ON COMPLETE CONVERGENCE FOR RANDOMLY INDEXED SUMS
FOR A CASE WITHOUT IDENTICAL DISTRIBUTIONS

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(Received December 26, 1995 and in revised form August 10, 1996)

ABSTRACT. In this note we extend the complete convergence for randomly indexed sums given
by Klesov (1989) to nonidentical distributed random variables.

KEY WORDS AND PHRASES: complete convergence, random indexed sums, regular cover,
array of rowwise independent random variables.

1991 AMS SUBJECT CLASSIFICATION CODES: 60F15, 60B12.

1. INTRODUCTION AND PRELIMINARIES

The following concept of complete convergence was given by Hsu and Robbins [1].

DEFINITION 1.1. A sequence \(\{X_n, n \geq 1\}\) of random variables converges completely to the
constant \(C\) if

\[
\sum_{n=1}^{\infty} P[|X_n - C| \geq \varepsilon] < \infty, \quad \forall \varepsilon > 0.
\]

The main result of Hsu and Robbins [1] states that for a sequence \(\{X_n, n \geq 1\}\) of i.i.d. random
variables with zero expectation and \(EX_n^2 < \infty\), we have

\[
\sum_{n=1}^{\infty} P[|S_n| \geq n\varepsilon] < \infty, \quad \forall \varepsilon > 0,
\]

where \(S_n = \sum_{k=1}^{n} X_k\), i.e. the sequence of arithmetic means \(S_n/n, n \geq 1\), completely convergence
to 0. Erdős [2] proved the converse statement.

Extensions and generalizations of those results were summarized by A. Gut [3]. Extensions
of (1.1) to randomly indexed sums of i.i.d. random variables one can find in Szynal [4], Gut
In this note we extend results on the complete convergence for randomly indexed sums in spirit of Gut [5] and Klesov [8] to nonidentical distributed random variables.

We use the following concept of regular cover of (the distribution of) a random variable.

**DEFINITION 1.2.** (See Pruss [11]). Let $X_1, \ldots, X_n$ be random variables and let $\xi$ be a random variable possible defined on a different probability space. Then $X_1, \ldots, X_n$ are said to be a regular cover of (the distribution of) $\xi$ provided we have

\[
E[G(\xi)] = \frac{1}{n} \sum_{k=1}^{n} E[G(X_k)],
\]

for any measurable function $G$ for which both sides make sense. If $X_1, \ldots, X_n$ are in addition independent, then we say they form an independent regular cover of $\xi$.

2. RESULTS.

The following theorem contains as a particular case the main result of Klesov [8].

**THEOREM 2.1.** Let $\{X_{nk}, n \geq 1, k \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0, E|X_{nk}|^r < \infty$, for some $r \geq 1$, and $n \geq 1, k \geq 1$, such that $X_{n1}, X_{n2}, \ldots, X_{nk}, n \geq 1, k \geq 1$, form an independent regular cover of a random variable $\xi$ with $E\xi = 0, E|\xi|^r < \infty$, for some $r \geq 1$. Suppose that $\{\nu_k, k \geq 1\}$ is a sequence of positive integer-valued random variables. Then for $S_{\nu_n} = \sum_{k=1}^{\nu_n} X_{nk}$ we have:

\[
\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[S_{\nu_n} \geq \epsilon \nu_n^\alpha\right] < \infty, \quad \forall \epsilon > 0,
\]

for $\alpha > 1/2, \alpha r > 1$ and $\beta \geq 1$, whenever

\[
\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\nu_n < n^\beta\right] < \infty,
\]

and (2.1) holds true for $\alpha > 1/2, \alpha r > 1$, and $0 < \beta < 1$, whenever additionally with (2.2) the condition

\[
\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \epsilon \nu_n^\alpha\right] < \infty, \quad \forall \epsilon > 0,
\]

is satisfied.

