NOTES ON $(\alpha, \beta)$-DERIVATIONS

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ABSTRACT. Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero ideal of $R$ and $0 \neq d$ a $(\alpha, \beta)$-derivation of $R$ where $\alpha$ and $\beta$ are automorphisms of $R$. i) $[d(U), a] = 0$ then $a \in Z$ ii) For $a, b \in R$, the following conditions are equivalent (I) $\alpha(a)d(x) = d(x)\beta(b)$, for all $x \in U$ (II) Either $\alpha(a) = \beta(b) \in C_R(d(U))$ or $C_R(a) = C_R(b) = R'$ and $a[a, x] = [a, x]b$ (or $a[b, x] = [b, x]b$) for all $x \in U$. Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$. iii) Let $d$ be a $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$. Suppose that $dg$ is a $(\alpha \gamma, \beta \delta)$-derivation and $g$ commutes both $\gamma$ and $\delta$ then $g(x)U \alpha^{-1}d(y) = 0$, for all $x, y \in U$. iv) Let $Ann(U) = 0$ and $d$ be an $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$. If for all $x, y \in U$, $\beta^{-1}(d(x))Ug(y) = 0 = g(x)U \alpha^{-1}(d(y))$ then $dg$ is a $(\alpha \gamma, \beta \delta)$-derivation on $R$

KEY WORDS AND PHRASES: Derivation, semiprime ring, prime ring, commutative

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1. INTRODUCTION

Let $R$ be a ring and $X$ be a subset of $R$. Let $Ann_+(X) = \{a \in R \mid xa = 0 \text{ all } x \in X\}$ and $Ann_-(X) = \{a \in R \mid ax = 0 \text{ all } x \in X\}$ be the right and left annihilators, respectively, of the subset $X$ of $R$. If $R$ is a semiprime ring then the left and right and two-sided annihilators of an ideal $X$ coincide. It will be denoted by $Ann(X)$. Let $U$ be an ideal of $R$. Note that if $\sigma$ is an automorphism of $R$ and $Ann(U) = 0$ then $Ann(\sigma(U)) = 0$. Let $R$ be a ring and $\alpha, \beta$ be two automorphisms of $R$. An additive mapping $d : R \to R$ is called an $(\alpha, \beta)$-derivation if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ holds for all pairs $x, y \in R$. Throughout this note $R$ will represent an associative ring. Let $R' = \{x \in R \mid d(x) = 0\}$. The centralizer of a subset $A$ of $R$ is $C_R(A) = \{y \in R \mid ay = ya, \forall a \in A\}$, $C_R(R) = Z$, the center of $R$.

There are two motivations for this research. Herstein [1] has proved that if $R$ is a prime ring of characteristic not 2, and $0 \neq d$ be a derivation of $R$, then any element $a \in R$ satisfying $ad(x) = d(x)a$ for all $x \in R$, should be central. In [2], Daif has proved the following theorem. Let $R$ be a prime ring and $a, b \in R$. Then the following conditions are equivalent

(i) $ad(x) = d(x)b$, $\forall x \in R$

(ii) Either $a = b \in C_R(d(R))$ or $C_R(a) = C_R(b) = R'$ and $a[a, x] = [a, x]b$ (or $a[b, x] = [b, x]b$) for all $x \in R$. In the first part of this note we generalized these two theorems for an ideal $U$ and $(\alpha, \beta)$-derivation of $R$.
In the second part, Bresar and Vukman [3] give some results concerning two derivations in semiprime rings. We will generalize some of these results by taking an ideal of \( R \) instead of \( R \) and extend to more general mappings. As a result of this, we will give a generalization of a well-known result of Posner which states that if \( R \) is a prime ring of characteristic not 2 and \( d, g \) are nonzero derivation of \( R \) then \( dg \) cannot be a derivation.

2. RESULTS

**Lemma 1.** Let \( R \) be a prime ring of characteristics not 2, \((0) \neq U\) an ideal of \( R \), \( 0 \neq d : R \to R \) a \((\alpha, \beta)\)-derivation such that \( \alpha d = d \alpha, \beta d = \beta d \) and \( \alpha, \beta, \alpha R, \beta R \in R \). If \( a \in C_R(d(U)) \) then \( a \in Z \).

