ASYMPTOTIC EQUIVALENCE OF SEQUENCES AND SUMMABILITY

JINLU LI
Department of Mathematics
Shawnee State University
Portsmouth, OH 45662

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ABSTRACT: For a sequence-to-sequence transformation $A$, let $R_m Ax = \sum_{n \geq m} |(Ax)_n|$ and $\mu_m Ax = \sup_{n \geq m} |(Ax)_n|$. The purpose of this paper is to study the relationship between the asymptotic equivalence of two sequences $(\lim_n x_n/y_n = 1)$ and the variations of asymptotic equivalence based on the ratios $R_m Ax/R_m Ay$ and $\mu_m Ax/\mu_m Ay$.

KEY WORDS: Asymptotically regular, Asymptotic equivalence.

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1. INTRODUCTION.

Let $x = (x)_n$ and $y = (y)_n$ be infinite sequences, and let $A$ be a sequence-to-sequence transformation. We write $x \sim y$ if $\lim_n x_n/y_n = 1$. In order to compare rates of convergence of sequences, in [2] Pobyvanets introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative sequences, that is $x \sim y$ implies $Ax \sim Ay$. Furthermore, in [1] Fridy introduced new ways to compare rates by using the ratios $R_m x/R_m y$, $\mu_m x/\mu_m y$ when they tend to zero. In [2] Marouf studied the relationship of these ratios when they have limit one. In the present study we investigate some further properties involved with the ratios such $\mu Ax/\mu Ay$, $RAx/RAy$ when they have limit one.

2. NOTATIONS AND BASIC THEOREMS.

For a summability transformation $A$, we use $DA$ to denote the domain of $A$:

$$DA = \{x : \sum_{k=0}^{\infty} a_n x_k \text{ converges for such } n \geq 0\}$$

and $CA$ to denote the summability field:

$$CA = \{x : x \in DA, \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n x_k \text{ converges.}\}$$

Also

$$P_\delta = \{x : x_n \geq \delta > 0 \text{ for all } n\}$$

and

$$P = \{x : x_n > 0 \text{ for all } n\}$$

For a sequence $x$ in $\ell^1$ or $\ell^\infty$, we also define $R_m x = \sum_{n \geq m} |x_n|$ and $\mu_m x = \sup_{n \geq m} |x_n|$ for $m \geq 0$.

We list the following results without proof.
THEOREM 1. (Pobyvanets [2]). A nonnegative matrix $A$ is asymptotically regular if and only if for each fixed integer $m$, $\lim_{n \to \infty} a_{nm}/\sum_{k=0}^{\infty} a_{nk} = 0$.

THEOREM 2. A matrix $A$ is a $c_0 - c_0$ matrix (i.e. $A$ preserves zero limits) if and only if

(a) $\lim_{n \to \infty} a_{nk} = 0$ for $k = 0, 1, 2, \ldots$

(b) There exists a number $M > 0$ such that for each $n$, $\sum_{k=0}^{\infty} |a_{nk}| < M$.

3. ASYMPTOTIC EQUIVALENCE PROPERTIES.

THEOREM 3. Let $A$ be a nonnegative matrix. Suppose $x \sim y$, and $x, y \in P_\delta$ for some $\delta > 0$. Then $\mu Ax \sim \mu Ay$ if and only if for each $i = 0, 1, 2, \ldots$

\[ \lim_{n \to \infty} a_{ni}/\sum_{i=0}^{\infty} a_{nj} = 0. \]

PROOF. If $\lim_{n \to \infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} = 0$, $i = 0, 1, 2, \ldots$, we want to prove that $\mu Ax \sim \mu Ay$.

Since $x \sim y$, there exists a null sequence $\zeta$, such that

\[ x_n = y_n(1 + \zeta_n) \quad n = 0, 1, 2, \ldots \]

then

\[ \frac{(\mu Ax)_n}{(\mu Ay)_n} = \frac{\sup_{k > n} (Ax)_k}{\sup_{k > n} (Ay)_k} \]

\[ = \frac{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} x_i}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} \]

\[ = \frac{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki}(y_i + y_i \zeta_i)}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} \]

\[ \leq 1 + \frac{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i |\zeta_i|}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} \]

\[ \leq 1 + \frac{\sup_{k > n} \sum_{i=0}^{N-1} a_{ki} y_i |\zeta_i|}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} + \frac{\sup_{k > n} \sum_{i=N}^{\infty} a_{ki} y_i |\zeta_i|}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} \]

where $N$ is a positive integer.

