u \in \chi_1$, with respect to the topology on $C_{E_\alpha}([-r, T])$ and let $g \in T_1(u)$ Since $F$ is measurable, then for all $n \geq 1$ the multivalued function

$$t \to B_n(t) = \{ z \in F(t, s_iu_n) : ||g(t) - z|| = d(g(t), F(t, s_iu_n)) \}$$

has a measurable selection $g_n : I \to E$. Thus, by Lemma 3.2, for all $t \in I$,

$$\lim_{n \to \infty} ||g(t) - g(t_n)|| \leq \limsup_{n \to \infty} d(g(t), F(t, s_iu_n)) \leq d(g(t), \inf_{n \to \infty} F(t, s_iu_n)) = d(g(t), F(t, s_iu)) = 0.$$

This means that $T_1$ is lower semicontinuous and hence there exists a continuous function $V_1 : \chi_1 \to K_\alpha$ such that $V_1(x) \in T_1(x), \forall x \in \chi_1$

(3) Define a function $\theta : \chi_1 \to \chi_1$ by $\theta(x) = u_f, f = V_1(x)$. By (iii) of the first step, $\theta$ is continuous. Hence, by Tichonoff's fixed point theorem, there exists $u \in \chi_1$ such that $u = u_f, f = V_1(u) \in T_1(u)$. This means that $u'(t) \in -A(t, u(t)) + f(t)$ and $f(t) \in F(t, s_iu)$ a.e. on $I$. The theorem is thus proved.

**THEOREM 3.2.** Let $H$ be a Hilbert space and $F$ be a measurable multivalued function from $I \times C_H([-r, 0])$ to $F_{ac}(H)$ satisfying conditions (F1), (F2) and (F3). Let $\Gamma$ be a multivalued function from $I$ to the family of nonempty closed convex subsets of $H$, with compact graph $G$ and satisfies the following conditions.

$(\Gamma_1)$ There exists $\gamma > 0$ such that $||x - \text{proj}_{\Gamma(t)}x|| \leq \gamma(\tau - t)$ for all $(t, x) \in G$ and all $t \in I, (t < \tau)$

$(\Gamma_2)$ The function $(t, x) \to \delta_\Gamma(t, x, \Gamma(t)) = \sup \{ (x, y) : y \in \Gamma(t) \}$ is lower semicontinuous on $I \times B_\sigma$, where $B_\sigma$ is the relative weak topology.

Then for all $\psi \in C_E([-r, 0])$ with $\psi(0) \in \Gamma(0)$, the problem (Q) has a solution.

**PROOF.** We split the proof into the following three steps.

(1) Let $f \in K_\alpha$. Since $\Gamma$ has a compact graph and satisfies conditions $(\Gamma_1)$ and $(\Gamma_2)$ then by Theorem 3.1 [11], there exists a unique absolutely continuous function $u_f : I \to H$ such that

(i) $u'(t) \in -N_{\Gamma(t)}(u(t)) + f(t)$ a.e. on $I$,

(ii) $u_f(0) = \psi(0), u_f(t) \in \Gamma(t), \forall t \in I$,

(iii) $||u_f(t)|| \leq \beta_2 = T(\gamma + \alpha), \forall t \in I$ and the function $f \to u_f$ is continuous from $K_\alpha$ to $C_{H_\alpha}$.

(2) Set $\chi_2 = \left\{ u \in C_H([-r, T]) : u = \psi$ on $[-r, 0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)\,ds, f \in K_\alpha \right\}$ and define a multivalued function $T_2$ on $\chi_2$ by $T_2(u) = \{ f \in K_\alpha : f(t) \in F(t, s_iu) \text{ a.e. on } I \}$. As in the second step of the proof of Theorem 3.1 we can show that $T_2$ has a continuous selection $V_2 : \chi_2 \to K_\alpha$.

