\section*{\textbf{\textit{\theta}}} - \textit{ALMOST SUMMABLE SEQUENCES}

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\textbf{ABSTRACT} King\textsuperscript{3} introduced and examined the concepts of almost $A$-summable sequence, almost conservative matrix and almost regular matrix. By following King, in this paper we introduce and examine the concepts of $\theta$-almost $A$-summable sequence, $\theta$-almost conservative matrix and $\theta$-almost regular matrix.

\textbf{KEY WORDS AND PHRASES}\ Lacunary sequence, $\theta$-almost summable sequence, $\theta$-almost conservative matrix, $\theta$-almost regular matrix

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\section*{1. INTRODUCTION}

Let $A=(a_{nk})$ be an infinite matrix of complex numbers and let $x=\{x_n\}$ be a sequence of complex numbers. The sequence $\{A_n(x)\}$ defined by

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$$

is called the $A$-transform of $x$ whenever the series converges for $n=1,2,3,\ldots$. The sequence $x$ is said to be $A$-summable to $L$ if $\{A_n(x)\}$ converges to $L$.

Let $m$ denote the linear space of bounded sequences. A sequence $x \in m$ is said to be almost convergent to $L$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k = L$$ uniformly in $m$.

A sequence $x \in m$ is said to be almost $A$-summable to $L$ if the $A$-transform of $x$ is almost convergent to $L$. The matrix $A$ is said to be almost conservative if $x \in e$ implies that the $A$-transform of $x$ is almost convergent. A is said to be almost regular if the $A$-transform of $x$ almost convergent to the limit of $x$ for each $x \in c$, where $c$ is the linear space of convergent sequences.

In the sequel the following notation is used: $C$ denotes the complex numbers, $Z$ denotes the integers and $N$ denotes positive integers. If $x=\{x_k\}$ is an element of $e$, then $\|x\|$ is defined by $\|x\| = \sup \{|x_k| \mid k \in N\}$. The linear space of all continuous linear functionals on $e$ is denoted by $e^*$. We use the special sequences $e=(1,1,\ldots), e_k=(0,\ldots,0,1,0,\ldots)$ (with 1 in rank $k$) and denote $\Delta=\{e,e_0,e_1,e_2,\ldots\}$.

By a lacunary sequence we shall mean an increasing integer sequence $\theta=(k_r)$ such that $k_0=0$ and $h_r=k_r-k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by $\theta=(k_r)$ will be denoted by $I_r=(k_{r-1},k_r]$.

Freedman, Sember and Raphael\textsuperscript{2} defined the space $N_{\theta}$ in the following manner: For any lacunary sequence $\theta=(k_r)$,

$$N_{\theta} = \left\{ x=(x_k) \mid \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L \right\}$$

Quite recently, the space $M_{\theta}$ was introduced by Das and Mishra\textsuperscript{1} as below

$$M_{\theta} = \left\{ x=(x_k) \mid \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0, \text{ for some } L, \text{ uniformly in } m \right\}$$

King[3], introduced and examined the concepts of almost A-summable sequence, almost conservative matrix and almost regular matrix.

In the present note, we introduce and examine the concepts of \( \theta \)-almost A-summable sequence, \( \theta \)-almost conservative matrix and \( \theta \)-almost regular matrix.

2. DEFINITIONS AND THEOREMS

Definition 1 A sequence \( x \) is said to be \( \theta \)-almost convergent to \( L \) if, for any lacunary sequence \( \theta \),
\[
\lim_{r \to \infty} h_r \sum_{k=0}^{\infty} x_{m} = L \quad \text{uniformly in } m
\]

Definition 2 A sequence \( x \) is said to be \( \theta \)-almost A-summable to \( L \) if, for any lacunary sequence \( \theta \), the A-transform of \( x \) is \( \theta \)-almost convergent to \( L \).

Definition 3 The matrix \( A \) is said to be \( \theta \)-almost conservative if \( x \) implies that the A-transform of \( x \) is \( \theta \)-almost convergent. \( A \) is said to be \( \theta \)-almost regular if the A-transform of \( x \) is \( \theta \)-almost convergent to the limit of \( x \) for each \( x \in c \).