**Proof.** Since \( a \in C_R(d(U)) \), \( ad(x) = d(x)a \) for all \( x \in U \). Replacing \( x \) by \( xy \), \( y \in U \), we obtain \( \alpha(x)d(y) + ad(x)\beta(y) = \alpha(x)d(y)a + d(x)\beta(y)a \). Using hypothesis we have

\[
d(x)[\alpha(x), \beta(y)] = [\alpha(x), \beta(y)]d(y).
\]

Taking \( yr, r \in R \), instead of \( y \), we obtain

\[
d(x)\beta(y)[\alpha(x), \beta(r)] = [\alpha(x), \beta(y)]d(r) \quad \text{for all } x, y \in U, r \in R.
\]

If we replace \( r \) by \( \beta^{-1}(d(z)), z \in U \), we get \( d(x)\beta(y)[\alpha(a), d(z)] = [\alpha(x), \alpha(y)]\beta^{-1}(d^2(z)) \). Since \( a \in C_R(d(U)) \) we have \( [\alpha(x), a] \alpha(y)\beta^{-1}(d^2(z)) = 0 \) for all \( x, y, z \in U \). Since \( a(U) \) is an ideal of \( R \) and \( R \) is prime we get \( a \in Z \) or \( d^2(U) = 0 \). If \( d^2(U) = 0 \) then \( d^2(xy) = \alpha^2(x)d^2(y) + 2d(\alpha(x))d(\beta(y)) \) and so \( d(\alpha(x))d(\beta(y)) = 0 \). By [4, Lemma 3] we have a contradiction. Thus \( a \in Z \).

**Theorem 1.** Let \( R \) be a prime ring of characteristics not 2, \((0) \neq U\) an ideal of \( R \), \( 0 \neq d : R \to R \) a \((\alpha, \beta)\)-derivation, \( 0 \neq d : R \to R \) a \((\alpha, \beta)\)-derivation, \( (0) \neq U \) and ideal of \( R \) and \( a, b \in R \). Then the following conditions are equivalent:

(I) \( \alpha(d(x)) = d(x)\beta(b) \), for all \( x \in U \).

(II) Either \( \beta(b) = \alpha(a) \in C_R(d(U)) \) or \( C_R(a) = C_R(b) = R' \) and \( a \times (a, x) = a \times (a, x) \) for all \( x \in U \).

**Proof.** (I) \( \Rightarrow \) (II) If \( a \in C_R(d(U)) \) then by Lemma 1 we get \( \alpha(a) \in Z \). (I) gives \( d(x)(\beta(b) - \alpha(a)) = 0 \), for all \( x \in U \). By [4, Lemma 3] it implies that \( \beta(b) = \alpha(a) \). Similarly, if \( \beta(b) \in C_R(d(U)) \) then \( \beta(b) = \alpha(a) \).

We assume henceforth that neither \( \alpha(a) \) nor \( \beta(b) \) in \( C_R(d(U)) \). Let in (I) \( x = rx \), \( r \in R \), and using (I), we have \( \alpha(a)\alpha(r) \beta(x) + \alpha(a) \beta(r) \beta(x) = \alpha(r) \beta(x) \beta(b) + d(r)\beta(x) \beta(b) \) and so

\[
\alpha([a, r])d(x) = d(r)\beta(xb) - \alpha(a)d(r)\beta(x).
\]

Taking \( y \) instead of \( r \) where \( y \in U \), in (2.1) and using (I) we obtain

\[
\alpha([a, y])d(x) = d(y)\beta([x, b]), \quad \text{for all } x, y \in U.
\]

(2.2)

Now if \( d(x) = 0 \) then (2.2) gives us \( d(y)\beta([x, b]) = 0 \) for all \( y \in U \). By [4, Lemma 3], we get \( x \in C_R(b) \). Conversely, if \( x \in C_R(b) \), then (2.2) gives us \( \alpha([y, a])d(x) = 0 \). Since by [4, Lemma 3] \( a \notin Z \), we have \( d(x) = 0 \). Therefore \( C_R(b) = R' \). Similarly, we can show that \( C_R(a) = R' \). In particular, \( d(a) = d(b) = 0 \) and \( ab = ba \).

