SUBCLASSES OF UNIVALENT FUNCTIONS SUBORDINATE TO CONVEX FUNCTIONS

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ABSTRACT. In this paper, we define a new subclass $M_\alpha(A, B)$ of univalent functions and investigate several interesting characterization theorems involving a general class $S^*[A, B]$ of starlike functions.

KEY WORDS AND PHRASES: Univalent function, subordination, $\alpha$-convex function

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1. INTRODUCTION AND DEFINITIONS

Let $A$ denote the class of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, let $S$ denote the class of all functions in $U$ which are univalent in $U$.

A function $f(z)$ belonging to $S$ is said to be starlike of order $\alpha$ ($0 \leq \alpha < 1$) if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in U; 0 \leq \alpha < 1).$$

We denote by $S^*(\alpha)$ the subclass of $S$ consisting of functions which are starlike of order $\alpha$.

A function $f(z)$ belonging to $S$ is said to be convex of order $\alpha$ ($0 < \alpha < 1$) if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in U; 0 < \alpha < 1).$$

We denote by $K(\alpha)$ the subclass of $S$ consisting of functions which are convex of order $\alpha$.

We note that

$$S^*(\alpha) \subseteq S^*(0) \equiv S^* \quad (0 \leq \alpha < 1) \quad (1.4)$$

and

$$K(\alpha) \subseteq K(0) \equiv K \quad (0 \leq \alpha < 1). \quad (1.5)$$

With a view to introducing an interesting family of analytic functions, we should recall the concept of subordination between analytic functions. Given two functions $f(z)$ and $g(z)$, which are analytic in $U$, the function $f(z)$ is said to be subordinate to $g(z)$ if there exists a function $h(z)$, analytic in $U$ with

$$h(0) = 0 \quad \text{and} \quad|h(z)| < 1,$$

such that

$$f(z) = g(h(z)) \quad (z \in U). \quad (1.6)$$

We denote this subordination by

$$f(z) \prec g(z).$$
In particular, if \( g(z) \) is univalent in \( \mathcal{U} \), the subordination (1.8) is equivalent to

\[
 f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).
\]

Janowski [1] introduced the class \( \mathcal{P}[A, B] \). For \(-1 < B < A < 1\), a function \( p \), analytic in \( \mathcal{U} \) with \( p(0) = 1 \), belongs to the class \( \mathcal{P}[A, B] \) if \( p(z) \) is subordinate to \( (1 + Az)/(1 + Bz) \). Also \( \mathcal{S}^*[A, B] \) and \( \mathcal{K}[A, B] \) denote the subclasses of \( \mathcal{S} \) consisting of all functions \( f(z) \) such that

\[
 \frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B] \quad \text{and} \quad \left( \frac{zf'(z)}{f(z)} \right) ' = \frac{-zf''(z)}{f'(z)} \in \mathcal{P}[A, B],
\]

respectively. We note that \( \mathcal{S}^*[-1, 1] = \mathcal{S}^* \) and \( \mathcal{K}[-1, 1] = \mathcal{K} \).

**Definition 1.** Let \( \alpha \) be a real number. A function \( f(z) \) belonging to the class \( \mathcal{A} \) with \( f(z)/z \neq 0 \) is said to be \( \alpha \)-convex in \( \mathcal{U} \) if and only if

\[
 \Re \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf'(z)}{f(z)} \right) \right] > 0.
\]

Also we denote the class of \( \alpha \)-convex functions by \( \mathcal{M}_\alpha \). Then it is easy to see that

\[
 \mathcal{M}_\alpha = \left\{ f \in \mathcal{S} : \Re \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf'(z)}{f(z)} \right) \right] > 0, \quad z \in \mathcal{U} \right\}.
\]

See Eenigenberg and Miller [5] for further information on them.