Since $\zeta$ is a null sequence, $\sup_j |\zeta_j| < \infty$, and for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $i \geq N$, then $|\zeta_i| < \epsilon$. Hence

\[ \frac{(\mu Ax)_n}{(\mu Ay)_n} \leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{\sup_{k > n} a_{ki} y_i}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i} + \epsilon \sup_{k > n} \sum_{i=0}^{\infty} a_{ki} y_i \]

\[ \leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{y_i \sup_{k > n} a_{ki}}{\sup_{k > n} \sum_{i=0}^{\infty} a_{ki}} + \epsilon \]

\[ \leq 1 + \sup_j |\zeta_j| \sup_{0 \leq k \leq N} y_j \sum_{i=0}^{N} \frac{a_{ki}}{\sum_{i=0}^{\infty} a_{ki}} + \epsilon. \]

According to the hypothesis, there exists $N_1 \in \mathbb{N}$, such that if $k \geq N_1$, then $a_{ki}/\sum_{i=0}^{\infty} a_{ki} < \epsilon/N \sup_j |\zeta_j| \sup_{0 \leq i \leq N} y_i$. So if $n \geq N$, we have
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\[
\frac{(μAx)_n}{(μAy)_n} \leq 1 + \epsilon + \epsilon.
\]

This implies that \(\lim_{n→∞} \frac{(μAx)_n}{(μAy)_n} \leq 1\). Similarly, we may prove \(\lim_{n→∞} \frac{\sup_{k≥n} \sum_{i=0}^{∞} a_{ki}}{\sup_{k≥n} \sum_{i=0}^{∞} a_{ki}} \leq 1\) and the two inequalities yield \(\lim_{n→∞} \frac{(μAx)_n}{(μAy)_n} = 1\).

Next, suppose \(μAx ∼ μAy\) for any \(x ∼ y\) such that \(x, y \in P_δ\) for some \(δ > 0\). We take \(x = y = (1, 1, \ldots)\). Then \(μAx ∼ μAy\), i.e.,

\[
\lim_{n→∞} \frac{\sup_{k≥n} \sum_{i=0}^{∞} a_{ki}}{\sup_{k≥n} \sum_{i=0}^{∞} a_{ki}} = 1.
\]

Hence, there exists \(M > 0\), such that \(\{\sum_{i=0}^{∞} a_{ki}\}_{n=0}^{∞}\) is bounded by \(M\).

If \(\lim_{n→∞} a_{ni}/\sum_{j=0}^{∞} a_{nj} \neq 0\) for some \(i\). Then there exists \(λ > 0\) and a sequence \(n_1 < n_2 < \ldots\), such that \(a_{ni}/\sum_{j=0}^{∞} a_{nj} ≥ λ, u = 1, 2, 3, \ldots\). Take \(t > 0\), and define \(x\) and \(y\) by

\[
y_n = 1, n = 0, 1, 2, \ldots
\]

and

\[
x_n = \begin{cases} 1 & \text{if } n \neq i \\ 1 + t & \text{if } n = i \end{cases}
\]

It is clear that \(x ∼ y\) and \(x, y \in P_1\). Consider the following limit:

\[
\lim_{n→∞} \frac{\sup_{k≥n} \sum_{i=0}^{∞} a_{nj}x_j}{\sup_{k≥n} \sum_{i=0}^{∞} a_{nj}y_j} = \lim_{n→∞} \frac{\sup_{k≥n} \sum_{i=0}^{∞} a_{nj}x_j + nλ}{\sup_{k≥n} \sum_{i=0}^{∞} a_{nj}} = 1 + nλ.
\]

We can choose \(t = 1/λ\), which gives

\[
\lim_{n→∞} \frac{(μAx)_n}{(μAy)_n} \geq 2.
\]

This is a contradiction of \(μAx ∼ μAy\).

**THEOREM 4.** Suppose \(A\) is a nonnegative matrix; then \(μx ∼ μy\) implies \(μAx ∼ μAy\) for any bounded sequences \(x, y \in P_δ\), for some \(δ > 0\), if and only if \(A\) satisfies the following three conditions:

(i) \(\{\sum_{j=0}^{∞} a_{kj}\}_{k=0}^{∞}\) is a bounded sequence dominated by some \(B\);

(ii) For any \(j = 0, 1, 2, \ldots\)

\[
\lim_{n→∞} \frac{\sup_{k≥n} a_{kj}}{\sup_{k≥n} \sum_{i=0}^{∞} a_{ki}} = 0;
\]

(iii) For any infinite sequence \(j_1 < j_2 < j_3 \ldots\)
Before we prove this theorem, we shall give some examples of $A$ which satisfy the above conditions (i), (ii), and (iii).

