A CHARACTERIZATION OF $B^*$-ALGEBRAS

A. K. GAUR

Department of Mathematics
Duquesne University,
Pittsburgh, PA 15282

(Received December 11, 1995 and in revised form August 16, 1996)

Abstract. A characterization of $B^*$-algebras amongst all Banach algebras with bounded approximate identities is obtained.

1991 AMS Subject Classification: 46J99, 46J15

Key Words and Phrases: Approximate identity; $B^*$-algebra; self-adjoint elements; Hermitian elements.

1. Introduction.

We recall that an approximate identity in a Banach algebra $A$ is a net $\{e_\alpha : \alpha \in I\}$ in $A$ where $I$ is a directed set such that $\lim_{\alpha} e_\alpha x = x = \lim_{\alpha} x e_\alpha$ for every $x$ in $A$. If there is a finite constant $M$ such that $\|e_\alpha\| \leq M$ for all $\alpha$, then the approximate identity is said to be bounded.

Let $A$ be a Banach algebra. For each $x$ in $A$, let

$$D_A(x) = \{f \in A' : \|f\| = 1 = f(x)\}.$$ 

By a corollary of the Hahn–Banach theorem, $D_A(x)$ is non-empty. We denote $S(A) = \{x \in A : \|x\| = 1\}$.

For each $a \in A$, we call the set $V_A(a) = \{f(ax) : f \in D_A(x), x \in S(A)\}$ the spatial numerical range of $a$.

We recall [5] that the relative numerical range of $a$ in $A$ with respect to $x \in A$, is defined as

$$\overset{\circ}{V}_x(A,a) = \{f(ax) : f \in D_A(x)\}.$$ 

Thus we see that $V_A(a) = \bigcup \left\{\overset{\circ}{V}_x(A,a) : x \in S(A)\right\}$, which is a bounded subset of the complex numbers bounded by $\|a\|$.

If $A$ has an approximate identity of norm less than or equal to one then $A$ can embedded, isometrically and isomorphically, in a unital Banach algebra $A^+$ in such a way that for each $a$ in $A$

$$V(A^+,a) = \overline{V}_A(a),$$

where $V(A^+,a) = \{f(a) : f \in (A^+)', \|f\| = 1 = f(a) = \|a\|\}$. For details see [4], Theorem 2.3.
An element $h$ of a Banach algebra $A$ is said to be Hermitian if $V_A(h) \subset R$. We denote by $H(A)$ the set of all Hermitian elements of $A$. A $B^*$-algebra is a Banach algebra $A$ with an involution, $a \rightarrow a^*$ satisfying the following conditions:

1. $(a + b)^* = a^* + b^*$;
2. $(ab)^* = b^*a^*$;
3. $(aa)^* = \bar{a}a^*$;
4. $a^{**} = a$; and
5. $|a^*a| = |a|^2$

for all $a, b$ in $A$ and $\alpha$ in $C$.

An element $a$ in a $B^*$-algebra is said to be self-adjoint if $a = a^*$. The set of all self adjoint elements will be denoted by $S(A)$. Each element $a \in A$ can be written uniquely in the form $a = h + ik$ where $h, k \in S(A)$. Some of the well known properties of $S(A)$ are the following:

a) The set $S(A)$ is a real partially ordered Banach space,

b) each of its elements has real spectrum,

c) if $h, k \in S(A)$ then $i(hk - kh) \in S(A)$, and

d) for each $h \in S(A)$, the spectral radius $\rho(h) = ||h||$.

It is clear that the set of Hermitian elements, $H(A)$, of a Banach algebra with a bounded approximate identity of norm less than or equal to one has many of the properties of $S(A)$ in a $B^*$-algebra.

In this note we prove that in an arbitrary $B^*$-algebra $A$, $H(A) = S(A)$ in Theorem 2.1. This results mimics a result by Bohnenblust and Karlin [2].

In [8], Vidav has shown that a unital Banach algebra $A$ with the following conditions:

1. $A = H(A) + iH(A)$;
2. for each $h$ in $H(A)$ there exists $h_1, h_2$ in $H(A)$ such that $h_1 + ih_2 = h^2$ and $h_1h_2 = h_2h_1$

is a $B^*$-algebra with Vidav-involution. Combining the results of Vidav [8], Berkson [1], and Glickfeld [6] we obtain the result that if $A$ is a unital Banach algebra such that $A = H(A) + iH(A)$ then $A$ is a $B^*$-algebra under the Vidav-involution. Here, we extend this result to the nonunital case in the form of Lemma 3.1.

Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have a characterization of $B^*$-algebras with bounded approximate identities.

2. Some Results.

We now prove the following theorem.

**Theorem 2.1** Let $A$ be a $B^*$-algebra with a bounded approximate identity of norm less than or equal to one. An element of $A$ is Hermitian if and only if it is self-adjoint.