**PROOF.** Firstly we prove that (2.2) and (2.3) with $\alpha > \frac{1}{2}, \alpha r > 1$, and $\beta > 0$ imply (2.1). Taking into account

\[
\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[|S_{\nu_n}| \geq \epsilon \nu_n^\alpha\right] \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[|S_{\nu_n}| \geq \epsilon \nu_n^\alpha, \nu_n \geq n^\beta\right] + \sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\nu_n < n^\beta\right]
\]

we see that we need only to show that

\[
\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[|S_{\nu_n}| \geq \epsilon \nu_n^\alpha, \nu_n \geq n^\beta\right] < \infty.
\]
Let $\delta > \frac{(\alpha r - 1)}{\beta}, \frac{1}{(\alpha r - 1)} < \gamma < 1$ and $q$ be a positive integer such that $q > \frac{(1 + \delta)}{(\alpha r - 1)}$. Define the sets (cf. Klesov [8]):

\begin{align*}
B_{n}^{(1)} &= \left\{ \exists k \leq \nu_{n} : |X_{nk}| \geq \frac{\epsilon \nu_{n}^{\alpha}}{q} \right\}, \\
B_{n}^{(2)} &= \left\{ \exists k \leq \nu_{n} : |X_{nk}| \geq \nu_{n}^{\alpha} \right\}, \\
B_{n}^{(3)} &= \left\{ \left\lceil \sum_{k \leq \nu_{n}} X_{nk} I(|X_{nk}| < \nu_{n}^{\alpha}) \right\rceil \geq \frac{\epsilon \nu_{n}^{\alpha}}{q} \right\},
\end{align*}

where $I[A]$ is the indicator function of an event $A$. Taking into account that

\[ |S_{\nu_{n}}| \geq \epsilon \nu_{n}^{\alpha} \leq B_{n}^{(1)} \cup B_{n}^{(2)} \cup B_{n}^{(3)} \]

we note that (2.4) will be proved if we show that

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_{n}^{(i)} \cap [\nu_{n} \geq n^{\beta}]] < \infty, \quad i = 1, 2, 3. \]  

For $i = 1$ we have

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_{n}^{(1)} \cap [\nu_{n} \geq n^{\beta}]] \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\exists k \leq \nu_{n} : |X_{nk}| \geq (\epsilon \nu_{n}^{\alpha})/q] \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\max_{k \leq \nu_{n}} |X_{nk}| \geq \epsilon' \nu_{n}^{\alpha}], \quad \epsilon' = \epsilon/q. \]

In the case $i = 2$ we state that

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_{n}^{(2)} \cap [\nu_{n} = j], \nu_{n} \geq n^{\beta}] \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j=1}^{\infty} P[\nu_{n} = j, |X_{n1}| \geq j^{\gamma_{1}}, \ldots, |X_{nk_{q}}| \geq j^{\gamma_{q}}, \nu_{n} \geq n^{\beta}] \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta} \sum_{j=1}^{\infty} j^{d - q \gamma_{\alpha}} \sum_{1 \leq k_{1} \leq k_{2} \leq \ldots \leq k_{q} \leq j} E|X_{n1}|^{r} \ldots E|X_{nk_{q}}|^{r} I[\nu_{n} = j, \nu_{n} \geq n^{\beta}] \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta} \sum_{j=1}^{\infty} j^{d - q \gamma_{\alpha}} \sum_{1 \leq k_{1} \leq k_{2} \leq \ldots \leq k_{q} \leq j} E|X_{n1}|^{r} \ldots E|X_{nk_{q}}|^{r} \]

\[ = \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta} \sum_{j=1}^{\infty} j^{d - q \gamma_{\alpha}} \sum_{k_{q} = q}^{k_{q-1} = q-1} \sum_{k_{q-1} = q-1}^{k_{q-2} = q-2} \sum_{k_{1} = q-1}^{k_{1} = 1} E|X_{nk_{q}}|^{r} \]

Now using the assumption (1.2) we get

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta} \sum_{j=1}^{\infty} j^{d - q \gamma_{\alpha}} \sum_{k_{q} = q}^{k_{q-1} = q-1} \sum_{k_{q-1} = q-1}^{k_{q-2} = q-2} \sum_{k_{1} = q-1}^{k_{1} = 1} E|X_{nk_{q}}|^{r} \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta} E|\xi|^{r} \sum_{j=1}^{\infty} j^{d - q \gamma_{\alpha} + 1} \sum_{k_{q} = q}^{k_{q-1} = q-1} \sum_{k_{q-1} = q-1}^{k_{q-2} = q-2} \sum_{k_{1} = q-1}^{k_{1} = 2} E|X_{nk_{q}}|^{r} \]

\[ \cdots \]
\[ \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta \delta} (E[\xi^n])^q \sum_{j=1}^{\infty} j^{q+1-(\gamma \alpha - \gamma)} < \infty \]

as \( \delta > \frac{\alpha r - 1}{\beta} \), \( \gamma > \frac{1}{\alpha r} \) and \( q > \frac{1 + \frac{q}{\gamma \alpha - 1}}{r - 1} \).

