WEIGHTED NORM INEQUALITIES FOR THE $\mathcal{H}$-TRANSFORMATION

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ABSTRACT. In this paper we establish weighted norm inequalities for an integral transform whose kernel is a Fox function.

KEY WORDS AND PHRASES: Fox function, integral transformation, weighted norm inequalities.

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1. INTRODUCTION

The transformations we will investigate in this paper are the ones called $\mathcal{H}$-transformations. These transformations are defined by

$$\mathcal{H}(f)(x) = \int_0^\infty F_{P,q}^{m,n} \left( x t \right) \left( \alpha_1, \alpha_2, \ldots, \alpha_p, \alpha_q \right) f(t) dt, \quad f \in C_0, \quad (1)$$

where $F_{P,q}^{m,n}$ denotes the Fox function ([9]) and as usual $C_0$ represents the class of complex valued functions on $(0, \infty)$ which are continuous and compactly supported. In the last years, the $\mathcal{H}$-transformation has been studied by several authors (see [6], [7], [14] and [18]) and it reduces to important integral transforms (Laplace, Hankel, Meijer, Hardy, ...) by specifying the involved parameters. In a previous paper [5] the authors (simultaneously to A. A. Kilbas, M. Saigo and S. A. Shlapakov [15], [16] and [17]), investigated the behavior of transformation (1) in certain weighted $L_p$ spaces introduced by P. G. Rooney [21].

Weighted Fourier transform norm inequalities have been exhaustively studied (see [2], [3], [4], [10], [13], [20], amongst others). Inspired by the above works our aim in this paper is to give conditions on a positive Borel measure $\Omega$ on $(0, \infty)$, and on a measurable nonnegative function $\nu$ on $(0, \infty)$ which are sufficient in order that the inequality

$$\left\{ \int_0^\infty \left| \mathcal{H}(f)(x) \right|^r d\Omega(x) \right\}^{\frac{1}{r}} \leq C \left\{ \int_0^\infty \nu(x) \left| f(x) \right|^s dx \right\}^{\frac{1}{s}}, \quad f \in C_0, \quad (2)$$

holds where $1 \leq r, s \leq \infty$ and $C$ is a suitable positive constant. Also we analyze some special cases of (2). Moreover we establish some properties on $\Omega$ and $\nu$ that are implied by (2).

We now introduce some notations that will be used throughout this paper. We need consider some parameters related to the $F$-function. Let $m, n, p, q \in \mathbb{N}$ being $0 \leq m \leq s$, $0 \leq n \leq r$ and $r + s \geq 1$. Assume that $\alpha_j, \beta_j, j = 1, \ldots, r$ and $\alpha_j, \beta_j, j = 1, \ldots, s$, are real numbers and $\alpha_j, \beta_j, j = 1, \ldots, r$, and $\beta_j, j = 1, \ldots, s$, are positive real numbers. We define

$$\alpha = \begin{cases} \max \left\{ -\frac{b_j}{\beta_j}, j = 1, \ldots, m \right\}, & \text{for } m > 0 \\ -\infty, & \text{for } m = 0 \end{cases}$$
\[ \beta = \min \left\{ \frac{1-a_j}{\alpha_j}, j = 1, \ldots, n \right\}, \quad \text{for} \quad n > 0 \\
+\infty \] 
\[ \mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j \]
\[ \nu = \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} \alpha_j \]
\[ \xi = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j \]
\[ \eta = \prod_{j=1}^{p} \alpha_j^{-a_j} \prod_{j=1}^{q} \beta_j^{b_j}. \]

Also we remember a result that was established in [5] and that will be very useful in the sequel.

**THEOREM A (Corollary 1 of [5])**. If \( \alpha < \gamma < \beta \) and if either

(a) \( \xi > 0 \) or
(b) \( \xi = 0, \mu \neq 0 \) and \( \nu + \mu \gamma - \frac{1}{2} (q - p) < -1 \)

holds, then the function \( \mathcal{F} \) is defined by

\[ \mathcal{F}(x) = h_{p,q}^{m,n} \left( x \left| \begin{array}{c}
(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\
(b_1, \beta_1), \ldots, (b_q, \beta_q)
\end{array} \right| s = \frac{1}{2\pi i} \int_{-\infty}^{\gamma+i\infty} x^{-s} h(s) ds \] (3)

for every \( x > 0 \), where

\[ h(s) = h_{p,q}^{m,n} \left( \begin{array}{c}
(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\
(b_1, \beta_1), \ldots, (b_q, \beta_q)
\end{array} \right) s = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j s) \Gamma(1 - \gamma s)}. \]

