ON SUBORDINATION FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

LIU JINLIN
Water Conservancy College
Yangzhou University
Yangzhou 225009, P.R. CHINA

(Received August 17, 1995 and in revised form July 9, 1996)

ABSTRACT. In the present paper the class $P_n[\alpha, M]$ consisting of functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k (n \geq 1)$, which are analytic in the unit disc $E = \{ z : |z| < 1 \}$ and satisfy the condition $|f'(z) + \alpha z f''(z) - 1| < M$ is introduced. By using the method of differential subordination the properties of the class $P_n[\alpha, M]$ are discussed.

KEY WORDS AND PHRASES: Analytic, starlike, convex univalent, subordination

1991 AMS SUBJECT CLASSIFICATION CODES: 30C45

1. INTRODUCTION

Let $A_n (n \geq 1)$ denote the class of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ which are analytic in the unit disc $E = \{ z : |z| < 1 \}$. A function $f(z)$ in $A_n$ is said to be in $P_n[\alpha, M]$ for some $\alpha (\alpha \geq 0)$ and $M (M > 0)$ if it satisfies the condition

$$|f'(z) + \alpha z f''(z) - 1| < M, \quad (z \in E).$$

(1.1)

Let $f(z)$ and $g(z)$ be analytic in $E$. Then we say that the function $g(z)$ is subordinate to $f(z)$ in $E$ if there exists an analytic function $w(z)$ in $E$ such that $|w(z)| < 1 (z \in E)$ and $g(z) = f(w(z))$. For this relation the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in $E$ we have that the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(E) \subset f(E)$.

In this paper, we shall use the method of differential subordination [2] to obtain certain properties of the class $P_n[\alpha, M]$.

2. MAIN RESULTS

In order to give our main results, we need the following lemma.

LEMMA [1]. Let $p(z) = a + p_n z^n + \ldots$ ($n \geq 1$) be analytic in $E$ and let $h(z)$ be convex univalent in $E$ with $h(0) = a$. If $p(z) + \frac{1}{\zeta} z p'(z) \prec h(z)$, where $c \neq 0$ and $\Re c \geq 0$, then $p(z) \prec \frac{\xi}{n} z^{-\xi} \frac{\xi}{n} \int h(t) t^{\xi-1} dt$

Applying the above lemma, we derive

THEOREM 1. Let $f(z) \in P_n[\alpha, M]$, then

$$|f'(z)| \leq 1 + \frac{M}{1 + n \alpha} |z|^n,$$  \hspace{1cm} (2.1)

$$\Re f'(z) \geq 1 - \frac{M}{1 + n \alpha} |z|^n,$$  \hspace{1cm} (2.2)
\[ |f(z)| \leq |z| + \frac{M}{(1 + n)(1 + n\alpha)} |z|^{n-1}, \quad (2.3) \]

\[ \text{Re } f(z) \geq |z| - \frac{M}{(1 + n)(1 + n\alpha)} |z|^{n-1}. \quad (2.4) \]

The results are sharp.

**Proof.** Since \( f(z) \in P_n[\alpha, M] \), it follows from (1.1) that

\[ f'(z) + az f''(z) < 1 + Mz. \quad (2.5) \]

With the help of the lemma, (2.5) yields

\[ f'(z) < \frac{1}{n\alpha} z^{-\frac{1}{n\alpha}} \int_0^z (1 + Mt)t^{-\frac{1}{n\alpha}-1} dt = 1 + \frac{M}{1 + n\alpha} z. \quad (2.6) \]

Using (2.6), we get

\[ f'(z) = 1 + \frac{M}{1 + n\alpha} w(z), \quad (2.7) \]

where \( w(z) \) is analytic in \( E \) and \( |w(z)| \leq |z|^n \). Thus, from (2.7) we obtain (2.1) and (2.2) immediately.

Further, using (2.1) and (2.2) we can arrive at (2.3) and (2.4) by integration, as follows

\[ f(z) = \int_0^z f'(t) dt = \int_0^z f'(te^{i\theta}) e^{i\theta} dt, \]

\[ |f(z)| \leq \int_0^z |f'(te^{i\theta})| dt \]

\[ \leq \int_0^z \left( 1 + \frac{M}{1 + n\alpha} t^n \right) dt = |z| + \frac{M}{(1 + n)(1 + n\alpha)} |z|^{n+1}, \]

\[ \text{Re } f(z) \geq \int_0^z \text{Re } f'(te^{i\theta}) dt \]

\[ \geq \int_0^z \left( 1 - \frac{M}{1 + n\alpha} t^n \right) dt = |z| - \frac{M}{(1 + n)(1 + n\alpha)} |z|^{n+1}. \]

By considering the function

\[ f(z) = z + \frac{M}{(1 + n)(1 + n\alpha)} z^{n+1}, \quad (2.8) \]

we can show that all estimates of this theorem are sharp.