To prove (2.5) for \( i = 3 \) we write

\[ Y_{k,j} = X_{nk}I[|X_{nk}| < j^{\gamma \alpha}] - EX_{nk}I[|X_{nk}| < j^{\gamma \alpha}], \]

\( 1 \leq k \leq j, j \geq 1 \) and \( n \geq 1 \).

Then we see that

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_n^{(3)} \cap [\nu_n \geq n^\delta]] \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} P[\sum_{k \leq j} X_{nk}I[|X_{nk}| < j^{\gamma \alpha}] \geq \frac{\varepsilon j^{\gamma \alpha}}{q}, \nu_n = j] \]

\[ \leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s} E[\sum_{k \leq j} X_{nk}I[|X_{nk}| < j^{\gamma \alpha}]]^s \]

\[ \leq \text{const} \left( \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s} E[\sum_{k \leq j} Y_{k,j}^n]^s + \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s} \sum_{k \leq j} EX_{nk}I[|X_{nk}| < j^{\gamma \alpha}]^s \right) \]

for every \( s > 0 \) and a positive constant \( c \).

We note that the second term in the last inequality is finite as

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s} \sum_{k \leq j} EX_{nk}I[|X_{nk}| < j^{\gamma \alpha}]^s \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s} \left( \sum_{k \leq j} E[X_{nk}^s I[|X_{nk}| \geq j^{\gamma \alpha}] \right)^s \]

\[ \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\delta]} j^{-\alpha s - \alpha \gamma(r-1) s + \alpha \gamma} \sum_{k \leq j} E[X_{nk}^r] \]

\[ \leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s + \alpha \gamma(r-1) s + \alpha \gamma} \sum_{k \leq j} E[X_{nk}^r] \]

\[ \text{const} \sum_{j=1}^{\infty} j^{-\alpha s + \alpha \gamma(r-1) s + \alpha \gamma} (E[\xi^n]^s) \sum_{j=1}^{\infty} j^{-s(\alpha + \alpha \gamma - 1)} < \infty \]

for \( s > \frac{c}{\alpha(1+\gamma)+\gamma \alpha-1} \).

Now we can write

\[ E[\sum_{k \leq j} Y_{k,j}^n]^s = \int_0^{\infty} z^{s-1} P[\sum_{k \leq j} Y_{k,j}^n \geq z] \, dz \]

\[ = \int_0^{j^{\gamma \alpha}} z^{s-1} P[\sum_{k \leq j} Y_{k,j}^n \geq z] \, dz + \int_{j^{\gamma \alpha}}^{\infty} z^{s-1} P[\sum_{k \leq j} Y_{k,j}^n \geq z] \, dz \]

\[ \leq j^{\gamma \alpha s} + \int_{j^{\gamma \alpha}}^{\infty} z^{s-1} P[\sum_{k \leq j} Y_{k,j}^n \geq z] \, dz. \]

But the Fuk-Nagaev inequality (cf. Fuk and Nagaev [12]):

\[ P[\sum_{i=1}^{n} X_i \geq z] \]
\[
\sum_{i=1}^{n} P[X_i \geq \eta z] + \frac{1}{(\eta z)^t} \sum_{i=1}^{n} \int_0^{\eta z} |u|^t dF_{X_i}(u) + \exp\left(-\frac{(1-\eta)^2 z^2}{2e^t \sum_{i=1}^{n} E[X_i^2]}\right),
\]

where \( t \geq 2, \eta = \frac{1}{t+2}, \) allows us to show that

\[
\int_{j^\gamma a}^{\infty} z^{s-1} P[\{\sum_{k \leq j} Y_{k}^n \geq \eta z\}] dz
\]

\[
\leq 2 \left( \sum_{j=1}^{J} \int_{j^\gamma a}^{\infty} z^{s-1} \left( \sum_{k \leq j} P[Y_{k}^n \geq \eta z] \right) dz + \frac{2}{\eta^t} \sum_{j=1}^{J} \int_{j^\gamma a}^{\infty} z^{s-t-1} \int_0^{\eta z} |u|^t dF_{Y_k^n}(u) dz \right)
\]

\[
+ \int_{j^\gamma a}^{\infty} z^{s-1} \exp\left(-\frac{(1-\eta)^2 z^2}{2e^t \sum_{k=1}^{n} E(Y_{k}^n)^2}\right) dz.
\]