Here the empty products as usual are understood as 1. Moreover

\[ |\mathcal{F}(x)| \leq C \gamma x^{-\gamma} \] (4)

for every \( x > 0 \), \( C \gamma \) being a positive constant. Furthermore if \( \alpha < \gamma < \beta \), \( \xi = 0 \) and \( \nu - \frac{1}{2} (q - p) < -1 \) then (3) and (4) hold for every \( x > 0 \) except for \( x = \eta \). ■

In view of the above considerations we will assume in the sequel that our parameters satisfy one of the following four conditions, namely

(i) \( \xi > 0 \)

(ii) \( \xi = 0, \mu > 0 \) and \( \beta < -\frac{1}{\mu} [\nu + 1 + \frac{1}{2} (p - q)] \)

(iii) \( \xi = 0, \mu < 0 \) and \( \alpha > -\frac{1}{\mu} [\nu + 1 + \frac{1}{2} (p - q)] \)

(iv) \( \xi = 0, \mu = 0 \) and \( \nu + \frac{1}{2} (p - q) < -1 \)

Throughout this paper for every \( 1 \leq r \leq \infty \) we denote by \( r' \) the conjugate of \( r \) (that is, \( r' = \frac{\gamma}{\gamma - 1} \)).

Also when some of the exponents in our weighted inequality are infinite said inequality must be understood in the obvious form.

### 2. WEIGHTED NORM INEQUALITIES FOR THE \( \mathcal{H} \)-TRANSFORM

We shall firstly give sufficient conditions on a positive function \( v \) on \((0, \infty)\) and on a positive Borel measure \( \Omega \) on \((0, \infty)\) in order that the inequality

\[ \left\{ \int_0^{\infty} |\mathcal{H}(f)(x)|^r d\Omega \right\}^{\frac{1}{r}} \leq C \left\{ \int_0^{\infty} v(x)|f(x)|^s dx \right\}^{\frac{1}{s}}, \quad f \in C_0, \]

holds, where \( 1 \leq r, s < \infty \) and \( C \) denotes a certain positive constant. When either \( r = \infty \) or \( s = \infty \) inequality (2) takes the obvious form. The employed procedure here is inspired by the one used by J. J. Benedetto and H. P. Heinig ([2] and [3]) in their studies about Fourier transforms.
PROPOSITION 1. Assume that $\Omega$ is a positive Borel measure on $(0, \infty)$ and that $v$ is a nonnegative measurable function on $(0, \infty)$ belonging to $L^1_{loc}(0, \infty)$.

If $1 \leq r \leq s \leq \infty$ and there exist $\alpha < a, b < \beta$ such that

$$B_1 = \sup_{x > 0} \left\{ \int_0^x t^{-as} d\Omega(t) \right\}^{\frac{1}{r}} \left\{ \int_0^x t^{-ar}v(t)^{1-r} dt \right\}^{\frac{1}{s}} < \infty$$

and

$$B_2 = \sup_{x > 0} \left\{ \int_x^\infty t^{-bs} d\Omega(t) \right\}^{\frac{1}{r}} \left\{ \int_x^\infty t^{-bs}v(t)^{1-r} dt \right\}^{\frac{1}{s}} < \infty,
$$

then (2) holds for every $f \in C_0$

Also if $1 < s < r < \infty$ and there exist $\alpha < a, b < \beta$ such that

$$B'_1 = \int_0^\infty \left\{ \int_0^x t^{-as} d\Omega(x) \right\}^{\frac{1}{r}} \left\{ \int_0^x t^{-ar}v(x)^{1-r} dx \right\}^{\frac{1}{s}} x^{-as} v(x)^{1-r} dx < \infty$$

and

$$B'_2 = \int_0^\infty \left\{ \int_x^\infty t^{-bs} d\Omega(x) \right\}^{\frac{1}{r}} \left\{ \int_x^\infty t^{-bs}v(x)^{1-r} dx \right\}^{\frac{1}{s}} x^{-bs} v(x)^{1-r} dx < \infty,$

where $\frac{1}{s} = \frac{1}{r} - \frac{1}{t}$, then (2) holds for every $f \in C_0$

**PROOF.** First we consider the case $1 < r < \infty$

Let $f \in C_0$. By virtue of (4) for every $\alpha < a, b < \beta$ there exists $C_{a,b} > 0$ such that

$$|\mathcal{H}(f)(z)| \leq C_{a,b} \left\{ \int_0^1 (xt)^{-\alpha}|f(t)|dt + \int_0^\infty (xt)^{-b}|f(t)|dt \right\}, \quad z > 0.$$ 

