SUBCONTRA-CONTINUOUS FUNCTIONS

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ABSTRACT. A weak form of contra-continuity, called subcontra-continuity, is introduced. It is shown that subcontra-continuity is strictly weaker than contra-continuity and stronger than both subweak continuity and sub-LC-continuity. Subcontra-continuity is used to improve several results in the literature concerning compact spaces.

KEY WORDS AND PHRASES: subcontra-continuity, contra-continuity, subweak continuity, sub-LC-continuity.

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1. INTRODUCTION

In [1] Dontchev introduced the notion of a contra-continuous function. In this note we develop a weak form of contra-continuity, which we call subcontra-continuity. We show that subcontra-continuity implies both subweak continuity and sub-LC-continuity. We also establish some of the properties of subcontra-continuous functions. In particular it is shown that the graph of a subcontra-continuous function into a $T_1$-space is closed. Finally, we show that many of the applications of contra-continuous functions to compact spaces established by Dontchev [1] hold for subcontra-continuous functions. For example, we establish that the subcontra-continuous, nearly continuous image of an almost compact space is compact and that the subcontra-continuous, $\beta$-continuous image of an $S$-closed space is compact.

2. PRELIMINARIES

The symbols $X$ and $Y$ denote topological spaces with no separation axioms assumed unless explicitly stated. The closure and interior of a subset $A$ of a space $X$ are signified by $Cl(A)$ and $Int(A)$, respectively. A set $A$ is regular open (semi-open, nearly open) provided that $A = Int(Cl(A)) \subseteq Cl(Int(A))$ and $A$ is regular closed (semi-closed) if its complement is regular open (semi-open). A set $A$ is locally closed provided that $A = U \cap F$, where $U$ is an open set and $F$ is a closed set.

**DEFINITION 1.** Dontchev [1]. A function $f : X \to Y$ is said to contra-continuous provided that for every open set $V$ in $Y$, $f^{-1}(V)$ is closed in $X$.

**DEFINITION 2.** Rose [2]. A function $f : X \to Y$ is said to be subweakly continuous if there is an open base $B$ for the topology on $Y$ such that $Cl(f^{-1}(V)) \subseteq f^{-1}(Cl(V))$ for every $V \in B$. 


DEFINITION 3. Ganster and Reilly [3]. A function \( f : X \rightarrow Y \) is said to be sub-LC-continuous provided there is an open base \( B \) for the topology on \( Y \) such that \( f^{-1}(V) \) is locally closed for every \( V \in B \).

DEFINITION 4. A function \( f : X \rightarrow Y \) is said to be semi-continuous (Levine [4]) (nearly continuous (Ptk [5]), \( \beta \)-continuous (Abd El-Monsef et al. [6])) if for every open set \( V \) in \( Y \), 
\[
\overline{f^{-1}(V)} \subseteq \overline{\text{Int}(f^{-1}(V))} \quad (f^{-1}(V) \subseteq \text{Int}(\overline{f^{-1}(V)}), \quad f^{-1}(V) \subseteq \overline{\text{Int}(\overline{f^{-1}(V)})}).
\]

DEFINITION 5. Gentry and Hoyle [7]. A function \( f : X \rightarrow Y \) is said to be \( c \)-continuous if, for every \( x \in X \) and every open set \( V \) in \( Y \) containing \( f(x) \) and with compact complement, there exists an open set \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \).

3. SUBCONTRA-CONTINUOUS FUNCTIONS

We define a function \( f : X \rightarrow Y \) to be subcontra-continuous provided there exists an open base \( B \) for the topology on \( Y \) such that \( f^{-1}(V) \) is closed in \( X \) for every \( V \in B \). Obviously contra-continuity implies subcontra-continuity. The following example shows that the reverse implication does not hold.

EXAMPLE 1. Let \( X \) be a nondiscrete \( T_1 \)-space and let \( Y \) be the set \( X \) with the discrete topology. Finally let \( f : X \rightarrow Y \) be the identity mapping. If \( B \) is the collection of all singleton subsets of \( Y \), then \( B \) is an open base for the topology on \( Y \). Since \( X \) is \( T_1 \), \( f \) is subcontra-continuous with respect to \( B \). Obviously \( f \) is not contra-continuous.

Subcontra-continuity is independent of continuity. The function in Example 1 is subcontra-continuous but not continuous. The next example shows that continuity does not imply subcontra-continuity.