Now we see that

\[
\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \sum_{k \leq j} \int_{j^\gamma a}^{\infty} z^{s-1} P[Y_{k}^n \geq \eta z] dz
\]

\[
= \left( \frac{1}{\eta^t} \right) \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \sum_{k=1}^{j} E[Y_{k}^n] \leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s+\gamma j(j^{\gamma a})} < \infty
\]

for \( s > \frac{\alpha+2}{\alpha(1-\gamma)} \).

Moreover, using the assumption on a regular cover (cf. Definition 1.2), we have

\[
\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \sum_{k=1}^{j} \int_{j^\gamma a}^{\infty} z^{s-t-1} \left( \int_0^{\eta z} |u|^t dF_{Y_k^n}(u) \right) dz
\]

\[
\leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \sum_{k=1}^{j} E[X_{nk}] \left( \sum_{k=1}^{j} E[Y_{k}^n] \right) \left( \sum_{k=1}^{j} E[Y_{k}^n] \right)
\]

\[
\leq \text{const} E[\xi]^t \sum_{j=1}^{\infty} j^{-\alpha s+\gamma (s-r)+\gamma (s-r)+t} < \infty
\]

for \( s > \frac{\alpha+2}{\alpha(1-\gamma)} \).

Further on, we note that

\[
\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \int_{j^\gamma a}^{\infty} z^{s-1} \exp\left(-\frac{(1-\eta)^2 z^2}{2e^t \sum_{k=1}^{n} E(Y_{k}^n)^2}\right) dz
\]

\[
\leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \left( \sum_{k=1}^{j} E(Y_{k}^n)^2 \right)^{s/2} \int_0^{\infty} y^{s/2-1} e^{-y} dy
\]

\[
\leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j=1}^{j^{\gamma a}} \left( \sum_{k=1}^{j} E(Y_{k}^n)^2 \right)^{s/2}.
\]
Assume now that \( r > 2 \). Then we have

\[
\sum_{n=1}^{\infty} n^{\alpha - r} \sum_{j \geq [n^p]} j^{-\alpha s} \left( \sum_{k=1}^{j} E(Y_k^n)^2 \right)^{s/2} \leq \text{const} \sum_{n=1}^{\infty} n^{\alpha - r} \sum_{j \geq [n^p]} j^{-\alpha s} \left( jE|\xi|^2 \right)^{s/2}
\]

\[
\leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s + c + s/2} < \infty \quad (2.13)
\]

for \( s > \frac{c+1}{\alpha - 1/2} \).

Similarly it can be proved that for \( r < 2 \)

\[
\sum_{n=1}^{\infty} n^{\alpha - r} \sum_{j \geq [n^p]} j^{-\alpha s} \left( \sum_{k=1}^{j} EY_k^2 \right)^{s/2} \leq \text{const} \sum_{j=1}^{\infty} j^{-s[\alpha - 1/2 - \gamma (2-r)/2] + c} < \infty \quad (2.14)
\]

whenever \( s > \frac{\alpha - 1/2 + \gamma (2-r)/2}{\alpha - 1/2} \) and \( \gamma \) is such that \( \gamma < \frac{2\alpha - 1}{2-r} \).

Collecting the estimates (2.7) - (2.14) we see that the series in (2.6) converges which completes the proof of (2.1) for \( \beta > 0 \).

But for the stronger requirement \( \beta > 1 \) we note that the condition (2.3) is fulfilled under the assumption \( E|X_{nk}|^{\alpha} < \infty, r \geq 1, k \geq 1, n \geq 1 \).