By using the Minkowski inequality we obtain

$$\left\{ \int_0^\infty |\mathcal{H}(f)(z)|^s d\Omega(x) \right\}^{\frac{1}{s}} \leq C_{a,b} \left\{ \left\{ \int_0^\infty \left\{ \int_0^1 t^{-\alpha}|f(t)|dt \right\}^{s} x^{-as} d\Omega(x) \right\}^{\frac{1}{s}} + \left\{ \int_0^\infty \left\{ \int_0^\infty t^{-b}|f(t)|dt \right\}^{s} x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}} \right\} = C_{a,b}(J_1 + J_2).$$

A straightforward change of variable leads to

$$J_1 = \left\{ \int_0^\infty \left\{ \int_0^1 t^{-\alpha}|f(t)|dt \right\}^{s} x^{-as} d\Omega(x) \right\}^{\frac{1}{s}} = \left\{ \int_0^\infty \left\{ \int_z^\infty h(u)du \right\}^{s} x^{-as} d\Omega(x) \right\}^{\frac{1}{s}}$$

where $h(u) = u^{s-2}|f(\frac{1}{u})|$, $u > 0$

Therefore from Theorem 4 (1.3.1) [19] one infers

$$J_1 \leq C_1 \left\{ \int_0^\infty h(t)^{\frac{1}{s}} v(t) dt \right\}^{\frac{1}{s}} = C_1 \left\{ \int_0^\infty |f(t)|^s v(t) dt \right\}^{\frac{1}{s}} \tag{6}$$

with $C_1 > 0$ and $v_1(t) = v(\frac{1}{t}) t^{2s-2-\alpha}$, $t > 0$, provided that $B_1 < \infty$.

On the other hand, we have

$$J_2 = \left\{ \int_0^\infty \left\{ \int_0^\infty t^{-b}|f(t)|dt \right\}^{s} x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}} = \left\{ \int_0^\infty \left\{ \int_0^\infty g(t)dt \right\}^{s} x^{-bs} d\Omega(x) \right\}^{\frac{1}{s}}$$

where $g(t) = t^{b-2}|f(\frac{1}{t})|$, $t > 0$. Then by invoking Theorem 1 (1.3.1) [19] it follows
\[ J_s \leq C_2 \left\{ \int_0^\infty g(t) \nu_2(t) \, dt \right\}^{\frac{1}{2}} = C_2 \left\{ \int_0^\infty |f(t)|^r v(t) \, dt \right\}^{\frac{1}{r}} \]  

(7)

where \( \nu_2(t) = v\left(\frac{1}{2}\right)t^{2r-2\beta}, \ t > 0, \) when \( B_2 < \infty \)

By combining (5), (6) and (7) we can immediately deduce (2)

When either \( r = \infty \) or \( s = \infty \) the proof can be made in a similar way

In the case \( 1 \leq s < r \leq \infty \) (2) can be established as the above case by invoking the Theorem 2 (1.3.2) [19].

In the sequel we present some special cases of inequality (2). The following results are related to known weighted norm inequalities for other integral transforms due to P. Heywood and P. G. Rooney ([11],[12]), N. E. Aguilera and E. O. Harboyre [1], B. Muckenhoupt [20] and S. A. Emara and H. P. Heinig [8].

A generalization of Theorem 2.1 of [12] is the following.

**PROPOSITION 2.** Let \( \alpha < 1 - \eta < \beta \) and \( 1 \leq s \leq \infty \). Then

\[ \left\{ \int_0^\infty |x^{1-\eta} \mathcal{H}(f)(x)|^s \, \frac{dx}{x} \right\}^{\frac{1}{s}} \leq C \int_0^\infty x^{\eta-1} |f(x)| \, dx, \ f \in C_0, \]

(8)

for certain \( C > 0 \).

**PROOF.** This result, that also can be proved in a similar way to Theorem 2.1 of [12], is a consequence of Proposition 1. In effect if \( 1 - \eta < \alpha < \beta \) we have

\[ \left\{ \int_x^\infty t^{\gamma(1-\eta-\alpha)-1} \, dt \right\}^{\frac{1}{\gamma}} \left\| t^{-\alpha-\eta+1} \chi_{[\frac{1}{t}, \infty)}(t) \right\|_{t^{-\gamma-1} \, dt} = (s(1-\eta-\alpha))^{\frac{1}{\gamma}}, \ x > 0 \ 	ext{and} \ 1 \leq s < \infty \]

where \( \| \cdot \|_{t^{-\gamma-1} \, dt} \) denotes the essential supremum respect to the measure \( t^{-\gamma-1} \, dt \) and \( \chi_E \) represents as usual the characteristic function associated to the measure set \( E \).