Example 1. $A = I$.

Example 2.

$$A = \begin{pmatrix}
1 & 1 \\
\frac{1}{2} & \frac{3}{2} \\
\frac{1}{3} & \frac{3}{3} & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
\frac{1}{(n+1)!} & \frac{1}{(n+1)!} & \cdots & \frac{1}{(n+1)!} & 1
\end{pmatrix}$$

PROOF OF THEOREM 4. First, assume that for any bounded sequences $x, y \in P_1$, for some $\delta > 0$, $\mu x \sim \mu y$ implies $\mu Ax \sim \mu Ay$; we wish to prove that $A$ satisfies the conditions (i), (ii) and (iii). Take $x = (1, 1, \ldots)$; then $x, y$ are bounded, $x, y \in P_1$, and $\mu x \sim \mu y$; so $\mu Ax \sim \mu Ay$. But $(\mu Ax)_n = \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}$. Hence, $(\sum_{i=0}^{\infty} a_{ki})^{\infty}_{i=0}$ should be bounded. This proves (i). To prove (ii) suppose there is a $j$ such that

$$\lim_{n \to \infty} \sup_{k \geq n} a_{kj} = \lambda$$

for some $\lambda > 0$. As in the proof of Theorem 3, take $t > 0$ and define $y = (1, 1, \ldots)$ and

$$x_n = \begin{cases} 
1 & \text{if } n \neq j, \\
1 + t & \text{if } n = j.
\end{cases}$$

Then $x, y \in P_1, x, y$ are bounded, and $\mu x \sim \mu y$; so we have $\mu Ax \sim \mu Ay$. But

$$\lim_{n \to \infty} \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} x_i = \lim_{n \to \infty} \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} y_i = \lim_{n \to \infty} \frac{\sup_{k \geq n} (t a_{kj} + \sum_{i=0}^{\infty} a_{ki})}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} \geq \lim_{n \to \infty} \frac{t \sup_{k \geq n} a_{kj}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki}} - 1 = t \lambda - 1.$$

By choosing $t = \frac{2}{\lambda}$, we get

$$\lim_{n \to \infty} \sup_{k \geq n} \sum_{i=0}^{\infty} a_{ki} x_i \geq 3 - 1 = 2.$$

This is a contradiction $\mu Ax \sim \mu Ay$, so (ii) must hold.

Finally, we are going to prove (iii). For any given infinite sequence $j_1 < j_2 < \ldots$, we define $x$ and $y$ by

$$y_n = 2 \text{ for every } n,$$

and

$$x_n = \begin{cases} 
2, & \text{if } n = j_u \text{ for } u = 1, 2, \ldots, \\
1, & \text{otherwise}.
\end{cases}$$
It is easy to see that $x, y$ are bounded, $x, y \in P_1$ and $\mu x \sim \mu y$. This implies $\mu A_x \sim \mu A_y$. Hence we have

$$1 = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_j}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} y_j} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \left( \sum_{j \in J} a_{kj} x_j + \sum_{j \in J} a_{kj} x_j \right)}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}$$

where $J = \{j_1, j_2, j_3, \ldots\}$

$$= \lim_{n \to \infty} \frac{\sup_{k \geq n} \left( \sum_{j \in J} a_{kj} x_j + \sum_{j=0}^{\infty} a_{kj} \right)}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \leq \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj} + \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}}{2 \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} + \frac{1}{2}.$$

Hence

$$1 \leq \frac{1}{2} \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} + \frac{1}{2}.$$

This implies

$$\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \geq 1.$$

On the other hand, it is clear that

$$\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} \leq 1.$$

Combining the last two inequalities together, we get

$$\lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{j \in J} a_{kj}}{\sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj}} = 1,$$

which proves (iii).