Indeed, we see that

\[
\sum_{n=1}^{\infty} n^{\alpha - r} P\left[ \max_{k \leq v_n} |X_{nk}| \geq \nu_n \right]
\]

\[
\leq \sum_{n=1}^{\infty} n^{\alpha - r} P\left[ \nu_n < n^\beta \right] + \sum_{n=1}^{\infty} n^{\alpha - r} P\left[ \max_{k \leq v_n} |X_{nk}| \geq \nu_n, \nu_n \geq n^\beta \right],
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} \left( 2m \right)^{\alpha - r - 1} \sum_{j=m}^{\infty} P\left[ \max_{k \leq v_n} \left| X_{2m-k} \right| \geq \nu_n, (2j)^{\beta} \leq \nu_n < (2j+1)^{\beta} \right]
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} \left( 2m \right)^{\alpha - r - 1} \sum_{j=m}^{\infty} P\left[ \max_{k \leq (2m+1)^\beta} \left| X_{2m-k} \right| \geq (2j)^{\beta} \right]
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} P\left[ \max_{k \leq (2m+1)^\beta} \left| X_{2m-k} \right| \geq \varepsilon (2m)^{\alpha \beta} \right]
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} (2m)^{\alpha - r - 1} \sum_{j=m}^{\infty} \left( 2m \right)^{\alpha \beta}
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} (2m)^{\alpha - r - 1} \sum_{k \leq (2m+1)^\beta} \left| X_{2m-k} \right| \geq \varepsilon (2m)^{\alpha \beta}
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} (2m)^{\alpha - r - 1} \sum_{k \leq (2m+1)^\beta} \left| X_{2m-k} \right| \geq \varepsilon (2m)^{\alpha \beta}
\]

\[
\leq \text{const} \sum_{m=1}^{\infty} \sum_{k \leq (2m+1)^\beta} \frac{E\left| X_{2m-k} \right|^r}{(2m)^{\alpha \beta r}}
\]

\[
= \text{const} E|\xi|^r \sum_{m=1}^{\infty} (2m)^{\alpha - r - 1 - \beta (ar-1)} < \infty
\]
for $\beta \geq 1$, which gives (2.3) and ends the proof of Theorem 2.1.

Now we note that the condition (2.3) ($0 < \beta < 1$) is fulfilled under a stronger moment condition than that of Theorem 2.1.

**COROLLARY.** Let $\{X_{nk}, n \geq 1, k \geq 1\}$ be an array of rowwise independent random variables such that $X_{n1}, X_{n2}, \ldots, X_{nk}, n \geq 1, k \geq 1$, form an independent regular cover of a random variable $\xi$, and assume that $E[X_{nk}] = 0$, $E[X_{nk}^{r+2}] < \infty$, $n \geq 1, k \geq 1$, $E[\xi] = 0$, and $E[|\xi|^{r+2}] < \infty$ for $r \geq 1, \alpha > 1/2, \alpha > 1, 0 < \beta < 1$.

If $\{\nu_n, n \geq 1\}$ is a sequence of positive integer-valued random variables such that

$$\sum_{n=1}^{\infty} n^{ar^2} P[\nu_n < n^\beta] < \infty,$$

then for any given $\epsilon > 0$

$$\sum_{n=1}^{\infty} P[|S_{\nu_n}| \geq \epsilon \nu_n] < \infty.$$

**PROOF.** It is enough to see that under the considered case the condition (2.3) is satisfied. Since

$$\sum_{n=1}^{\infty} n^{ar^2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \epsilon \nu_n]$$

$$\leq \sum_{n=1}^{\infty} n^{ar^2} P[\nu_n < n^\beta] + \sum_{n=1}^{\infty} n^{ar^2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \epsilon \nu_n, \nu_n \geq n^\beta],$$

then we need only to note that

$$\sum_{n=1}^{\infty} n^{ar^2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \epsilon \nu_n, \nu_n \geq n^\beta]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{j=m}^{\max_{k \leq \nu_n}} P[\max_{k \leq \nu_n} |X_{2^m k}| \geq \epsilon \nu_n, \nu_n \geq n^\beta]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{j=m}^{\max_{k \leq \nu_n}} P[\max_{k \leq \nu_n} |X_{2^m k}| \geq \epsilon (2^j)^{ar^2}]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{j=m}^{\max_{k \leq \nu_n}} P[\max_{k \leq \nu_n} |X_{2^m k}| \geq \epsilon (2^j)^{ar^2}]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{k \leq \nu_n} P[|X_{2^m k}| \geq \epsilon (2^m)^{ar^2}]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{k \leq \nu_n} P[|X_{2^m k}| \geq \epsilon (2^m)^{ar^2}]$$

$$\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{ar^2-1} \sum_{k \leq \nu_n} P[|\xi|^{ar^2} \geq \epsilon (2^m)^{ar^2}]$$