In a similar way we can see that if \( \alpha < b < 1 - \eta \) and \( 1 \leq s < \infty \). Then

\[ \sup_{x > 0} \left\{ \int_0^x s^{\gamma(1-\eta-b)-1} \, dt \right\}^{\frac{1}{\gamma}} \left\| t^{-b-\eta+1} \chi_{[0, \frac{1}{t})}(t) \right\|_{t^{-\gamma-1} \, dt} < \infty. \]

Hence according to Proposition 1 (8) holds for every \( 1 \leq s < \infty \)

When \( s = \infty \) the result can be proved analogously.

We now investigate the inequality (2) when \( d\Omega = u(x) \, dx \) being \( u \) is a measurable nonnegative function on \( (0, \infty) \), \( \nu = 1 \) and \( r = s \).

**PROPOSITION 3.** Let \( 1 \leq r \leq 2, \alpha < 0 \) and \( \frac{1}{2} < \beta \). If \( u \) is a locally integrable nonnegative function on \( (0, \infty) \) for which there exists a constant \( M > 0 \) such that for every measurable set \( E \int_E u(x) \, dx \leq M |E|^{r-1} \) is satisfied, then

\[ \int_0^\infty u(x) |\mathcal{H}(f)(x)|^r \, dx \leq C \int_0^\infty |f(x)|^r \, dx, \ f \in C_0, \]

(9)

for a certain \( C > 0 \).

**PROOF.** Our proof is essentially the same one given in Theorem 1 of [1]. Let \( 1 < r < 2 \) we define the operator

\[ (Tf)(x) = \begin{cases} u^{-\frac{1}{r}}(x) \mathcal{H}(f)(x) & \text{if } u(x) \neq 0 \\ 0 & \text{if } u(x) = 0 \end{cases} \quad f \in C_0 \]

where \( b = \frac{2}{2-r} \)

Since \( \alpha < 0 < \beta \), then by (4) \( \mathcal{H} \) is a bounded function on \( (0, \infty) \) Hence, according to Theorem 2 of [1] we obtain
\[
\int_{\{x \mid Tf(x) > \lambda\}} u^b(x) \, dx \leq \int_{x \leq \frac{C_2}{\lambda} \int_0^x |f(z)\, dz} u^b(x) \, dx \leq \frac{C_2}{\lambda} \int_0^\infty |f(x)| \, dx
\]

where \(C_i, i = 1, 2\), are positive constants. Thus \(T\) is a weak type \((1,1)\) operator, on measure spaces \(((0, \infty), dx)\) and \(((0, \infty), u^b(x) \, dx)\).

Moreover, by virtue of Proposition 3 of [5], \(T\) is a bounded operator from \(L_2(0, \infty)\) into itself because \(\alpha < \frac{1}{2} < \beta\). Therefore

\[
\int_0^\infty |Tf(x)|^2 u^b(x) \, dx \leq C \int_0^\infty |f(x)|^2 \, dx
\]

with \(C > 0\), and \(T\) is a strong type \((2,2)\) operator between the spaces under consideration.

Now, by the Marcinkiewicz interpolation theorem we obtain the desired result for \(1 < r < 2\).

Finally, note that if \(r = 1\) then \(\int_0^\infty u(x) \, dx < \infty\) and (9) holds trivially because \(\alpha < 0 < \beta\) and by (4).

Moreover, if \(r = 2\) then \(u\) is bounded function on \((0, \infty)\) and since \(\alpha < \frac{1}{2} < \beta\) (4) leads to (9).

By proceeding as in §7 of [1] we can deduce from Proposition 3 conditions for a function \(v\) that imply inequality (2) holds when \(\Omega\) is the Lebesgue measure on \((0, \infty)\) and \(r = s\).

We now give conditions for \(u\) that are deduced from (9).

**Proposition 4.** Let \(1 \leq r < \infty\). Assume that one of the following two conditions is satisfied:

(i) There exists \(j_0 \in \mathbb{N}, 1 \leq j_0 < p\), such that \(a_j > \max\{\alpha, 1 - \frac{1}{r}\}\) and

\[
\inf_{0 < \varepsilon < 1} \left| \mathcal{F}^{m,n}_{p,q} \left( (a'_1, \alpha_1), \ldots, (a'_p, \alpha_p) \left| \frac{x}{\varepsilon} \right| \right) \right| = K_1 > 0
\]

where \(a'_j = a_j + 1\) and \(a'_j = a_j, 1 \leq j \leq p, j \neq j_0\).

(ii) There exists \(j_0 \in \mathbb{N}, 1 \leq j_0 < q\), such that \(\frac{1 - b_0}{b_0} > \max\{\beta, 1 - \frac{1}{r}\}\) and

\[
\inf_{0 < \varepsilon < 1} \left| \mathcal{F}^{m,n}_{p,q} \left( (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \left| \frac{x}{\varepsilon} \right| \right) \right| = K_2 > 0
\]

where \(b'_0 = b_0 - 1\) and \(b'_j = b_j, 1 \leq j \leq q, j \neq j_0\).