**Proof.** Case 1. Suppose that $A$ has a unit element $1$. Let $f \in D_A(1)$. Then it is known that such a functional has the property that $f(h^*) = \overline{f(h)}$, for every $h$ in $A$. Thus if $h$ is a self-adjoint element of $A$, $f(h) = f(h^*) = \overline{f(h)}$ and hence $f(h)$ is real for all $f$ in $D_A(1)$. Hence, $S(A) \subseteq H(A)$.

Case 2. If $A$ has no identity element then it will have an approximate identity of norm less than or equal to one. Also, with the involution defined by $(a, \alpha)^* = (a^*, \bar{\alpha})$ for $(a, \alpha) \in A^+$, and
by Theorem 2.3 in [4], $A^+$ becomes a unital $B^*$-algebra containing as a sub-$B^*$-algebra, ([3], 1.3.8).

Let $h$ be a self-adjoint element of $A$. Then $(h, 0)$ is self-adjoint and hence Hermitian in the unital $B^*$-algebra $A^+$. Hence $h \in H(A)$. We have therefore for any $B^*$-algebra, $S(A) \subseteq H(A)$.

Suppose conversely that $h \in H(A)$. Then for $h_1$ and $h_2$ in $S(A)$, $h = h_1 + ih_2$. This implies that $\nu(h_2) = 0$ (where $\nu(x) = \sup\{|\lambda| : \lambda \in V_A(x)\}$ and is called numerical radius of $x$ in $A$) and hence $h = 0$. Thus $h = h_1$ so that $h$ is self-adjoint. Hence $H(A) \subseteq S(A)$ and hence the theorem.

**Remark 2.1** The above theorem shows that in a $B^*$-algebra the Hermitian elements generate the whole algebra in the sense that each element $a$ may be written in the form $a = h_1 + ih_2$ with $h_1$ and $h_2$ in $H(A)$. In an arbitrary Banach algebra $A$ this is not true. We therefore consider the set $J(A) = H(A) + \i H(A)$. Since $H(A)$ is a real space it follows that $J(A)$ is a complex linear space. If $A$ has no unit element then by Theorem 2.3, [4], $J(A) \times C = J(A^+)$. We define a map $a \to a^*$ from $J(A)$ into itself by

$$(h_1 + ih_2)^* = h_1 - ih_2, \text{ for all } h_1, h_2 \in H(A).$$

The linear map $a \to a^*$ is known as the Vidav-involution on $J(A)$.

**Remark 2.2** If $A$ has no unit element then it is a simple matter to verify that the Vidav-involution on $J(A^+)$ is an extension of the Vidav-involution on $J(A)$. The space $J(A)$ is a complex Banach space and $a \to a^*$ is a continuous linear involution on $J(A)$. In general, the Banach space $J(A)$ is not an algebra, and if $J(A)$ is an algebra under some conditions, then the Vidav-involution has the additional property

$$(ab)^* = a^*b^*, \text{ for all } a, b \in J(A).$$


Vidav has shown in [8] that a unital Banach algebra $A$ with the following conditions:

(V1) $A = H(A) + \i H(A),$

(V2) for each $h$ in $H(A)$ there exists $h_1, h_2$ in $H(A)$ such that $h_1 + ih_2 = h$ and $h_1h_2 = h_2h_1$, is a $B^*$-algebra with Vidav-involution and a norm equivalent to the original norm on $A$.

According to Palmer [7], the condition (V1) implies (V2). Also Berkson [1], Glickfeld [6], and Palmer [7] have shown that if (V1) is satisfied by the algebra $A$ the equivalent norm by Vidav is equal to the original norm on $A$. So by these results we have the result that if $A$ is a unital Banach algebra satisfying (V1) then $A$ is $B^*$-algebra under the Vidav-involution. The following lemma extends this result to the non-unital case.

**Lemma 3.1** Let $A$ be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Suppose that every $a$ in $A$ has the form $a = h_1 + ih_2$, for all $h_1, h_2$ in $H(A)$. Then with the Vidav-involution, $A$ is a $B^*$-algebra.

**Proof.** From Remark 2.1 we have that $J(A^+) = J(A) \times C$. Since $J(A) = A$ (by the hypothesis) we have $J(A^+) = A^+$. Therefore $A^+$ is a unital $B^*$-algebra under the Vidav-involution. Furthermore, $A$ is a closed and self-adjoint subalgebra of $A^+$, and is therefore a $B^*$-algebra under the Vidav-involution.
Finally, combining the results of Theorem 2.1 and Lemma 3.1 we have the following:

**Theorem 3.2** Let $A$ be a Banach algebra with a bounded approximate identity of norm less than or equal to one. Then $A$ is a $B^*$-algebra under some involution if and only if each element $a$ of $A$ can be written in the form $a = h_1 + ih_2$ where $h_1$ and $h_2$ are Hermitian elements of $A$.

4. Acknowledgement.

The author expresses his appreciation to the referee for his or her valuable suggestions which improved the clarity of this presentation.

**References**


