INEQUALITIES VIA LAGRANGE MULTIPLIERS

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ABSTRACT. An easy method is obtained to prove many inequalities using Lagrange multipliers.

KEY WORDS AND PHRASES: Inequalities.


1. INTRODUCTION

Let us assume that \( d_1, d_n \) are unit perpendicular vectors in an \( n \)-dimensional space \( X \). In particular \( d_1, d_2, \) and \( d_3 \) are the unit perpendicular vectors \( i, j, \) and \( k \) in the 3-dimensional space. Any vector \( v \) in \( X \) is usually uniquely written in the form

\[ v = \sum_{i=1}^{n} \lambda_i d_i, \]

for scalars \( \lambda_i \). We define

\[ \nabla f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) d_i, \quad f_x = \frac{\partial}{\partial x}. \]

Kapur and Kumar (1986), have used the principle of dynamic programming to prove major inequalities due to Shannon, Renyi, and Holder, see [1]. In this note we give a new method using Lagrange multipliers.

2. SHANNON'S INEQUALITY

THEOREM 2.1. Given \( \sum_{i=1}^{n} p_i = a, \sum_{i=1}^{n} q_i = b \), then

\[ a \ln(a/b) \leq \sum_{i=1}^{n} p_i \ln(p_i/q_i), \quad p_i, q_i \geq 0. \]

The equality holds iff \( p_i = q_i \), for each \( i \).

PROOF. Let the \( q_i \)'s and \( a \) be fixed, set

\[ f(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i \ln(p_i/q_i); \quad p_i, q_i \geq 0, \]

we aim to minimize \( f \) subject to the constraint

\[ g(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i - a = 0. \]
There is a minimum achieved where $\nabla f = \lambda \nabla g$ because $g$ is linear and $f$ is convex, since its second order partials are all non-negative

$$\nabla f = \lambda \nabla g \Rightarrow \sum_{i=1}^{n} (1 + \ln(p_i/q_i))d_i = \lambda \sum_{i=1}^{n} d_i$$

$$\Rightarrow 1 + \ln(p_i/q_i) = \lambda$$

$$\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2} = \ldots = \frac{p_n}{q_n} = \frac{\sum a_i}{\sum b_i} = \frac{a}{b}.$$

Therefore

$$\min \sum_{i=1}^{n} p_i \ln(p_i/q_i) = \ln(a/b) \sum_{i=1}^{n} p_i = a \ln(a/b),$$

or

$$a \ln(a/b) \leq \sum_{i=1}^{n} p_i \ln(p_i/q_i).$$

If $a = b = 1$, we get Shannon's inequality

$$\sum_{i=1}^{n} p_i \ln(p_i/q_i) \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i \ln(p_i/q_i) = 0 \iff p_i = q_i \quad \text{for each} \quad i.$$

3. RENYI'S INEQUALITY

**Theorem 3.1.** Given $\sum a_i = a$, $\sum b_i = b$, then

$$\frac{1}{\alpha - 1} (a^\alpha b^{1-\alpha} - a) \leq \sum_{i=1}^{n} \frac{1}{\alpha - 1} (p_i^\alpha q_i^{1-\alpha} - p_i), \quad p_i, q_i \geq 0, 0 < \alpha \neq 1.$$

The equality holds iff $p_i = q_i$ for each $i$.

**Proof.** Let the $q_i$'s and $a$ be fixed and write

$$f(p_1, \ldots, p_n) = \sum_{i=1}^{n} \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}, \quad g(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i - a = 0$$

$$\nabla f = \lambda \nabla g \Rightarrow \sum_{i=1}^{n} \frac{\alpha}{\alpha - 1} p_i^{\alpha-1} q_i^{1-\alpha}d_i = \lambda \sum_{i=1}^{n} d_i$$

$$\Rightarrow (p_i/q_i)^{\alpha-1} = \lambda \left( \frac{\alpha - 1}{\alpha} \right)$$

$$\Rightarrow \frac{p_1}{q_1} = \ldots = \frac{p_n}{q_n} = \frac{a}{b}$$

$$\Rightarrow \min f(p_1, \ldots, p_n) = \frac{1}{\alpha - 1} a^\alpha b^{1-\alpha},$$

by the convexity of $f$ and linearity of $g$. Hence

$$\frac{1}{\alpha - 1} a^\alpha b^{1-\alpha} \leq \sum_{i=1}^{n} \frac{1}{\alpha - 1} p_i^\alpha q_i^{1-\alpha}.$$

If $a = b = 1$, we get Renyi's inequality

$$\frac{1}{\alpha - 1} \left( \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} - 1 \right) \geq 0.$$

4. HOLDER'S INEQUALITY

**Theorem 4.1.** Given $\sum a_i^p = A$, $\sum b_i^q = B$, $\sum a_i b_i = C$, $a_i, b_i \geq 0$, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$,
then
\[ C \leq A^{1/p} B^{1/q}. \]  