Then there exists a positive constant \(L\) such that

\[
\int_0^a u(x) \, dx \leq C a^{1-r}, \quad \text{holds for every} \quad a > 0,
\]

provided that (9) holds.

**Proof.** We will establish the result when (i) is satisfied with \(n + 1 \leq j_0 \leq p\). The proof in the other cases can be made in a similar way.

It is easy to see that

\[
\frac{d}{dx} \left[ x^{-a_0} \mathcal{F}^{m,n}_{p,q} \left( (a'_1, \alpha_1), \ldots, (a'_p, \alpha_p) \left| \frac{x^{a_0}}{\varepsilon} \right| \right) \right]
\]

\[
= - x^{-(a_0 + 1)} \mathcal{F}^{m,n}_{p,q} \left( (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \left| \frac{x^{a_0}}{\varepsilon} \right| \right), \quad x > 0
\]

being \(a'_j = a_j + 1\) and \(a'_j = a_j, j = 1, \ldots, p, j \neq j_0\).

For \(a > 0\) fixed, define
By using (12) we can write

\[
(\mathcal{H}f_{\alpha})(x) = \int_{0}^{\frac{1}{x}} t^{-\frac{\alpha}{\alpha_0} - \frac{2\beta}{\alpha_0}} f_{p,q}^{m,n} \left( \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right) dt
\]

because

\[
\lim_{v \to +0} v^{-\frac{\alpha}{\alpha_0}} \int_{0}^{\frac{1}{v}} t^{-\frac{\alpha}{\alpha_0} - \frac{2\beta}{\alpha_0}} f_{p,q}^{m,n} \left( \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right) dt = 0. \tag{13}
\]

Since \( \alpha < -\frac{2\beta}{\alpha_0} \) to see (13) it is sufficient to take into account (4) Hence, by virtue of (10)

\[
\int_{0}^{a} u(x) dx \leq K_{1}^{-r} \int_{0}^{a} u(x) \left( \frac{a}{x} \right)^{r} dx = \left( K_{1} a_{\alpha} a_{\beta}^{-\frac{2\beta}{\alpha_0}} \right)^{-r} \int_{0}^{a} u(x) |\mathcal{H}(f_{\alpha})(x)|^{r} dx.
\]

Similarly from (9) one deduces

\[
\int_{0}^{a} u(x) dx \leq C \left( K_{2} a_{\alpha} a_{\beta}^{-\frac{2\beta}{\alpha_0}} \right)^{-r} \int_{0}^{a} \left( \frac{a_{\alpha} a_{\beta}^{-\frac{2\beta}{\alpha_0}}}{x} \right)^{r} dx = C(K_{2} a_{\alpha})^{-r} a_{\beta}^{r-1}.
\]

Thus the proof is finished □

Note that if \( r = 1 \) (11) implies that \( u \) is integrable over \((0, \infty)\) When \( r = 2 \), \( u \) is bounded on \((0, \infty)\) provided that (11) holds Also if \( r > 2 \) and (11) is satisfied then \( u = 0 \), a.e \((0, \infty)\)

B Muckenhoupt [20] investigated sufficient conditions for the measurable functions \( u \) and \( v \) that guarantee that the inequality (2), with \( d\Omega(x) = u(x)dx \), holds when the \( \mathcal{H} \)-transformation is replaced by the Fourier transform Also he studied the converse problem proving that, in some cases, the above cited conditions are necessary Later P Heywood and P.G Rooney [11] analyzed weighted norm inequalities for the Hankel transformation in a similar way We now use an analogous procedure to extend the results in [11] to the \( \mathcal{H} \)-transformation (note that this transform reduces to the Hankel transformation when the parameters take on suitable values)

It will be used to recall some definitions of [11]. For every \( \eta \in \mathbb{R} \), \( 1 \leq r < \infty \) and for every \( v \) nonnegative measurable function on \((0, \infty)\), the space \( \mathcal{L}_{\eta,v,r} \) is constituted by all those measurable functions \( f \) on \((0, \infty)\) such that

\[
\|f\|_{\eta,v,r} = \left( \int_{0}^{\infty} \frac{x^{\eta} v(x) |f(x)|^{r} dx}{x} \right)^{\frac{1}{r}} < \infty.
\]

The space \( \mathcal{L}_{\eta,v,r} \) is a Banach space when it is endowed with the topology associated to the norm \( \|f\|_{\eta,v,r} \). Also, if \( u \) and \( v \) are nonnegative measurable functions on \((0, \infty)\) we say that \((u, v) \in A(r, s, \delta)\) with \( \delta \in \mathbb{R} \) and \( 1 < r, s < \infty \) when there exist positive constants \( B \) and \( C \) for which
\[
\left[ \int_{\omega(z) > B \omega} \left\{ \frac{x^2 u(z)}{x} \right\} \; \frac{dz}{x} \right]^{\frac{1}{2}} \left[ \int_{\omega(z) < \omega} \left\{ \frac{x^2}{v(z)} \right\} \; \frac{dx}{x} \right]^{\frac{1}{2}} \leq C
\]

for every \( \omega > 0 \).