Theorem 1. Let \( A=(a_{nk}) \) be an infinite matrix and let \( \theta \) be a lacunary sequence. Then the matrix \( A \) is \( \theta \)-almost conservative if and only if

(i) \( \sup_{r} \left( \sum_{k=0}^{\infty} a_{n,k} \right) \infty \), \( n=0,1,2, \ldots \),

(ii) there exists an \( a \in c^* \) such that
\[
\lim_{r \to \infty} h_r \sum_{k=0}^{\infty} a_{n,k} = a \quad \text{uniformly in } n, \text{ and}
\]

(iii) there exists an \( a_k \in c, k=0,1,2, \ldots \), such that
\[
\lim_{r \to \infty} h_r \sum_{k=0}^{\infty} a_{n,k} = a_k \quad \text{uniformly in } n.
\]

Proof. Suppose that \( A \) is \( \theta \)-almost conservative. Fix \( n \in N \). Let
\[
t_m(x) = \frac{1}{h_r} \sum_{j=0}^{\infty} w_{j,n}(x),
\]

where \( w_{j,n}(x) = \sum_{k=0}^{\infty} a_{j,n,k} x_k \).

It is clear that \( w_{j,n} \in c^* \), \( j=0,1,2, \ldots \). Hence \( t_m \in c^* \). Since \( A \) is \( \theta \)-almost conservative
\[
\lim_{r \to \infty} t_m(x) = t(x) \quad \text{uniformly in } n.
\]

It follows that \( \{t_m(x)\} \) is bounded for \( x \in c \) and fixed \( n \).

Hence \( \|t_m\| \) is bounded by the uniform boundedness principle.

For each \( p \in N \), define the sequence \( v=(v(n,r)) \) by
\[
v_k = \begin{cases} 
\text{sgn} \sum_{j=0}^{p} a_{j,n,k}, & 0 \leq k \leq p \\
0, & p < k
\end{cases}
\]

Then \( v \in c \), \( \|v\| = 1 \), and
\[
|f_m(v)| = \frac{1}{h_r} \sum_{k=0}^{p} \sum_{j=0}^{\infty} a_{j,n,k}
\]

Hence \( |f_m(v)| \leq \|f_m\| = \|t_m\| \). Therefore \( \frac{1}{h_r} \sum_{k=0}^{\infty} \sum_{j=0}^{p} a_{j,n,k} \leq \|t_m\| \), so that (i) follows.

Since \( e \) and \( e_k \) are convergent sequences, \( k=0,1, \ldots \), \( \lim_{n \to \infty} t_m(e) \) and \( \lim_{n \to \infty} t_m(e_k) \) must exist uniformly in \( n \).

Hence (ii) and (iii) must hold.

Now assume that
\[
t_m(x) = \frac{1}{h_r} \sum_{j=0}^{\infty} a_{j,n,k} x_k = \frac{1}{h_r} \sum_{k=0}^{\infty} a_{n,k} x_k
\]

so that
\[
|f_m(x)| \leq \frac{1}{h_r} \sum_{k=0}^{\infty} |a_{n,k}||x_k| \quad j,n=0,1,2, \ldots
\]
Therefore, \( |t_m(x)| \leq K_n \|x\| \) by (ii), where \( K_n \) is a constant independent of \( n \). Hence \( t_m \in c^* \), and the sequence \( \{t_m(n)\} \) is bounded for each \( n \) (ii) and (iii) imply that \( \lim_{n \to \infty} t_m(e) \) and \( t_m(e_n) \) exist for \( n = k = 1, 2 \). Since \( \Delta \) is a fundamental set in \( \ell \) it follows from an elementary result of functional analysis that \( \lim_{n \to \infty} t_m(x) = t(x) \) exists and \( t(x) \in \ell^* \). Therefore \( t \) has the form

\[
t_m(x) = b - \sum_{k=0}^{x} t_m(e_k) + \sum_{k=1}^{x} x_k t_m(e_k),
\]

where \( b = \lim_{n \to \infty} x_k \). But \( t_m(e_k) = a \) and \( t_m(e_n) = a_{kn, k=0,1,2, \ldots} \), by (ii) and (iii), respectively. Hence

\[
t_m(x) = b - \sum_{k=0}^{x} a_k + \sum_{k=1}^{x} x_k t_m(e_k),
\]