We now define the class \( \mathcal{M}_\alpha(A, B) \) as follows: If \( \alpha \) is a real number, then

\[
 \mathcal{M}_\alpha(A, B) = \left\{ f \in \mathcal{S} : \Re \left[ (1 - \alpha) \frac{zf''(z)}{f'(z)} + \alpha \left( 1 + \frac{zf'(z)}{f(z)} \right) \right] < \frac{1 + Az}{1 + Bz}, \right\}
\]

where \(-1 < B < A < 1, z \in \mathcal{U} \) (1.13).

Clearly, we have

\[
 \mathcal{M}_\alpha(1, -1) = \mathcal{M}_\alpha, \quad \mathcal{M}_1(A, B) = \mathcal{K}[A, B],
\]

and

\[
 \mathcal{M}_0(A, B) = \mathcal{S}^*[A, B].
\]

**2. MAIN RESULTS**

Applying the method of the integral representation [2] for functions in \( \mathcal{M}_\alpha(A, B) \) (\( \alpha > 0 \)), it is not difficult to deduce

**Lemma 1.** The function \( f(z) \) is in \( \mathcal{M}_\alpha(A, B) \), \( \alpha > 0 \), if and only if there exists a function \( g(z) \) belonging to the class \( \mathcal{S}^*[A, B] \) such that

\[
 f(z) = \left[ \frac{1}{\alpha} \int_0^1 \{g(t)\}^{1/\alpha} t^{-1} dt \right]^\alpha.
\]

**Proof.** Setting \( g(z) = f(z) \left( \frac{zf'(z)}{f(z)} \right)^\alpha \), so that (2.1) is satisfied, we observe that

\[
 \frac{zg'(z)}{g(z)} = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf'(z)}{f(z)} \right).
\]

Hence \( f \in \mathcal{M}_\alpha(A, B) \) if and only if \( g \in \mathcal{S}^*[A, B] \).

Before stating our first theorem, we need the following definition

**Definition 2.** Let \( c \) be a complex number such that \( \Re c > 0 \), and let
If $h$ is the univalent function $h(z) = 2Nz/(1 - z^2)$ and $b = h^{-1}(c)$, then we define the "open door" (cf [3]) function $Q_c$ as

$$Q_c(z) = h[(z + b)/(1 + b)]/z, \quad z \in \mathcal{U}. \quad (2.3)$$

**Theorem 1.** Let $f \in \mathcal{M}_\alpha(A, B)$ ($\alpha > 0$), and let

$$z + b z^2 z f'(z) f(z) = 1, \quad (2.7)$$

Then $f \in S^*$

**Proof.** Since $f \in \mathcal{M}_\alpha(A, B)$ ($\alpha > 0$), it follows that there exists a function $g \in S^*[A, B]$ such that

$$f(z) = \left[ \frac{1}{\alpha} \int_0^z \{g(t)\}^{1/\alpha} t^{-1} dt \right] \in S^*. \quad (2.5)$$

by using Lemma 1. By the hypothesis, we also have

$$1 + \frac{zg'(z)}{g(z)} = 1 + \frac{1 + Az}{1 + Bz} \in Q_b(z). \quad (2.6)$$

Thus, by a result of Miller and Mocanu ([3], Corollary 3.1), we have

$$f(z) = \left[ \frac{1}{\alpha} \int_0^z \{g(t)\}^{1/\alpha} t^{-1} dt \right] \in S^*. \quad (2.5)$$

**Lemma 2.** (Mocanu [4]) Let $\mathcal{P}$ be an analytic function in $\mathcal{U}$ satisfying $\mathcal{P} < Q_c$. If $p$ is analytic in $\mathcal{U}$, $p(0) = 1/c$, and

$$zp'(z) + P(z)p(z) = 1, \quad (2.7)$$

then $\text{Re} \, p(z) > 0$ in $\mathcal{U}$

Making use of Lemma 2, we now prove

**Theorem 2.** Let $f \in \mathcal{M}_\alpha(A, B)$ ($\alpha > 0$), and let

$$zf'(z) f(z) = 1 - Q_1. \quad (2.8)$$

Then $f \in S^*[A, B]$.