In Propositions 4-8 [5] we established some conditions on the parameters involved in the \( \mathcal{H} \)-function in order that the \( \mathcal{H} \)-transformation can be extended to the space \( \mathcal{L}_{n,r} \) as a bounded operator from \( \mathcal{L}_{n,r} \) into \( \mathcal{L}_{1-\eta} \). In the following Proposition the above results are improved. We prove that under suitable conditions the \( \mathcal{H} \)-transformation can be extended to \( \mathcal{L}_{n,n,v} \) as a bounded operator from \( \mathcal{L}_{n,n,v} \) into \( \mathcal{L}_{1-\eta,u,s} \). We only stated the result corresponding Proposition 8 of [5] although similar results corresponding to Propositions 4-7 of [5] can be established.

**Proposition 5.** Let \( 1 < r < s < \infty, \eta > 0 \) and \( \alpha < 1 - \eta < \beta \). Suppose that \( (u,v) \in A(r, s, 1 - \eta - \sigma) \), with \( \alpha < \sigma < \beta \). Then the \( \mathcal{H} \)-transformation can be extended to \( \mathcal{L}_{n,n,v} \) as a bounded operator from \( \mathcal{L}_{n,n,v} \) into \( \mathcal{L}_{1-\eta,u,s} \).

**Proof.** This result can be proved as Theorem 1 of [11]. It is sufficient to take into account that \( |\mathcal{H}(x)| \leq C_\sigma x^{-\sigma}, x > 0 \), with \( \alpha < \sigma < \beta \) and for certain \( C_\sigma > 0 \). By using this inequality instead of (2.5) of [11] and Proposition 8 of [5] instead of Lemma 1 of [11] the proof of our result follows as the one of Theorem 1 of [11].

On the other hand this result can be proved also by invoking Proposition 1 because if \( (u,v) \in A(r, s, 1 - \eta - \sigma) \) being \( \alpha < 1 - \eta, \sigma < \beta \) then the conditions \( B_i < \infty, i = 1,2 \), in Proposition 1 are satisfied when \( d\Omega \) and \( v \) are replaced by \( x^{(1-\eta-\sigma)r-1} u(x)^s dx \) and \( x^{(1-\eta-\sigma)r-1} v(x)^r \), respectively.

Our next objective is to establish a partial converse to Proposition 5.

**Lemma 1.** Let \( 1 < r < s < \infty \) and \( \eta > 0 \). Assume that \( u \) and \( v \) are nonnegative measurable functions on \( (0, \infty) \) such that \( u \) is decreasing, \( \lim_{x \to \infty} u(x) = 0 \) and \( v \) is increasing. Also suppose that

\[
\inf_{0 < x < 1} S_{p,q}^{n,n} \left( \begin{array}{c}
(a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\
(b_1, \beta_1), \ldots, (b_q, \beta_q)
\end{array} \right) = C_1 > 0.
\]

Then there exists a positive constant \( B > 0 \) for which

\[
\sup \{ x : u(x) > B \omega \} \cdot \sup \{ x : v(x) < \omega \} \leq 1,
\]

for every \( \omega > 0 \), provided that \( \mathcal{H} \) is a bounded operator from \( \mathcal{L}_{n,n,v} \) into \( \mathcal{L}_{1-\eta,u,s} \).

**Proof.** This result will be proved when we see that if

\[
\sup \{ x : u(x) > B \omega \} \cdot \sup \{ x : v(x) < \omega \} > 1,
\]

for some \( \omega > 0 \), then \( B \) is less than a positive constant only depending on \( r \), \( s \) and \( \eta \), the lemma then holds with any larger value of \( B \).