(1)

Since \( t_m \in c^* \) for each \( r \) and \( n \), it has the form

\[
t_m(x) = b - \sum_{k=0}^{x} a_k + \sum_{k=1}^{x} x_k t_m(e_k),
\]

(2)

It is easy to see from (1) and (2) that the convergence of \( \{t_m(x)\} \) to \( t(x) \) is uniform in \( n \), since \( t_m(e_k) \to a \) and \( t_m(e_n) \to a \), uniformly in \( n \). Therefore \( A \) is \( \theta \)-almost conservative and the theorem is proved.

**Theorem 2** Let \( A = (a_{kh}) \) be an infinite matrix and let \( \theta \) be a lacunary sequence. Then the matrix \( A \) is \( \theta \)-almost regular if and only if

\[
\sup_{r \in \mathbb{N}} \left\{ \sum_{k=0}^{r} a_{n,r,k} \right\} < \infty, \quad n = 0, 1, 2, \ldots
\]

(iv)

\[
\lim_{r \to \infty} \frac{1}{r} \sum_{k=0}^{r} a_{n,r,k} = 1 \quad \text{uniformly in } n, \quad k = 0, 1, 2, \ldots
\]

(v)

\[
\lim_{r \to \infty} \frac{1}{r} \sum_{k=0}^{r} a_{n,r,k} = 0 \quad \text{uniformly in } n, \quad k = 0, 1, 2, \ldots
\]

(vi)

**Proof.** Suppose that \( A \) is \( \theta \)-almost regular. Then \( A \) is \( \theta \)-almost conservative so that (iv) must hold by Theorem 1 (v) and (vi) must hold since the \( A \)-transform of the sequences \( e_k \) and \( e \) must be \( \theta \)-almost convergent to 0 and 1, respectively.

Now suppose that (iv),(v) and (vi) hold. Then \( A \) is \( \theta \)-almost conservative by Theorem 1. Therefore \( \lim_{n \to \infty} t_m(x) = t(x) \) uniformly in \( n \) for each \( x \in c \). The representation (1) gives \( t(x) = \lim_{n \to \infty} x_k \). Hence \( A \) is \( \theta \)-almost regular. This proves the theorem.

In the applications of summability theory to function theory it is important to know the region in which the sequence of partial sums of the geometric series is \( \theta \)-summable to \( 1/(1-z) \) for a given matrix \( A \). The following theorem is helpful in determining the region in which the sequence of partial sums of the geometric series is \( \theta \)-almost \( A \)-summable to \( 1/(1-z) \).

**Theorem 3.** Let \( A \) be an infinite matrix such that (v) holds. The sequence of partial sums of the geometric series is \( \theta \)-almost \( A \)-summable to \( 1/(1-z) \) if and only if \( z \in \Omega \) where

\[
\Omega = \left\{ z : \lim_{r \to \infty} \frac{1}{h_r} \sum_{j=1}^{r} a_{n,r,k} z^k = 0 \quad \text{uniformly in } n \right\}
\]

**Proof.** Let \( \{s_r(z)\} \) denote the sequence of partial sums of the geometric series. Then

\[
t_m(x) = \frac{1}{h_r} \sum_{j=1}^{r} a_{n,r,k} s_r(z) = \frac{1}{h_r} \sum_{j=1}^{r} a_{n,r,k} \left[ \frac{1 - z^{r+1}}{1 - z} \right]
\]

Hence,

\[
\lim_{r \to \infty} t_m(x) = \frac{1}{1-z} - \lim_{r \to \infty} \frac{z}{h_r (1-z)} \sum_{j=1}^{r} a_{n,r,k} z^k
\]

Therefore

\[
\lim_{r \to \infty} t_m(x) = \frac{1}{1-z} \quad \text{uniformly in } n \text{ if and only if } z \in \Omega
\]


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