(3) Define the function $\theta : \chi_2 \to \chi_2$ by $\theta(x) = u_f, f = V_2(x)$. As in the third step of the proof of Theorem 3.1, we can show that there exists a unique $u \in \chi_2$ such that $u = u_f, f \in T_2(u)$. Clearly $u$ is a solution of (Q).
4. EXISTENCE OF INTEGRAL SOLUTIONS FOR THE PROBLEM \( P \)
WHEN THE OPERATOR \( A \) IS INDEPENDENT OF TIME

In this section \( A \) denotes a multivalued operator from \( E \) to \( 2^E - \{ \phi \} \). Consider the evolution equation

\[
\begin{cases}
  u'(t) \in -A(u(t)) + f(t) & \text{a.e. on } I \\ 
  u(0) = x_0 \in D(A),
\end{cases}
\]

where \( f \in L^1_b(I) \). By an integral solution of \( P^* \) we mean a continuous function \( u : I \to \overline{D(A)} \) with \( u(0) = x_0 \) such that

\[
\|u(t) - z\| \leq \|u(s) - z\| + \int_s^t \|u(r) - z, f(r) - y\|_+ dr,
\]

for each \( z \in D(A) \), \( y \in A(z) \) and \( 0 \leq s \leq t \leq T \), where

\[
[x_1, x_2]_+ = \lim_{h \to 0^+} \frac{(\|x_1 + hx_2\| - \|x_1\|)}{h}, \forall x_1, x_2 \in E.
\]

It is known that [7] if \( A \) is an m-accretive operator then for each \((x_0, f) \in \overline{D(A)} \times L^1_b(I)\), the problem \((P^*)\) has a unique integral solution \( u_f \), such that the function \( f \to u_f \) is continuous. In this section we are concerned with the existence of integral solutions of the functional evolution equation

\[
\begin{cases}
  u'(t) \in -A(u(t)) + F(t, su) & \text{a.e. on } I \\
  u \equiv \psi & \text{on } [-r, 0],
\end{cases}
\]

where \( F \) is a multivalued function from \( I \times C_E([-r, 0]) \) to \( 2^E - \{ \phi \} \). \( S_t; t > 0 \) is the operator of translation defined in section 1 and \( \psi \) is a given function, belongs to \( C_E([-r, 0]) \) with \( \psi(0) = \overline{D(A)} \).

By an integral solution of \((P^{**})\) we mean a continuous function \( u : [-r, T] \to E \) with \( u \equiv \psi \) on \([-r, ]0 \), such that \( u \) is an integral solution of the evolution equation \( u'(t) = -A(u(t)) + f(t), u(0) = \psi(0) \), where \( f \in L^1_b(I) \) and \( f(t) \in F(t, su), a.e. \) on \( I \)

We say that the operator \( A : E \to 2^E - \{ \phi \} \) has the \((M)\)-property ([7], [12]) if for each \( x_0 \in D(A) \) and each uniformly integrable subset \( Q \) of \( L^1_b(I) \), the set \( \{u_g : g \in Q\} \) is a relatively compact subset of \( C_E(I) \) where \( u_g \) is the unique integral solution of the evolution equation \( u'(t) = -A(u(t)) + g(t), a.e. \) on \( I \); \( u(0) = x_0 \). It is well known that ([7], [12]) if the proper operator \(-A\) generates a compact semigroup (via Crandall-Liggett's exponential formula [3], [13]), then \( A \) has the property \((M)\)

**Theorem 4.1.** Let \( E \) be a Banach space and \( A \) an m-accretive multivalued operator from \( E \) to \( 2^E - \{ \phi \} \) having the \((M)\)-property. Let \( F \) be a measurable multivalued function from \( I \times C_E([-r, 0]) \) to the non-empty closed subsets of \( E \) satisfying the condition \((F_3)\) together with the following conditions

\[
(F_4) \text{ There exists a function } h \in L^1_b(I) \text{ such that } \\
\sup\{\|z\| : z \in F(t, u)\} \leq h(t), \forall (t, u) \in I \times C_E([-r, 0]).
\]

\[
(F_5) \text{ For all } t \in I, F(t, \cdot) : C_E([-r, 0]) \to E \text{ is lower semicontinuous in the Kuratowski sense.}
\]