Let \( B, \omega > 0 \). For simplicity denote

\[
M = M(B, \omega) = \sup \{ x : u(x) > B \omega \}.
\]

Since \( \lim_{x \to \infty} u(x) = 0 \), \( M(B, \omega) < \infty \). Assume now \( M(B, \omega) \cdot \sup \{ x : v(x) < \omega \} > 1 \) and define the function

\[
f(x) = \begin{cases} 
1, & \text{if } 0 < x < \frac{1}{M} \\
0, & \text{if } x > \frac{1}{M}.
\end{cases}
\]

It is clear that \( f \in \mathcal{L}_{n,n,v} \) and one has

\[
\|f\|_{n,n,v} = \left\{ \int_0^B \left[ x^n v(x)^s \right] \; \frac{dx}{x} \right\}^{\frac{1}{s}} \leq \left\{ \int_0^B \omega^s x^{n-s-1} dx \right\}^{\frac{1}{s}} = \frac{\omega}{C_{\eta}^n M_{\eta}}
\]

because \( v(x) \leq \omega \), for every \( x \in (0, \frac{1}{M}) \). Since \( \mathcal{H} f \in \mathcal{L}_{1-\eta,u,s} \) then by virtue of (14) and since
For every $z \in (0, M)$ we can write
\[ \int |x|^{-\eta} \frac{dx}{x} > \int_{z}^{M} x^{-(1-\eta)} \frac{dx}{x} \]
for a suitable $K > 0$.

Moreover for a certain $C > 0$
\[ \|Hf\|_{L^{1-n,u,s}} \leq C\|f\|_{n,v,r}. \]  

By combining (15), (16) and (17) one concludes that
\[ B < \frac{C}{C_{1}(\eta)} \]
Note that the constant in the right hand side of the last inequality is positive since $0 < \eta < 1$ Thus the proof is complete.

**PROPOSITION 6.** Let $1 < r \leq s < \infty$ and $0 < \eta < 1$ Assume that $u$ and $v$ are measurable nonnegative functions on $(0, \infty)$ such that $u$ is decreasing, $\lim_{x \to \infty} u(x) = 0$, $v$ is increasing and $f(x) < \infty$, for every $v > 0$. Then $(u, v) \in A(r, s, 1-\eta)$ provided that $H$ is a bounded operator from $L^{n,v,r}$ into $L^{1-n,u,s}$ and (14) holds

**PROOF.** We define for every $\omega > 0$ the function
\[ f_{\omega}(x) = \begin{cases} x^{1-\eta} \frac{v(x)}{x} \text{ if } 0 < v(x) < \omega \\ x < \sup \{ x : u(x) > B \omega \} \end{cases} \]
It is not hard to show that
\[ \|f_{\omega}\|_{n,v,r} = \left\{ \int_{v(x) < \omega} \left( \frac{x^{1-\eta}}{v(x)} \right)^{r} \frac{dx}{x} \right\}^{\frac{1}{r}} \]
and $f_{\omega} \in L^{n,v,r}$, for every $\omega > 0$

But since $H$ is a bounded operator from $L^{n,v,r}$ into $L^{1-n,u,s}$, there exists a positive constant $C > 0$ such that
\[ \|Hf_{\omega}\|_{1-n,u,s} \leq C\|f_{\omega}\|_{n,v,r}, \omega > 0. \]
Hence
\[ \int_{u(x) > B \omega} |x^{1-\eta} u(x) H(f_{\omega}(x))|^{s} \frac{dx}{x} \]
\[ \leq \|Hf_{\omega}\|_{1-n,u,s} \leq C \int_{v(x) < \omega} \left( \frac{x^{1-\eta}}{v(x)} \right)^{r} \frac{dx}{x} \}
where $B$ denotes the constant given in Lemma 1.

Moreover, according to Lemma 1, if $\omega, x, t > 0$, $u(x) > B \omega$ and $v(t) < \omega$, then
\[ x \leq \sup \{ x : u(x) > B \omega \} \sup \{ t : v(t) < \omega \} \leq 1. \]
Hence (14) leads to
\[ \int_{u(x) > B \omega} |x^{1-\eta} u(x) H(f_{\omega}(x))|^{s} \frac{dx}{x} \]
\[ = \int_{u(x) > B \omega} \left( x^{1-\eta} u(x) \right) \int_{v(t) < \omega} t^{1-\eta} f_{\omega}(x) v(t) \frac{dt}{t} \]
\[ \geq C_{1} \int_{u(x) > B \omega} \left( x^{1-\eta} u(x) \right) \int_{v(t) < \omega} \frac{dt}{t}, \omega > 0. \]

\[ (19) \]
By combining (18) and (19) we conclude that \((u, v) \in A(r, s, 1 - \eta)\).