EXAMPLE 2. Let \( X = \{a, b\} \) be the Sierpinski space with the topology \( T = \{X, \emptyset, \{a\}\} \) and let \( f : X \rightarrow X \) be the identity mapping. Obviously \( f \) is continuous. However, any open base for the topology on \( X \) must contain \( \{a\} \) and \( f^{-1}(\{a\}) \) is not closed. It follows that \( f \) is not subcontra-continuous.

Since closed sets are locally closed, subcontra-continuity implies sub-LC-continuity. We see from the following theorem that subcontra-continuity also implies subweak continuity.

THEOREM 1. Every subcontra-continuous function is subweakly continuous.

PROOF. Assume \( f : X \rightarrow Y \) is subcontra-continuous. Let \( B \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in B \). Then for \( V \in B \), 
\[
\overline{f^{-1}(V)} = \overline{f^{-1}(\overline{f^{-1}(V)})} \subseteq f^{-1}(\overline{f^{-1}(V)}) \quad \text{and hence } f \text{ is subweakly continuous.} \]

Since a subweakly continuous function into a Hausdorff space has a closed graph (Baker [8]), a subcontra-continuous function into a Hausdorff space has a closed graph. However, the following stronger result holds for subcontra-continuous functions.

THEOREM 2. If \( f : X \rightarrow Y \) is a subcontra-continuous function and \( Y \) is \( T_1 \), then the graph of \( f \), \( G(f) \), is closed.

PROOF. Let \( (x, y) \in X \times Y - G(f) \). Then \( y \neq f(x) \). Let \( B \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in B \). Since \( Y \) is \( T_1 \), there exists \( V \in B \) such that \( y \in V \) and \( f(x) \notin V \). Then we see that \( (x, y) \in (X - f^{-1}(V)) \times V \subseteq X \times Y - G(f) \). It follows that \( G(f) \) is closed.

COROLLARY 1. If \( f : X \rightarrow Y \) is contra-continuous and \( Y \) is \( T_1 \), then the graph of \( f \) is closed.

Long and Hendrix [9] proved that the closed graph property implies \( c \)-continuity. Therefore we have the following corollary.

COROLLARY 2. If \( f : X \rightarrow Y \) is subcontra-continuous and \( Y \) is \( T_1 \), then \( f \) is \( c \)-continuous.
The next two results are also implied by the closed graph property (Fuller [10]).

**COROLLARY 3.** If \( f : X \to Y \) is subcontra-continuous and \( Y \) is \( T_1 \), then for every compact subset \( C \) of \( Y \), \( f^{-1}(C) \) is closed in \( X \).

**COROLLARY 4.** If \( f : X \to Y \) is subcontra-continuous and \( Y \) is \( T_1 \), then for every compact subset \( C \) of \( X \), \( f(C) \) is closed.

For a function \( f : X \to Y \), the graph function of \( f \) is the function \( g : X \times X \times Y \) given by \( g(x) = (x, f(x)) \). We shall see in the following example that the graph function of a subcontra-continuous function is not necessarily subcontra-continuous.

**EXAMPLE 3.** Let \( X = \{a, b\} \) be the Sierpinski space with the topology \( T = \{X, \emptyset, \{a\}\} \) and let \( f : X \to X \) be given by \( f(a) = b \) and \( f(b) = a \). Obviously \( f \) is subcontra-continuous, in fact contra-continuous. Let \( \mathcal{B} \) be any open base for the product topology on \( X \times X \). Then there exists \( V \in \mathcal{B} \) for which \( (a, b) \in V \subseteq \{(a, a), (a, b)\} \). We see that \( V = \{(a, a), (a, b)\} \) and that, if \( g : X \times X \) is the graph function for \( f \), then \( g^{-1}(V) = \{a\} \) which is not closed. Thus the graph function of \( f \) is not subcontra-continuous.

However, the following result does hold for the graph function.

**THEOREM 3.** The graph function of a subcontra-continuous function is sub-LC-continuous.

**PROOF.** Assume \( f : X \to Y \) is subcontra-continuous and let \( g : X \times X \times Y \) be the graph function of \( f \). Let \( \mathcal{B} \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in \mathcal{B} \). Then \( \{U \times V : U \text{ is open in } X, V \in \mathcal{B}\} \) is an open base for the product topology on \( X \times Y \).

Since \( g^{-1}(U \times V) = U \cap f^{-1}(V) \), we see that \( g \) is sub-LC-continuous.