According to the proof of Theorem 1, we have

**Corollary.** Let \( f(z) \in P_n[\alpha, M] \), then

\[ |f'(z) - 1| < \frac{M}{1 + n\alpha}, \quad (2.9) \]

\[ \left| \frac{f(z)}{z} - 1 \right| < \frac{M}{(1 + n)(1 + n\alpha)}. \quad (2.10) \]

The results are sharp.

**Theorem 2.** Let \( f(z) \in P_n[\alpha, M] \). If \( M \leq 1 + n\alpha \), then \( \text{Re} \left( e^{i\beta} f'(z) \right) > 0 \) \((z \in E)\), where \( \beta \) is real and \( |\beta| \leq \frac{\pi}{2} - \arcsin \frac{M}{1 + n\alpha} |z|^n \). The result is sharp in the sense that the range of \( \beta \) cannot be increased.

**Proof.** From the proof of Theorem 1, we have
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\[ |\arg \{e^{i\theta} f'(z)\}| \leq |\theta| + |\arg f'(z)| \leq |\theta| + \arcsin \frac{M}{1 + n\alpha} |z|^n \leq \frac{\pi}{2} \]

for \(|\theta| \leq \frac{\pi}{2} - \arcsin \frac{M}{1 + n\alpha} |z|^n\)

The result is sharp and the extremal function has the form of (2.8)

**THEOREM 3.** Let \( f(z) \in P_n[\alpha, M] \) if \( M \leq \frac{(1+n)(1+n\alpha)}{\sqrt{1+(1+n)^2}} \), then \( f(z) \) is univalent starlike in \( E \)

**PROOF.** According to the corollary and the assumption of Theorem 3, it follows immediately that \( \Re f'(z) > 0 (z \in E) \) and \( \Re \frac{f'(z)}{z} > 0 (z \in E) \)

On the other hand, we see that

\[ |\arg f'(z)| < \arcsin \frac{M}{1 + n\alpha} \leq \arcsin \frac{1 + n}{\sqrt{1 + (1 + n)^2}}, \quad (2.11) \]

and

\[ |\arg \frac{f(z)}{z}| < \arcsin \frac{M}{(1 + n)(1 + n\alpha)} \leq \arcsin \frac{1}{\sqrt{1 + (1 + n)^2}}, \quad (2.12) \]

Using (2.11) and (2.12), we obtain

\[ |\arg \frac{zf'(z)}{f(z)}| \leq |\arg f'(z)| + \left| \arg \frac{f(z)}{z} \right| \]
\[ < \arcsin \frac{1 + n}{\sqrt{1 + (1 + n)^2}} + \arcsin \frac{1}{\sqrt{1 + (1 + n)^2}} \]
\[ = \frac{\pi}{2} \quad (z \in E), \]

which implies that \( f(z) \) is univalent starlike in \( E \).

**THEOREM 4.** Let \( c > -1 \) and let \( f(z) \in P_n[\alpha, M] \). Then the function \( F(z) \) defined by

\[ F(z) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t)dt \quad (2.13) \]

belongs to \( P_n[\frac{1}{c+1}, \frac{M}{1+n\alpha}] \). The result is sharp.

**PROOF.** By (2.13) and (2.6), we have

\[ F'(z) + \frac{1}{c+1} zF''(z) = f'(z) \times 1 + \frac{M}{1 + n\alpha} z, \]

which shows that \( F(z) \in P_n[\frac{1}{c+1}, \frac{M}{1+n\alpha}] \)

This result is sharp and the extremal function has the form of (2.8).

**THEOREM 5.** Let \( c > -1 \) and \( \alpha > 0 \). If \( F(z) \in P_n[\alpha, M] \), then the function \( f(z) \) defined by (2.13) satisfies \( |f'(z) - 1| < M \) for \( z \in E \).

**PROOF.** Since \( F(z) \in P_n[\alpha, M] \), we have from (1.1), (2.5) and (2.6) that

\[ F'(z) + \alpha zF''(z) \leq 1 + Mz \quad (2.14) \]

and

\[ F'(z) < 1 + \frac{M}{1 + n\alpha} z. \quad (2.15) \]

From (2.13), we get

\[ f'(z) = \frac{1}{\alpha(c+1)} \{ [F'(z) + \alpha zF''(z)] + [\alpha(c+1) - 1]F'(z) \}. \quad (2.16) \]
On using (2.14) and (2.15), (2.16) yields

\[ f'(z) = \frac{1}{\alpha(c + 1)} \left\{ \left[ F'(z) + \alpha z F''(z) \right] + \left[ \alpha(c + 1) - 1 \right] F'(z) \right\} \]

\[ \leq \frac{1}{\alpha(c + 1)} \left\{ 1 + Mz + [\alpha(c + 1) - 1](1 + Mz) \right\} \]

\[ = 1 + Mz \]

which implies that \( |f'(z) - 1| \leq M|z| < M \) \( (z \in E) \).

ACKNOWLEDGMENT. The author expresses his grateful thanks to the referee for his useful suggestions.

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