Then for all \( \psi \in C_E([-r, 0]) \) with \( \psi(0) = \overline{D(A)} \), the problem \((P^{**})\) has an integral solution

**Proof.** Consider the set \( Q = \{f \in L^1_b(I) : \|f(t)\| \leq h(t) \text{ a.e. on } I\} \). One can easily show that \( Q \) is nonempty and uniformly integrable subset of \( L^1_b(I) \). As mentioned above, for each \( f \in Q \) there exists a unique continuous function \( u_f : I \to \overline{D(A)} \) such that \( u_f \) is the unique integral solution of the evolution equation \( u'(t) = A(u(t)) + f(t), u(0) = \psi(0) \) and the function \( f \to u_f \) is continuous from \( Q \) to \( C_E(I) \). Let \( \chi^* = \{u_f^* \in C_E([-r, T]) : f \in Q\} \), where \( u_f^* \equiv \psi \) on \([-r, 0] \) and \( u_f^* \equiv u_f \) on \( I \). Since \( a \) has the property \((M)\), \( \chi^* \) is compact in the metric space \( C_E([-r, T]) \). Now, define a multivalued function \( T \) on \( \chi^* \) by \( T(x) = \{f \in L^1_b(I) : f(t) \in F(t, su) \text{ a.e. on } I\} \). As in the second step of the proof of Theorem 3.1, we can show that \( T \) has a continuous selection \( V : \chi^* \to L^1_b(I) \).
Also, define a function $\Phi : \mathcal{X}^* \to \mathcal{X}^*, \Phi(x) = u^*_f, f = V(x)$ The function $\Phi$ is clearly continuous and hence has a fixed point $x \in \mathcal{X}^*$. It is obvious that $x$ is the desired solution.

5. EXAMPLES

In this section we give some examples illustrating the scope of the results developed in sections 3 and 4.

**EXAMPLE 1.** Let for all $t \in I$, $A(t) = B - h(t)$ where $h : I \to E$ is integrable and $B$ is an $m$-accretive operator on $E$. Clearly $A(t)$ is $m$-accretive for all $t \in I$. Let $\lambda > 0, s, t \in I$ and $x \in E$. Then

$$\|A_\lambda(t, x) - A_\lambda(s, x)\| \leq \frac{1}{\lambda} \|J_\lambda A(t, x) - J_\lambda A(s, x)\| \leq \|h(t) - h(s)\|.$$ 

Hence condition (C1) of Lemma 2.1 holds.

**EXAMPLE 2.** In [6] there are several examples for operators $A$ such that for every $t \in I$, $A(t)$ is $m$-accretive and satisfies condition (C1).

**EXAMPLE 3.** Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\Phi : H \to H$ be a proper lower semicontinuous convex function. The set $\partial \Phi(x) = \{z \in H : \Phi(x) \leq \Phi(y) + \langle x - y, z \rangle\}$ for each $y \in H$ is called the subdifferential of $\Phi$ at the point $x$. We recall that $D(\partial \Phi) = \{x \in H : \partial \Phi(x) \text{ is nonempty}\}$. Now if we define an operator $A : D(A) = D(\partial \Phi) \to 2^H$ by $A(x) = \partial \Phi(x)$, then $A$ is $m$-accretive and the following conditions are equivalent [7]:

(i) For each $\lambda > 0$, the resolvent $J_\lambda A$ is a compact operator.

(ii) The function $\Phi$ is of compact type.

(iii) The semigroup generated by the operator $-A$ is compact.

**EXAMPLE 4.** Take $E = L^p([0, \pi])$ and let us define $A : D(A) \subseteq E \to E$ by $Au = -u^{(2)}(t)$ for each $u \in D(A)$ where $D(A) = \{u \in E : u^{(2)} \in E, u(0) = u(\pi) = 0\}$. The operator $A$ is $m$-accretive and the semigroup $\{S(t) : t > 0\}$ generated by $-A(S(t) = \lim_{n \to \infty} (I + tA)^{-n})$ is compact [7].

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