The graph function of a subweakly continuous function is subweakly continuous (Baker [8]) and the graph function of a sub-LC-continuous function is sub-LC-continuous (Ganster and Reilly [3]). It follows that the graph function in Example 3 is subweakly continuous and sub-LC-continuous but not subcontra-continuous. Therefore subcontra-continuity is strictly stronger than sub-LC-continuity and subweak continuity.

**THEOREM 4.** If \( Y \) is a \( T_1 \)-space and \( f : X \to Y \) is a subcontra-continuous injection, then \( X \) is \( T_1 \).

**PROOF.** Let \( x_1 \) and \( x_2 \) be distinct points in \( X \). Let \( \mathcal{B} \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in \mathcal{B} \). Since \( Y \) is \( T_1 \) and \( f(x_1) \neq f(x_2) \), there exists \( V \in \mathcal{B} \) such that \( f(x_1) \notin V \) and \( f(x_2) \in V \). Then \( x_1 \in X - f^{-1}(V) \) which is open and \( x_2 \notin X - f^{-1}(V) \).

**THEOREM 5.** Let \( A \subseteq X \) and \( f : X \to X \) be a subcontra-continuous function such that \( f(X) = A \) and \( f|_A \) is the identity on \( A \). Then, if \( X \) is \( T_1 \), \( A \) is closed in \( X \).

**PROOF.** Suppose \( A \) is not closed. Let \( x \in \text{Cl}(A) - A \). Let \( \mathcal{B} \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed for every \( V \in \mathcal{B} \). Since \( x \notin A \), we have that \( x \neq f(x) \).

Since \( X \) is \( T_1 \), there exists \( V \in \mathcal{B} \) such that \( x \in V \) and \( f(x) \notin V \). Let \( U \) be an open set containing \( x \). Then \( x \in U \cap V \) which is open. Since \( x \in \text{Cl}(A) \), \( (U \cap V) \cap A \neq \emptyset \). Let \( y \in (U \cap V) \cap A \). Since \( y \in A \), \( f(y) = y \in V \). So \( y \in f^{-1}(V) \). Thus \( y \in U \cap f^{-1}(V) \) and hence \( U \cap f^{-1}(V) \neq \emptyset \). We see that \( x \in \text{Cl}(f^{-1}(V)) = f^{-1}(V) \) which is a contradiction. Therefore \( A \) is closed.

The next result follows easily for the definition.

**THEOREM 6.** If \( f : X \to Y \) is subcontra-continuous, then for every open set \( V \) in \( Y \), \( f^{-1}(V) \) is a union of closed sets in \( X \).

Obviously every function with a \( T_1 \)-domain satisfies the above condition. However, as we see in the following example, a function with a \( T_1 \)-domain can fail to be subcontra-continuous. It follows that the converse of Theorem 6 does not hold.
EXAMPLE 4. Let \( X = \mathbb{R} \) with the usual topology and let \( f : X \to X \) be the identity mapping. Since \( X \) is connected, \( f \) is not subcontra-continuous. However, since \( X \) is \( T_1 \), \( f \) has the property that the inverse image of every (open) set is a union of closed sets.

4. APPLICATIONS TO COMPACT SPACES

In [1] Dontchev establishes that the image of an almost compact space under a contra-continuous, nearly continuous mapping is compact and that the contra-continuous image of a strongly S-closed space is compact. In this section, we strengthen both of these results by replacing contra-continuity with subcontra-continuity. The proofs mostly follow Dontchev's.

DEFINITION 6. Dontchev [1]. A space \( X \) is almost compact provided that every open cover of \( X \) has a finite subfamily the closures of whose members cover \( X \).

THEOREM 7. The image of an almost compact space under a subcontra-continuous, nearly continuous mapping is compact.

PROOF. Let \( f : X \to Y \) be subcontra-continuous and nearly continuous and assume that \( X \) is almost compact. Let \( B \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) closed in \( X \) for every \( V \in B \). Let \( C \) be an open cover of \( f(X) \). For each \( x \in X \), let \( C_x \in C \) such that \( f(x) \in C_x \). Then let \( V_x \in B \) for which \( f(x) \in V_x \subseteq C_x \). Now \( f^{-1}(V_x) \) is closed and nearly open. It follows that \( f^{-1}(V_x) \) is clopen and hence that \( \{ f^{-1}(V_x) : x \in X \} \) is a clopen cover of \( X \). Since \( X \) is almost compact, there is a finite subfamily \( \{ f^{-1}(V_{x_i}) : i = 1, \ldots, n \} \) for which \( X = \bigcup_{i=1}^{n} C(f^{-1}(V_{x_i})) = \bigcup_{i=1}^{n} f^{-1}(V_{x_i}) \subseteq \bigcup_{i=1}^{n} f^{-1}(C_{x_i}) \). Thus we have that \( f(X) \subseteq \bigcup_{i=1}^{n} C_{x_i} \) and therefore that \( f(X) \) is compact. \( \square \)

DEFINITION 7. Dontchev [1]. A space \( X \) is strongly S-closed provided that every closed cover of \( X \) has a finite subcover of \( X \) has a finite subcover.