Note that the moment condition of Corollary is close to optimal which shows the following statement.
THEOREM 2.2. Let \( \{X_{nk}, n \geq 1, k \geq 1\} \) be an array of rowwise independent random variables such that \( X_{n1}, X_{n2}, \ldots, X_{nk}, n \geq 1, k \geq 1, \) form an independent regular cover of a random variable \( \xi, \) and assume that \( EX_{nk} = 0. \)

Then for \( r \geq 1, \alpha > 1/2, \alpha r > 1, \beta > 0, \) the convergence of the series

\[
\sum_{n=1}^{\infty} n^{\alpha r-2} P[|S_{[n^a]}| \geq \epsilon n^{\alpha \beta}] < \infty
\]  

implies \( E|\xi|^{\frac{\alpha r-1+\beta}{\alpha}} < \infty. \)

PROOF. Let \( \mu_n \) be a median of \( S_n, \) i.e. \( \mu_n = \{ t : P[S_n < t] \geq 1/2 \}. \) By the standard symmetrization inequalities (cf. Loève [13]) we have

\[
P[|S_{[n^a]}| \geq \epsilon n^{\alpha \beta}] \geq \frac{1}{2} P[|S_{[n^a]}| \geq 2\epsilon n^{\alpha \beta}] \geq \frac{1}{4} P[|S_{[n^a]}| - \mu_{[n^a]}| \geq 2\epsilon n^{\alpha \beta}]
\]

which by (2.15) gives

\[
\sum_{n=1}^{\infty} n^{\alpha r-2} P[|S_{[n^a]}| - \mu_{[n^a]} \geq 2\epsilon n^{\alpha \beta}] < \infty.
\]  

We note that \( \tau_n = \sup \{ \tau : P[|\xi| \geq \tau] \geq \frac{1}{4n^{\beta}} \}. \) We note that \( \tau_n \geq \tau_{n-1}, \) and

\[
P[|\xi| \geq \tau_n] \geq \frac{1}{4n^{\beta}}, \quad P[|\xi| \leq \tau_n] \geq 1 - \frac{1}{4n^{\beta}}.
\]  

If the \( \tau_n \) are all negative then \( P[|\xi| < 0] = 1 \) so \( E(\xi^+)^{\frac{\alpha r-1+\beta}{\alpha}} = 0 < \infty. \) Thus, assume that for \( n \) sufficiently large we have \( \tau_n \geq 0. \) Moreover, we note that by (2.17)

\[
P[X_{nk} > \tau_n] \leq P[X_{n1} > \tau_1] + \ldots + P[X_{n[n^a]} > \tau_n]
\]

\[
\leq n^{\beta} P[|\xi| > \tau_n] = n^{\beta} (1 - P[|\xi| \leq \tau_n]) \leq \frac{1}{4}.
\]

Furthermore, for \( k \in \{1, \ldots, [n^a]\} \) define \( \{\rho_{nk}, 1 \leq k \leq [n^a]\} \) with

\[
\rho_{nk} = \sup \{ \rho : P[S_{[n^a]}] \geq X_{nk} \geq \rho \} \geq \frac{1}{3}.
\]

Then we have

\[
P[S_{[n^a]}] - X_{nk} \geq \rho_{nk} \geq \frac{1}{3}, \quad P[S_{[n^a]}] - X_{nk} \leq \rho_{nk} \geq \frac{2}{3}.
\]  

Using the independence \( S_{[n^a]} - X_{nk} \) and \( X_{nk}, \) (2.18) and (2.19) we get

\[
P[S_{[n^a]}] \leq \tau_n + \rho_{nk} \geq P[X_{nk} \leq \tau_n], \quad S_{[n^a]} - X_{nk} \leq \rho_{nk}
\]

\[
= P[X_{nk} \leq \tau_n] P[S_{[n^a]} - X_{nk} \leq \rho_{nk}]
\]

\[
= (1 - P[X_{nk} > \tau_n]) P[S_{[n^a]} - X_{nk} \leq \rho_{nk}] \geq \frac{1}{2}.
\]