Conversely, assume $A$ satisfies the conditions (i), (ii) and (iii), and suppose $x, y$ are bounded by some $M > 0$, $x, y \in P_6$ for some $\delta > 0$, and $\mu x \sim \mu y$. For any $\epsilon > 0$, since $x, y$ are bounded, there exists $N_1 \in \mathbb{N}$ such that if $j \geq N_1$, then

$$y_j \leq \lim_{k \to \infty} \sup_{i \geq k} y_i + \epsilon$$

and also there exists an infinite sequence $j_1 < j_2 < \ldots$, such that

$$x_{j_i} \geq \lim_{k \to \infty} \sup_{j \geq k} x_j - \epsilon$$

for $i = 1, 2, 3, \ldots$. Therefore

$$\lim_{n \to \infty} \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} x_j \leq \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kj} y_j.$$
\[ \geq \lim_{n \to \infty} \frac{\sup_{k > n} \sum_{i=0}^{\infty} a_{kji} x_i}{\sup_{k \geq n} \sum_{j=0}^{N_1} a_{kji} x_i + \sum_{j=N_1+1}^{\infty} a_{kji} y_j} \]

\[ \geq \lim_{n \to \infty} \frac{\sup_{k > n} \sum_{i=0}^{\infty} a_{kji} (\lim_{\ell \to \infty} \sup_{\ell \geq \ell} x_i - \varepsilon)}{M \sup_{k > n} \sum_{j=0}^{N_1} a_{kji} + \sup_{k \geq n} \sum_{j=N_1+1}^{\infty} a_{kji} (\lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j + \varepsilon)} \]

\[ \geq \lim_{n \to \infty} \frac{(\sup_{k > n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} x_i - \varepsilon \sup_{k > n} \sum_{i=0}^{\infty} a_{kji}}{M \sup_{k > n} \sum_{j=0}^{N_1} a_{kji} + \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kji} + \sup_{k \geq n} (\sum_{j=N_1+1}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j} \]

\[ \geq \lim_{n \to \infty} \frac{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j}{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j} \]

\[ \varepsilon \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}}{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j} \]

\[ \frac{\varepsilon}{\delta} \]

(here, we used (iii) to deduce that

\[ \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}}{\sup_{k \geq n} \sum_{j=0}^{N_1} a_{kji} + \sum_{j=N_1+1}^{\infty} a_{kji}} = \lim_{n \to \infty} \frac{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}} = 1 = 1) \]

\[ \geq \lim_{n \to \infty} \frac{1}{B_1 + B_2 + B_3} - \varepsilon \]

\[ \geq \lim_{n \to \infty} \frac{\frac{M \sup_{k > n} \sum_{i=0}^{N_1} a_{kji}}{\sup_{k > n} \sum_{i=0}^{\infty} a_{kji}} + \frac{\varepsilon \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kji}}{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j}}{\sup_{k \geq n} \sum_{j=0}^{N_1} a_{kji} + \sum_{j=N_1+1}^{\infty} a_{kji}} - \frac{\varepsilon}{\delta} \]

where

\[ B_1 = \frac{M \sup_{k > n} \sum_{i=0}^{N_1} a_{kji}}{\sup_{k > n} \sum_{i=0}^{\infty} a_{kji}}, \quad B_2 = \frac{\varepsilon \sup_{k \geq n} \sum_{j=0}^{\infty} a_{kji}}{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j}, \]

\[ B_3 = \frac{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j}{(\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}) \lim_{\ell \to \infty} \sup_{\ell \geq \ell} y_j}. \]

For the fixed \( N_1 \), combining conditions (ii) and (iii), we can easily prove

\[ \sup_{k \geq n} \sum_{j=0}^{N_1} a_{kji} \to 0 \text{ as } n \to \infty. \]

Hence, for the given \( \varepsilon > 0 \), there is \( N_2 \in \mathbb{N} \), such that if \( n \geq N_2 \), then

\[ \frac{\sup_{k \geq n} \sum_{i=0}^{N_1} a_{kji}}{\sup_{k \geq n} \sum_{i=0}^{\infty} a_{kji}} < \varepsilon. \]
\[
\sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj}}{\sum_{i=1}^{\infty} a_{ki}} < 1 + \varepsilon \quad \text{(by (iii))},
\]

and
\[
\sup_{k \geq N_2} \frac{\sum_{j=0}^{N_1} a_{kj}}{\sum_{i=1}^{N_1} a_{ki}} < 1 + \varepsilon \quad \text{(by (iii))}.
\]

These imply that if \( n \geq N_2 \)
\[
\sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj}}{\sum_{i=1}^{\infty} a_{ki}} \leq \frac{1}{e} + \frac{1}{2} (1 + \varepsilon) + 1 + \varepsilon.
\]

Hence
\[
\lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj} x_j}{\sum_{j=0}^{\infty} a_{kj} y_j} \geq \frac{1}{e} + \frac{1}{2} (1 + \varepsilon) + 1 + \varepsilon - \frac{\varepsilon}{\delta}.
\]

Since \( \varepsilon \) is arbitrary, we have
\[
\lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj} x_j}{\sum_{j=0}^{\infty} a_{kj} y_j} \geq 1.
\]

Similarly, we can prove
\[
\lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj} x_j}{\sum_{j=0}^{\infty} a_{kj} y_j} \leq 1.
\]

Thus, we have finished the proof.