S A Emara and H P Heinig [8] established interpolation theorems (Theorems 1 and 2 of [8]) that they employed to study the behavior of the Hankel and \(K\)-transformations on weighted \(L^p\)-spaces. We can use such interpolation theorems to obtain new weighted norm inequalities for the \(\mathcal{H}\)-transform. The weight functions that appear in this inequality are in the class \(F^*_r\), that we are going to define. Let \(u\) and \(v\) be nonnegative measurable functions defined on \((0, \infty)\) and let \(u^*\) and \((\frac{1}{v})^*\) be the equimeasurable decreasing rearrangements of \(u\) and \(\frac{1}{v}\), respectively. We say that \((u, v) \in F^*_r\), if

\[
\sup_{w > 0} \left\{ \int_0^1 u^*(t)^s dt \right\}^{\frac{1}{s}} \left\{ \int_0^w \left[ \left( \frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} < \infty
\]

(20)

holds for every \(1 < r \leq s < \infty\), and when \(1 < s < r < \infty\) the conditions

\[
\int_0^\infty \left\{ \int_0^1 u^*(t)^s dt \right\}^{\frac{1}{s}} \left\{ \int_0^w \left[ \left( \frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} \left( \frac{1}{v} \right)^*(x)^r dx < \infty
\]

(21)

\[
\int_0^\infty \left\{ \int_0^\infty t^{-1} u^*(t)^s dt \right\}^{\frac{1}{s}} \left\{ \int_0^w \left[ t^{-1} \left( \frac{1}{v} \right)^*(t) \right]^r dt \right\}^{\frac{1}{r}} \left( \frac{1}{v} \right)^*(x x^{-1})^r dx < \infty
\]

(22)

hold, where \(\frac{1}{h} = \frac{1}{s} - \frac{1}{r}\). Moreover if (20), (21) and (22) hold when \(u^*\) and \((\frac{1}{v})^*\) are replaced by \(u\) and \((\frac{1}{v})\), respectively, then we write \((u, v) \in F^*,\).

**Proposition 7.** Assume that \(1 < r, s < \infty\), \(\alpha < 0\) and \(\frac{1}{2} < \beta\). Then

\[
\left\{ \int_0^\infty |u(x)\mathcal{H}(f)(x)|^r dx \right\}^{\frac{1}{r}} \leq C \left\{ \int_0^\infty |v(x)f(x)|^\alpha dx \right\}^{\frac{1}{\alpha}}, \quad f \in C_0,
\]

(23)

holds for a certain \(C > 0\), provided that \((u, v) \in F^*_r\).

**Proof.** Since \(\alpha < 0 < \beta\), according to (4) we can write

\[
\sup_{x > 0} |\mathcal{H} f(x)| \leq C \int_0^\infty |f(x)| dx, \quad f \in L_1(0, \infty)
\]

for a certain \(C > 0\), and then \(\mathcal{H}\) is a bounded operator from \(L_1(0, \infty)\) into \(L_\infty(0, \infty)\).

Moreover, \(\mathcal{H}\) is a bounded operator from \(L_2(0, \infty)\) into itself because \(\alpha < \frac{1}{2} < \beta\) (Proposition 3 of [5]).

Hence from Theorems 1 and 2 of [8] we can infer that the inequality (23) is satisfied.

We now prove a result that is a (partial) converse to Proposition 7. Note that here no monotonicity assumptions on the weights need be made.

**Proposition 8.** Let \(1 < r \leq s < \infty\) and let \(u\) and \(v\) be nonnegative measurable functions on \((0, \infty)\). Assume that (14) holds and that \(\int_0^\infty u(x)^{-r} dx < \infty\), for every \(\omega > 0\). Then \((u, v) \in F^*_r\), when (23) is satisfied.

**Proof.** Firstly we define for every \(\omega > 0\) the function

\[
f_\omega(x) = \begin{cases} u(x)^{-r}, & \text{if } 0 < x < \omega \\ 0, & \text{if } x > \omega. \end{cases}
\]

From (14) one deduces

\[
\int_0^\infty u(x) |\mathcal{H}(f_\omega)(x)|^r dx \geq \int_0^\infty u(x) \int_0^\infty \mathcal{H}(x \mathcal{H}(f_\omega))(t) dt \left( \int_0^1 (\frac{1}{v})^*(t) \right)^r dx \geq M \int_0^\infty u(x)^r dx \left( \int_0^\infty (\frac{1}{v})^*(t) dt \right)^r, \quad \omega > 0
\]

for a certain \(M > 0\). Moreover,
\[
\int_0^\infty |f_\omega(x)v(x)|^\omega \, dx = \int_0^\infty v(x)^{-\omega} \, dx, \quad \omega > 0.
\]

Since (23) holds we can write
\[
\left\{ M \int_0^1 u(x)^\omega \, dx \left\{ \int_0^w v(t)^{-\omega} \, dt \right\}^\omega \right\} \leq C \left\{ \int_0^w v(t)^{-\omega} \, dt \right\}^\omega, \quad \omega > 0.
\]

Thus we conclude that \((u, v) \in F_{r,s}\). 

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**REFERENCES**