Replace \( r \) by \( yb \), \( y \in U \), in (2.1) we have \( \alpha([a, y])\alpha(b)d(x) = d(y)\beta(b)(xb) - \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(bx) - \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(bx) = \alpha(a)[a, y]d(x) \) and so

\[
\alpha([a, y]b - a[a, y])d(x) = 0 \quad \text{for all } x, y \in U.
\]

By [4, Lemma 3] we obtain

\[
a[a, y] = [a, y]b \quad \text{for all } y \in U.
\]
Furthermore, replacing $x$ by $ax$ in (2.2) and using (2.2) and hypothesis we also have $a[b, x] = [b, x]b$

(II) $\Rightarrow$ (I) If $\alpha(a) = \beta(b) \in C_R(d(U))$ it is obviously $\alpha(a)d(x) = d(x)\beta(b)$ for all $x \in U$. Therefore it suffices to show that if $C_R(a) = C_R(b) = R'$ and $a[a, x] = [a, x]b$ for all $x \in U$ then $\alpha(a)d(x) = d(x)\beta(b)$ for all $x \in U$.

Since $d(a) = d(b) = 0$, $ab = ba$, $[a, ax - xb] = a[a, x] - [a, x]b = 0$ It gives $ax - xb \in R'$ and so $0 = d(ax - xb) = \alpha(a)d(x) - d(x)\beta(b)$. This proves the theorem.

For the second part we begin with

**Lemma 2** [3, Lemma 1]. Let $R$ be a 2-torsion free semiprime ring and $a, b$ the elements of $R$. Then the following conditions are equivalent:

(i) $axb = 0$ for all $x \in R$
(ii) $bxa = 0$ for all $x \in R$
(iii) $axb + bxa = 0$ for all $x \in R$

If one of these conditions is fulfilled then $ab = ba = 0$ too.

**Lemma 3.** Let $R$ be a semiprime ring and $U$ a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $d$ be an $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$. If $d(U)Ug(U) = 0$ then $d(R)Ug(R) = 0$.

**Proof.** For all $x, y, z \in U$, $d(x)yg(z) = 0$. Replace $x$ by $ax$, $s \in R$ we have $0 = d(xs)yg(z) = \alpha(x)d(s)yg(z) + d(x)\beta(s)yg(z)$ Since $\beta(s)y \in U$, the last equation implies that $\alpha(x)d(s)yg(z) = 0$, for all $x, y, z \in U$ and $s \in R$. Taking $t$ instead of $z$, where $t \in R$, we have $0 = \alpha(x)d(s)y\gamma(t)g(z) + \alpha(x)d(s)yg(t)\delta(z)$. Since $\gamma(t) \in U$, it gives $\alpha(x)d(s)yg(t)\delta(z) = 0$ for all $x, y, z \in U$ and $s, t \in R$. Therefore $d(s)yg(t)\delta(z) \in Ann(\alpha(U)) = 0$. Thus we get $d(s)yg(t)\delta(z) = 0$ for all $y, z \in U$ and $s, t \in R$. Hence $d(s)yg(t) \in Ann(\delta(U)) = 0$. As a result of this, it implies that $d(R)Ug(R) = 0$.

**Lemma 4.** Let $R$ be a semiprime ring and $U$ be a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $a, b \in R$ be such that $aUb = 0$ then $aRb = 0$.

**Proof.** For all $x \in U = axb$. Replace $x$ by $tbxrat$, where $t, r \in r$ we have $atbxratbx = 0$. Since $R$ is semiprime ring, this implies that $atbU = 0$ for all $t \in R$. Thus $atb \in Ann(U) = 0$ we get $aRb = 0$.

**Theorem 2.** Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ with $Ann(U) = 0$. Let $d$ be a $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$. Suppose that $dg$ is an $(\alpha\gamma, \beta\delta)$-derivation and $g$ commutes both $\gamma$ and $\delta$. Then $g(x)Ua^{-1}d(y) = 0$, for all $x, y \in U$.