**PROOF.** This follows from Renyi's inequality, taking \( \alpha = 1/p, \ a_i = p_i^p, \ b_i = q_i^q \), or, we prove the result directly as follows:

let the \( a_i \)'s and \( C \) be fixed and write

\[
\begin{align*}
    f(b_1, ..., b_n) &= A^{q/p} \sum_{i=1}^{n} b_i^q, \\
    g(b_1, ..., b_n) &= \sum_{i=1}^{n} a_i b_i - C = 0
\end{align*}
\]

\( \nabla f = \lambda \nabla g \Rightarrow q A^{q/p} \sum_{i=1}^{n} b_i^{q-1} d_i = \lambda \sum_{i=1}^{n} a_i d_i \)

\[ \Rightarrow A^{q/p} b_i^{q-1} = (\lambda/q)a_i \]  

\[ (4.2) \Rightarrow A^{q/p} = (\lambda/q)C, \]  

and

\[ A^q B = (\lambda/q)A, \quad \text{as} \quad p(q - 1) = q \]  

\[ (4.3) \& (4.4) \Rightarrow \lambda/q = C^{q-1}. \]

Therefore, by the convexity of \( f \) and linearity of \( g \),

\[ \min(A^{q/p} B) = C^q, \]

or

\[ C \leq A^{1/p} B^{1/q}. \]

5. **GENERALIZATIONS OF HOLDER'S INEQUALITY**

**THEOREM 5.1.** Given \( \sum_{i=1}^{n} a_i^p = A, \sum_{i=1}^{n} b_i^q = B, \sum_{i=1}^{n} c_i = C, \) and \( \sum_{i=1}^{n} a_i b_i c_i = D, \ a_i, b_i, c_i \geq 0, \)

\( \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \) then

\[ D \leq a^{1/p} B^{1/q} C^{1/r}. \]

**PROOF.** This follows by an easy application of Holder's inequality:

\[
\begin{align*}
    \sum_{i=1}^{n} a_i b_i c_i &\leq \left[ \sum_{i=1}^{n} (a_i b_i)^{p/q} \right]^{p/q} \left[ \sum_{i=1}^{n} (a_i b_i)^{q/r} \right]^{q/r} \\
    &= \left[ \sum_{i=1}^{n} (a_i b_i)^{p/r} \right]^{p/q} \left[ \sum_{i=1}^{n} (a_i b_i)^{q/r} \right]^{q/r} \\
    &\leq \left[ \left( \sum_{i=1}^{n} a_i^{p/r} c_i^{p/q} \right)^{p/q} \right] \left[ \left( \sum_{i=1}^{n} b_i^{q/r} c_i^{q/r} \right)^{q/r} \right] \ C^1 \\
    &= A^{1/p} B^{1/q} C^{1/r}. 
\end{align*}
\]

6. **MINKOWSKI'S INEQUALITY**

**THEOREM 6.1.** Given \( \sum_{i=1}^{n} a_i^p = A, \sum_{i=1}^{n} b_i^p = B, \) and \( \sum_{i=1}^{n} (a_i + c_i)^p = D, \ a_i, b_i, c_i \geq 0, \) then

\[ C^{1/p} \leq A^{1/p} + B^{1/p}. \]

**PROOF.** Let the \( b_i \)'s and \( A \) be fixed and write
\[ f(a_1, \ldots, a_n) = \sum_{i=1}^{n} (a_i + b_i)^p, \quad g(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i^p - A = 0 \]

\[ \nabla f = \mu \nabla g \Rightarrow \sum_{i=1}^{n} p(a_i + b_i)^{p-1} d_i = \mu \sum_{i=1}^{n} p a_i^{p-1} d_i \]

\[ \Rightarrow (a_i + b_i)^{p-1} = \mu a_i^{p-1} \]

\[ \Rightarrow \frac{b_1}{a_1} = \ldots = \frac{b_n}{a_n} = C. \]

Therefore,

\[ \max C^\frac{1}{p} = \left[ \sum_{i=1}^{n} (a_i + ca_i)^p \right]^\frac{1}{p} \]

\[ = (1 + c)A^\frac{1}{p} \]

\[ = A^\frac{1}{p} + cA^\frac{1}{p} \]

\[ = A^\frac{1}{p} + B^\frac{1}{p}, \]

or

\[ C^\frac{1}{p} \leq A^\frac{1}{p} + B^\frac{1}{p}. \]

7. ARITHMETIC-GEOMETRIC-MEAN INEQUALITY

THEOREM 7.1.

\[ \left( \prod_{i=1}^{n} x_i \right)^\frac{1}{n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i. \]

PROOF. Write

\[ f(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n = y, \quad g(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i - C = 0. \]

Let \( C \) be fixed, we have

\[ \nabla f = \mu \nabla g \Rightarrow \sum_{i=1}^{n} \frac{y}{x_i} d_i = \frac{\mu}{n} \sum_{i=1}^{n} d_i \]

\[ \Rightarrow x_i = \frac{n}{\mu} y \]

\[ \Rightarrow C = \frac{n}{\mu} y. \]

Therefore

\[ \max y^\frac{1}{n} = \frac{n}{\mu} y = C, \]

or

\[ \left( \prod_{i=1}^{n} x_i \right)^\frac{1}{n} \leq \frac{1}{n} \sum_{i=1}^{n} x_i. \]

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