A NOTE ON SEMIPRIME RINGS WITH DERIVATION

Dedicated to the memory of Professor H. Tominaga

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ABSTRACT. Let $R$ be a 2-torsion free semiprime ring, $I$ a nonzero ideal of $R$, $Z$ the center of $R$ and $d : R \to R$ a derivation. If $d [z, y] + [z, y] \in Z$ or $d [z, y] - [z, y] \in Z$ for all $z, y \in I$, then $R$ is commutative.

KEY WORDS AND PHRASES: Derivation, semiprime ring, 2-torsion free ring.
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1 INTRODUCTION.
Throughout, $R$ will represent a ring, $Z$ the center of $R$, $I$ a nonzero ideal of $R$, and $d : R \to R$ a derivation. As usual, for $x, y \in R$, we write $[x, y] = xy - yx$ and $x \circ y = xy + yx$. Given a subset $S$ of $R$, we put $VR(S) = \{ z \in R \mid [z, s] = 0 \text{ for all } s \in S \}$. In [1], Daif and Bell showed that a semiprime ring $R$ must be commutative if it admits a derivation $d$ such that (i) $d [x, y] = [x, y]$ for all $x, y \in R$, or (ii) $d [x, y] + [x, y] = 0$ for all $x, y \in R$. Our present objective is to generalize this result.

2 THE RESULTS.
As mentioned in §1, our present objective is to prove the following theorem which generalizes [1, Theorem 3].

THEOREM 1. Let $R$ be a 2-torsion free semiprime ring, and let $I$ be a nonzero ideal of $R$. Then the following conditions are equivalent:

1. $R$ admits a derivation $d$ such that $d [x, y] - [x, y] \in Z$ for all $x, y \in I$.
2. $R$ admits a derivation $d$ such that $d [x, y] + [x, y] \in Z$ for all $x, y \in I$.
3. $R$ admits a derivation $d$ such that $d [x, y] + [x, y] \in Z$ or $d [x, y] - [x, y] \in Z$ for all $x, y \in I$.
4. $I \subseteq Z$.

In preparation for proving our theorem, we state the following two lemmas.
LEMMA 1. Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$, and $a \in R$.

(1) Let $b \in I$. If $[b, z] = 0$ for all $z \in I$, then $b \in Z$. Therefore, if $I$ is commutative, then $I \subseteq Z$.

(2) If $[a, z] \in Z$ for all $z \in I$, then $a \in V_R(I)$.

(3) Let $R$ be a 2-torsion free ring and $[a, [x, y]] \in Z$ for all $x, y \in I$, then $a \in V_R(I)$.

**PROOF.** (1) is well known.

(2) For any $z \in I$, we have $a[a, x] = [a, ax] \in Z$, and so we get $0 = [a[a, x], z] = [a, x]^2$.

Since $R$ is semiprime and $[a, x] \in Z$, we obtain that $[a, x] = 0$ for all $x \in I$. Hence $a \in V_R(I)$.

(3) Since $Z \ni [a, [x, z]] = [a, x[z, y]] = [a, [x, y]] + [a, x][y, z]$ for all $x, y \in I$, we have $0 = [a, x[y, z]] + [a, x][y, z]] = 2[a, x][a, [y, z]] + [a, x][y, z]$. Now, substituting $az$ for $y$, we get $0 = 2[a, x][a, [y, z]] + [a, x][y, z] = -2[a, x][a, [y, z]] + [a, x][y, z]$. Substituting $[x, y]$ for $z (y \neq I)$, we have $2[a, [x, y]]^3 = 0$.

Since $R$ is a 2-torsion free semiprime ring and $[a, [x, y]] \in Z$, we get $[a, [x, y]] = 0$ for all $x, y \in I$. Hence we have $a \in V_R(I)$ by [1, Lemma 1].

**LEMMA 2.** Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$, and $d : R \rightarrow R$ a nonzero derivation such that $d[x, y] + [x, y] \in Z$ or $d[x, y] - [x, y] \in Z$ for all $x, y \in I$. If $d(I) \subseteq V_R(I)$, then $I$ is commutative, and so $I \subseteq Z$.

**PROOF.** Let $a \in I$. For any $z \in I$, we have $0 = [a, d[x, y] \pm [x, y]] = \pm [a, [x, y]]$, and so we get $a \in V_R(I)$ by [1, Lemma 1]. Therefore, $I$ is commutative, and so we obtain that $I \subseteq Z$ by Lemma 1 (1).

We are now ready to complete the proof of Theorem 1.