**Proof.** Since $g$ commutes both $\gamma$ and $\delta$, from the first part to the proof of [5, Lemma 1] there is no loss of generality in assuming $\beta = 1$ and $\delta = 1$ For all $x, y \in U$, $dg(xy) = d(\gamma(x))g(y) + g(x)y = \alpha\gamma(x)dg(y) + d(\gamma(x))g(y) + \alpha(g(x))d(y) + dg(x)y$. On the other hand, since $dg$ is an $(\alpha\gamma, 1)$-derivation we have $dg(xy) = \alpha\gamma(x)dg(y) + d(\gamma(x))g(y)$. Comparing the two expressions so obtained for $dg(xy)$, we see that

$$d(\gamma(x))g(y) + \alpha(g(x))d(y) = 0 \quad \text{for all} \quad x, y \in U. \tag{2.3}$$

Replacing $y$ by $yz$ where $z \in R$ in (2.3) we obtain $0 = d(\gamma(x))g(yz) + \alpha(g(x))d(yz) = d(\gamma(x))\gamma(y)g(z) + d(\gamma(x))g(y)z + \alpha(g(x))\alpha(y)d(z) + \alpha(g(x))d(y)z = d(\gamma(x))g(y) + \alpha(g(x))d(y)\gamma(z) + \alpha(g(x))\alpha(y)d(z)$. This relation reduces to

$$d(\gamma(x))\gamma(y)g(z) + \alpha(g(x))\alpha(y)d(z) = 0 \quad \text{for all} \quad x, y \in U, z \in R. \tag{2.4}$$

Replace $y$ by $yg(t)$, $t \in U$ and take $z \in U$ we have $d(\gamma(x))\gamma(y)g(t)g(z) + \alpha(g(x))\alpha(y)\alpha(g(t))d(z) = 0$. Considering this relation (2.4) and (2.3) we obtain $d(\gamma(x))\gamma(y)g(t)g(z) = -\alpha(g(x))\alpha(y)d(t)g(z) = \alpha(g(x))\alpha(y)\alpha(g(t))d(z)$ for all $x, y, z \in U$. Comparing the last two relations we get $2\alpha(g(x))\alpha(y)\alpha(g(t))d(z) = 0$. Since $R$ is 2-torsion free, it gives
Replacing $t$ by $tu, u \in U$ it follows $0 = g(x)y\gamma(t)g(u)\alpha^{-1}(d(z)) + g(x)yg(t)u\alpha^{-1}(d(z))$. Since $\gamma(t) \in U$ this relation reduces to $g(x)u\alpha^{-1}(d(z)) = 0$ for all $x, t, u, z \in U$. By Lemma 4 we have for all $x, t, u, z \in U, g(x)Rg(t)u\alpha^{-1}(d(z)) = 0$. In particular $g(x)u\alpha^{-1}(d(z))Rg(x)u\alpha^{-1}(d(z)) = 0$ for all $x, u, z \in U$. Since $R$ is semiprime we obtain $g(x)u\alpha^{-1}(d(z)) = 0$ for all $x, z \in U$.

**COROLLARY.** Let $R$ be a prime ring of characteristic not 2, $d$ be an $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$. If the composition $dg$ is a $(\alpha\gamma, \beta\delta)$-derivation then $d = 0$ or $g = 0$.

**THEOREM 3.** Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $d$ be a $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$. If for all $x, y \in U, \beta^{-1}(d(x))Ug(y) = g(x)U\alpha^{-1}(d(y)) = 0$ then $dg$ is a $(\alpha\gamma, \beta\delta)$-derivation on $R$

**PROOF.** From Lemma 3 and Lemma 4, we get $\beta^{-1}(d(x))yg(z) = 0 = g(x)y\alpha^{-1}(d(z))$ for all $x, y, z \in R$. On the other hand, since $\beta^{-1}(d(x))yg(z) = 0$ for all $x, y, z \in R$ and since $\gamma$ is an automorphism of $R$ we obtain $d(\gamma(x))\beta(y)\beta(g(z)) = 0$ for all $x, y, z \in R$. Since $R$ is a semiprime ring, by Lemma 2 we get $d(\gamma(x))\beta(g(z)) = 0$ for all $x, z \in R$. Similarly from $g(x)U\alpha^{-1}(d(y)) = 0$, we get $\alpha(g(x))d(\delta(y)) = 0$. Therefore $dg$ is an $(\alpha\gamma, \beta\delta)$-derivation on $R$.

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