**Proof.** If we set $p(z) = zf'(z)/f(z)$, then

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}. \quad (2.9)$$

Hence

$$(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) + \alpha \frac{zp'(z)}{p(z)}. \quad (2.10)$$

Since $f \in \mathcal{M}_\alpha(A, B)$,

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \leq \frac{1 + Az}{1 + Bz}. \quad (2.11)$$

Setting $\mathcal{P}(z) = p(z) + 1/p(z) - 1$, we obtain

$$zp'(z) + \mathcal{P}(z)p(z) = 1 \quad (2.12)$$
and \( \mathcal{P} \prec Q_1 \) by the hypothesis (2.8)

Thus, by Lemma 2, we have

\[
\text{Re} p(z) > 0 \quad (z \in \mathcal{U}).
\]  

(2.13)

Since \( \alpha > 0 \),

\[
\text{Re} \left\{ \frac{1}{\alpha} p(z) \right\} > 0 \quad (z \in \mathcal{U}).
\]  

(2.14)

Also \((1 + Az)/(1 + Bz)\) (with \(-1 \leq B < A \leq 1\)) is a convex univalent function. Therefore, by appealing to a known result ([6], Theorem 7), we conclude from (2.11) and (2.14) that

\[
p(z) < \frac{1 + Az}{1 + Bz}.
\]  

(2.15)

This evidently completes the proof of Theorem 2.

As an example of ([7], Corollary 3.2, see also [9]), consider the case when \( \alpha > 0 \), \(-1 \leq B < A \leq 1\), and \( A \neq B \). Then the differential equation

\[
q(z) + \alpha \frac{zq'(z)}{q(z)} = \frac{1 + Az}{1 + Bz}
\]  

has a univalent solution given by

\[
q(z) = \begin{cases} 
\frac{z \frac{1}{2} (1 + Bz)^{\frac{1}{2}} (A \frac{1}{2} B^2)}{\frac{1}{2} \int_0^1 t \frac{1}{2} e^{\frac{1}{2} t} (1 + Bt)^{\frac{1}{2}} (A \frac{1}{2} B^2) dt} & \text{if } B \neq 0 \\
\frac{z \frac{1}{2} e^{\frac{1}{2} z}}{\frac{1}{2} \int_0^1 t \frac{1}{2} e^{\frac{1}{2} t} dt} & \text{if } B = 0.
\end{cases}
\]  

(2.17)

If \( p(z) \) is analytic in \( \mathcal{U} \) and satisfies

\[
p(z) + \alpha \frac{zp'(z)}{p(z)} < \frac{1 + Az}{1 + Bz},
\]  

(2.18)

then

\[
p(z) < q(z) < \frac{1 + Az}{1 + Bz}.
\]  

(2.19)

Hence, by the equations (2.11) and (2.19), we obtain

\textbf{THEOREM 3.} Let \( \alpha > 0 \) and \( f \in \mathcal{M}_\alpha(A, B) \). Then

\[
\frac{zf'(z)}{f(z)} < q(z) < \frac{1 + Az}{1 + Bz},
\]  

(2.20)

where \( q(z) \) is given by (2.17).

\textbf{THEOREM 4.} \( \mathcal{K}(\alpha) \subset \mathcal{M}_\alpha(1 - 2\alpha, -1) \) \((0 \leq \alpha < 1)\).

\textbf{PROOF.} If we define

\[
h_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1),
\]  

(2.21)

then we can easily see that \( f \in \mathcal{K}(\alpha) \) if and only if

\[
1 + \frac{zf''(z)}{f'(z)} < h_\alpha(z)
\]  

(2.22)

(cf [10], Equation (9)). Hence, by Theorem 1 of [10], we have
Therefore we conclude from [8, Lemma 2.2] that

\[ \frac{zf'(z)}{f(z)} < h_\alpha(z). \] (2.23)

Therefore we conclude from [8, Lemma 2.2] that

\[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) < h_\alpha(z). \] (2.24)

This completes the proof of Theorem 4

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REFERENCES