**PROOF OF THEOREM 1.** (1) $\Rightarrow$ (4). Let $d$ be a derivation such that $d[x, y] - [x, y] \in Z$ for all $x, y \in I$. If $d = 0$, then $I \subseteq Z$ by Lemma 1 (1) and (2). Now we suppose that $d \neq 0$. For any $x, y, z \in I$, we have $Z \ni d[x, y, z] - [x, [y, z]] = d(x)[y, z] + [x, d[y, z]] - [x, [y, z]] = [d(x), [y, z]] + [x, d[y, z]] - [y, z] = [d(x), [y, z]] + [x, d[y, z]] - [y, z] = [d(x), [y, z]]$, and so we have $d(x) \in V_R(I)$ by Lemma 1 (3), that is, $d(I) \subseteq V_R(I)$. Therefore we have $I \subseteq Z$ by Lemma 2.

(2) $\Rightarrow$ (4). Let $d$ be a derivation such that $d[x, y] + [x, y] \in Z$ for all $x, y \in I$. Then the derivation $(-d)$ satisfies the condition $(-d)[x, y] - [x, y] \in Z$. And so we have $I \subseteq Z$ by (1).

(3) $\Rightarrow$ (4). For each $x \in I$, we put $I_x = \{ y \in I \mid d[x, y] - [x, y] \in Z \}$ and $I^*_x = \{ y \in I \mid d[x, y] + [x, y] \in Z \}$. Then $I = I_x \cup I^*_x$. By Brauer's Trick, we have $I = I_x$ or $I = I^*_x$. By the same method, we can see that $I = \{ z \in I \mid I = I_x \}$ or $I = \{ z \in I \mid I = I^*_x \}$. Therefore, by (1) and (2) we have $I \subseteq Z$.

(4) $\Rightarrow$ (1), (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are clear.

The next is a generalization of [1, Theorem 2].

**COROLLARY 1.** Let $R$ be a 2-torsion free semiprime ring, $Z$ the center of $R$ and $d : R \rightarrow R$ a derivation. If $d[x, y] + [x, y] \in Z$ or $d[x, y] - [x, y] \in Z$ for all $x, y \in R$, then $R$ is commutative.

**PROPOSITION 1.** Let $R$ be a 2-torsion free ring with identity 1. Then there is no derivation $d : R \rightarrow R$ such that $d(z \circ y) = z \circ y$ for all $z, y \in R$ or $d(z \circ y) + (z \circ y) = 0$ for all $z, y \in R$.

**PROOF.** If there exists a nonzero derivation $d : R \rightarrow R$ such that $d(z \circ y) = z \circ y$ or $d(z \circ y) + (z \circ y) = 0$ for $z, y \in R$, then we have $2z = z \circ 1 = \pm d(z \circ 1) = \pm 2d(z)$ for all $z \in R$. Since $R$ is 2-torsion free, we get $d(z) = \pm z$ for all $z \in R$. For any $z, y \in R$, we have $zy + yz = z \circ y = \pm d(z \circ y) = \pm d(zy + yz) = 2(zy + yz)$, and so we get $z \circ y = zy + yz = 0$. Since $R$ is 2-torsion free, we have $z^2 = 0$. Hence we have $0 = z \circ (z + 1) = 2z$, and so we
get \( z = 0 \) for all \( z \in R \); a contradiction. If there exists a zero derivation \( d: R \to R \) such that 
\[ d(x \circ y) = x \circ y \text{ or } d(x \circ y) + (x \circ y) = 0 \]
for all \( x, y \in R \), then we can easily see that \( z = 0 \) for all \( z \in R \); a contradiction.

**REMARK.** In Theorem 1 and Corollary 1, we can not exclude the condition “2-torsion free” as below.

**EXAMPLE.** We denote by \( Z \) the integer system. Let \( R = \left( \begin{array}{cc} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{array} \right), \quad a = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right), \) and \( d \) the inner derivation induced by \( a \), that is, \( d(x) = [a, x] \) for all \( x \in R \). Then \( R \) is a non-commutative prime ring with \( \text{char} \ R = 2 \), and \( d[x, y] \pm [x, y] \in Z \) for all \( x, y \in R \).

Finally, we state two questions.

Let \( R \) be a 2-torsion free semiprime ring, \( d: R \to R \) a nonzero derivation, and \( I \) a nonzero ideal of \( R \). And let \( n \) be a fixed positive integer.

**QUESTION 1.** Does the condition that \( d^n[x, y] + [x, y] \in Z \) or \( d^n[x, y] - [x, y] \in Z \) for all \( x, y \in I \) imply that \( I \subseteq Z \) ?

**QUESTION 2.** Does the condition that \( d^m[x, y] + d^n[x, y] \in Z \) or \( d^m[x, y] - d^n[x, y] \in Z \) for some positive integers \( m = m(x, y) \) and \( p = p(x, y) \), and for all \( x, y \in I \) imply that \( I \subseteq Z \) ?

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**REFERENCE**