THEOREM 8. The subcontra-continuous image of a strongly S-closed space is compact.

PROOF. Let \( f : X \to Y \) be subcontra-continuous and assume that \( X \) is strongly S-closed. Let \( B \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in B \). Let \( C \) be an open cover of \( f(X) \). For each \( x \in X \), let \( C_x \in C \) with \( f(x) \in C_x \). Then let \( V_x \in B \) for which \( f(x) \in V_x \subseteq C_x \). Since \( \{ f^{-1}(V_x) : x \in X \} \) is a closed cover of \( X \) and \( X \) is strongly S-closed, there is a finite subcover \( \{ f^{-1}(V_{x_i}) : i = 1, \ldots, n \} \) of \( X \). Then we see that \( f(X) = f \left( \bigcup_{i=1}^{n} f^{-1}(V_{x_i}) \right) = \bigcup_{i=1}^{n} f(f^{-1}(V_{x_i})) \subseteq \bigcup_{i=1}^{n} V_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i} \) and hence that \( f(X) \) is compact. \( \square \)

In [1] Dontchev also shows that the contra-continuous, \( \beta \)-continuous image of an S-closed space is compact. We extend this result by replacing contra-continuity with subcontra-continuity. The proof parallels that of Dontchev's.

DEFINITION 8. Mukherjee and Basu [11]. A space \( X \) is S-closed provided that every semi-open cover of \( X \) has a finite subfamily the closures of whose members covers \( X \).

From Herrmann [12], a space \( X \) is S-closed if and only if every regular closed cover of \( X \) has a finite subcover.

THEOREM 9. The subcontra-continuous, \( \beta \)-continuous image of an S-closed space is compact.

PROOF. Assume that \( f : X \to Y \) is subcontra-continuous and \( \beta \)-continuous and that \( X \) is S-closed. Let \( B \) be an open base for the topology on \( Y \) for which \( f^{-1}(V) \) is closed in \( X \) for every \( V \in B \). Let \( C \) be an open cover of \( f(X) \). Then for each \( x \in X \) there exists \( C_x \in C \) for which \( f(x) \in C_x \). For each \( x \in X \), let \( V_x \in B \) such that \( f(x) \in V_x \subseteq C_x \). Since \( f \) is subcontra-continuous, \( \{ f^{-1}(V_x) : x \in X \} \) is a closed cover of \( X \). The \( \beta \)-continuity of \( f \) implies that \( f^{-1}(V_x) \subseteq Cl(Int(Cl(f^{-1}(V_x)))) \) and therefore we see that \( f^{-1}(V_x) = Cl(Int(f^{-1}(V_x))) \) or that
$f^{-1}(V_x)$ is regular closed. Since $X$ is $S$-closed, the regular closed cover $\{f^{-1}(V_x) : x \in X\}$ has a finite subcover $\{f^{-1}(V_{x_i}) : i = 1, \ldots, n\}$. Then we have $f(X) = f\left(\bigcup_{i=1}^{n} f^{-1}(V_{x_i})\right) \subseteq \bigcup_{i=1}^{n} V_{x_i} \subseteq \bigcup_{i=1}^{n} C_{x_i}$, and therefore $f(X)$ is compact. □

In the above proof we showed that, if $f : X \to Y$ is subcontra-continuous and $\beta$-continuous, then there exists an open base $B$ for the topology on $Y$ such that for every $V \in B$, $f^{-1}(V)$ is regular closed and hence semi-open. Since unions of semi-open sets are semi-open (Arya and Bhamini [13]), it follows that inverse images of open sets are semi-open. Therefore we have the following theorem which strengthens the corresponding result for contra-continuous functions established by Dontchev [1].

**THEOREM 10.** Every subcontra-continuous, $\beta$-continuous function is semi-continuous.

**REFERENCES**