**REMARK.**

Let \( A \) be a nonnegative matrix, \( A = (a_{ij}) \). If \( A \) satisfies the following two conditions, then \( A \) satisfies the conditions (i), (ii), (iii) of theorem 4:

a) There exists \( \lambda > 0 \), such that
\[
\lim_{n \to \infty} a_{nn} = \lambda
\]

b) \( \lim_{n \to \infty} \sum_{j \neq n} a_{nj} = 0 \)

**PROOF OF THE REMARK.** If \( A \) satisfies the above conditions a and b, it is easy to see that \( A \) satisfies (i) in theorem 4. To prove (iii), let \( j_1, j_2, \ldots \) be an infinity sequence: \( j_1 < j_2 < \ldots \) Then
\[
\lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj}}{\sum_{j=0}^{\infty} a_{kj}} \geq \lim_{n \to \infty} \frac{\sup_{k \geq N_2} a_{j,n}}{\sum_{j=0}^{\infty} a_{kj}}
\]

\[
= \lim_{n \to \infty} \sup_{j \geq N_2} a_{j,n} = \frac{\lambda}{\lambda + 0} = 1
\]

This inequality gives that
\[
\lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j=0}^{\infty} a_{kj}}{\sum_{j=0}^{\infty} a_{kj}} = 1
\]

Next, let's prove (ii) of theorem 4. In fact, for any fixed \( j = 0, 1, 2, \ldots \)
\[
\lim_{n \to \infty} \frac{\sup_{k \geq N_2} a_{kj}}{\sup_{k \geq N_2} \sum_{i=0}^{\infty} a_{ki}} \leq \lim_{n \to \infty} \frac{\sup_{k \geq N_2} \sum_{j \leq k} a_{kj}}{\sum_{j \leq k} a_{kj}}
\]

\[
= \lim_{n \to \infty} \sup_{k \geq N_2} \frac{\sum_{j \leq k} a_{kj}}{\sum_{j \leq k} a_{kj}}
\]
Next, we give some examples to show that the conditions of theorem 4 are necessary.

Example 3. Let \( A \) be defined as follows:

\[
A = \begin{pmatrix}
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

It is easy to see that \( A \) satisfies the conditions (i) and (ii), not (iii).

Take

\[
x = (2, 2, 2, 2, \ldots) \\
y = (2, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 2, \ldots)
\]

\( x, y \) are bounded sequences and \( x, y \in P_1 \). For \( m = 1, 2, 3, \ldots \) we have \( \mu_m(x) = \mu_m(y) = 2 \).

Hence \( \frac{\mu_m(x)}{\mu_m(y)} = 1 \). But

\[
Ax = (8, 8, 8, \ldots) \\
Ay = (6, 3, \ldots) \\
y = (y_i), y_i \leq 6, i = 1, 2, \ldots
\]

This implies

\[
\frac{\mu_m Ax}{\mu_m Ay} \to \frac{8}{6} = \frac{4}{3} \neq 1, \text{ as } n \to \infty.
\]

Example 4.

Let

\[
A = \begin{pmatrix}
1 & \frac{1}{2} & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{8} \\
0 & \text{...}
\end{pmatrix}
\]

\( A \) satisfies (i) and (ii) not (iii).

Take

\[
x = (2, 2, 2, \ldots) \\
y = (2, 1, 2, 1, \ldots).
\]
\( z \) and \( y \) are bounded and \( z, y \in P_1 \). We also have

\[
\frac{\mu_m x}{\mu_m y} = 1, \quad m = 1, 2, \ldots
\]

\[
Ax = (2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)
\]

\[
Ay = (2, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots)
\]

Then, if \( m \) is odd,

\[
\frac{\mu Ax}{\mu Ay} = 2
\]

if \( m \) is even

\[
\frac{\mu Ax}{\mu Ay} = 1
\]

\( \Rightarrow \frac{\mu Ax}{\mu Ay} \) has no limit as \( m \to \infty \).

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