Now using

\[
T_{nk} := [X_{nk} > 2\epsilon n^{\alpha \beta} + \tau_n], \quad R_{nk} := [S_{[n^a]} - X_{nk} \geq \rho_{nk}]
\]
we see that

\[ P \left[ S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]} \right] \geq P \left[ S_{[n^\beta]} > 2\varepsilon n^{\alpha\beta} + \tau_n + \rho_{nk} \right] \geq P \left[ \bigcup_{k=1}^{[n^\beta]} (T_{nk} \cap R_{nk}) \right] \]

\[ = \sum_{k=1}^{[n^\beta]} P \left[ (T_{n1} \cap R_{n1})^c \cap \ldots \cap (T_{nk-1} \cap R_{nk-1})^c \cap (T_{nk} \cap R_{nk}) \right] \]

\[ \geq \sum_{k=1}^{[n^\beta]} \left[ P \left[ T_{n1}^c \cap \ldots \cap T_{nk-1}^c \cap T_{nk} \cap R_{nk} \right] \right. \]

\[ \geq \sum_{k=1}^{[n^\beta]} \left\{ P \left[ T_{nk} \cap R_{nk} \right] - P \left[ (T_{n1} \cup \ldots \cup T_{nk-1}) \cap R_{nk} \right] \right\} \]

\[ \geq \sum_{k=1}^{[n^\beta]} P \left[ T_{nk} \right] \left\{ P \left[ R_{nk} \right] - \sum_{k=1}^{[n^\beta]} P \left[ T_{nk} \right] \right\} \]

Having \( \tau_n \geq 0 \) for sufficiently large \( n \) we get

\[ \sum_{k=1}^{[n^\beta]} P \left[ T_{nk} \right] = \sum_{k=1}^{[n^\beta]} P \left[ X_{nk} \geq 2\varepsilon n^{\alpha\beta} + \tau_n \right] \]

\[ < n^\beta P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n] = n^\beta \left( 1 - P[\xi \leq 2\varepsilon n^{\alpha\beta} + \tau_n] \right) \leq \frac{1}{4}. \]

where we have used the covering identity (1.1) as well as (2.17).

Thus, (2.20) implies that

\[ P \left[ S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]} \right] \geq \frac{1}{12} [n^\beta] P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n] \]

for \( n \) sufficiently large.

Hence, by (2.16) we conclude that

\[ \sum_{n=1}^{\infty} n^{\alpha r - 2 + \beta} P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n] < \infty \]

which is equivalent to

\[ \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1 + \beta} P[\xi > 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2m}] < \infty. \]  

(2.21)

Similarly as in Pruss [11] (cf. Lemma 4) we can show that for \( m \) sufficiently large we have

\[ \tau_{2^m+1} \leq 2^m \varepsilon + \tau_{2^m}. \]

Assume that \( M \) is a positive integer number such that

\[ \tau_{2^m+1} \leq 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2^m} \quad \text{for} \quad m \geq M. \]

Iterating this inequality for \( m \geq M \) we obtain

\[ \tau_{2^m} < 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2^m}. \]
which gives $2\varepsilon (2^m)^{\alpha \beta} + \tau_2 m < 4\varepsilon (2^m)^{\alpha \beta} + \tau_2 M$.

Therefore, using (2.21), we have

$$\sum_{m=1}^{\infty} (2^m)^{\alpha r - 1 + \beta} P[\xi > 4\varepsilon (2^m)^{\alpha \beta} + \tau_2 M] < \infty$$

which proves that

$$\infty > \sum_{n=1}^{\infty} n^{\alpha r - 2 + \beta} P[\xi > 4\varepsilon n^{\alpha \beta} + \tau_2 M]$$

$$\geq \sum_{n=1}^{\infty} n^{\alpha r - 2 + \beta} P[\xi > (4\varepsilon + \tau_2 M)n^{\alpha \beta}] \geq \text{const} E(\xi^+) \frac{\alpha r - 1 + \beta}{\alpha \beta}.$$ 

Similarly one can show that $E(\xi^-) \frac{\alpha r - 1 + \beta}{\alpha \beta} < \infty$, which completes the proof of Theorem 2.2.

ACKNOWLEDGEMENT. We are very grateful to the referee for his helpful comments allowing us to improve the previous version of the paper.

REFERENCES


