ON A CONJECTURE OF VUKMAN

QING DENG
Department of Mathematics
Southwest China Normal University
Chongqing 630715, P.R CHINA

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ABSTRACT. Let $R$ be a ring. A bi-additive symmetric mapping $d : R \times R \to R$ is called a symmetric bi-derivation if, for any fixed $y \in R$, the mapping $x \mapsto D(x, y)$ is a derivation. The purpose of this paper is to prove the following conjecture of Vukman.

Let $R$ be a noncommutative prime ring with suitable characteristic restrictions, and let $D : R \times R \to R$ and $f : x \mapsto D(x, x)$ be a symmetric bi-derivation and its trace, respectively. Suppose that $f(x) \in Z(R)$ for all $x \in R$, where $f_{k+1}(x) \neq [f_k(x), x]$ for $k \geq 1$ and $f_1(x) = f(x)$, then $D = 0$.

KEY WORDS AND PHRASES: Prime ring, centralizing mapping, symmetric bi-derivation.

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1. INTRODUCTION

Throughout this paper, $R$ will denote an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and $I_a$ for the inner derivation deduced by $a$. A mapping $D : R \times R \to R$ will be called symmetric if $D(x, y)$ holds for all pairs $x, y \in R$. A symmetric mapping is called a symmetric bi-derivation, if $D(x + y, z) = D(x, z) + D(y, z)$ and $D(xy, z) = D(x, z)y + xD(y, z)$ are fulfilled for all $x, y \in R$. The mapping $f : R \to R$ defined by $f(x) = D(x, x)$ is called the trace of the symmetric bi-derivation $D$, and obviously, $f(x + y) = f(x) + f(y) + 2D(x, y)$. The concept of a symmetric bi-derivation was introduced by Gy. Maksa in [1,2]. Some recent results concerning symmetric bi-derivations of prime rings can be found in Vukman [3,4]. In [4], Vukman proved that there are no nonzero symmetric bi-derivations $D$ in a noncommutative prime ring $R$ of characteristic not two and three, such that $[[D(x, x), x], x] \in Z(R)$. The following conjecture was raised. Let $R$ be a noncommutative prime ring of characteristic different from two and three, and let $D : R \times R \to R$ be a symmetric bi-derivation. Suppose that for some integer $n \geq 1$, we have $f_n(x) \in Z(R)$ for all $x \in R$, where $f_{k+1}(x) = [f_k(x), x]$ for $k = 1, 2, ..., $ and $f_1(x) = D(x, x)$. Then $D = 0$.

The purpose of this paper is to prove this conjecture under suitable characteristic restrictions.

2. THE RESULTS

THEOREM 1. Let $R$ be a prime ring of characteristic different from two. Suppose that $R$ admits a nonzero symmetric bi-derivation. Then $R$ contains no zero divisors.

PROOF. It is sufficient to show that, $a^2 = 0$ for $a \in R$ implies $a = 0$. We need three steps to establish this.

LEMMA A. If $D(a, *) \neq 0$, then $D(a, *) = \mu I_a$, where $\mu \in C$, the extended centroid of $R$.

PROOF. Since $D(a^2, x) = D(0, x) = 0$, we have...
\[ aD(a, x) + D(a, x)a = 0 \quad \text{for all} \quad x \in R. \]

Replacing \( x \) by \( xy \), we obtain
\[ I_a(x)D(a, y) = D(a, x)I_a(y) \quad \text{for all} \quad x \in R; \]
and replacing \( y \) by \( yz \), we get
\[ I_a(x)yD(a, z) = D(a, x)yI_a(z), \quad x, y, z \in R. \quad (2.1) \]

Since \( D(a, *) \neq 0 \), we may suppose that \( D(a, z) \neq 0 \) for a fixed \( z \in R \). Obviously \( I_a(z) \neq 0 \). By (2.1), and by [5, Lemma 1.3.2], there exist \( \mu(x) \) and \( \nu(x) \) in \( C \), either \( \mu(x) \) or \( \nu(x) \) being not zero, such that
\[ \mu(x)I_a(x) + \nu(x)D(a, x) = 0. \]
If \( \nu(x) \neq 0 \) then \( D(a, x) = -\frac{\mu(x)}{\nu(x)} I_a(x) \); on the other hand, if \( \nu(x) = 0 \) then \( \mu(x)I_a(x) = 0 \) and \( I_a(x) = 0 \), using (2.1) and \( I_a(z) \neq 0 \), so \( D(a, x) = 0 \). In any event, we have
\[ D(a, x) = \mu(x)I_a(x) \quad \text{Hence (2.1) implies (} \mu(x) - \mu(z)\text{)} I_a(x)yI_a(z) = 0 \]
It follows that either \( I_a(x) = 0 \) or \( \mu(x) = \mu(z) \). By (2.1), the former implies \( D(a, x) = 0 \) and \( D(a, x) = \mu(z)I_a(x) \).

In both cases, we get \( D(a, x) = \mu(z)I_a(x) \) for all \( x \in R \), and \( \neq \mu(z) \) being fixed.

The fixed element \( \mu \) in Lemma A is somewhat dependent on \( a \), we write it as \( \mu_a \). For any given \( r \in R \), \( ara \) satisfies our original hypotheses on \( a \); therefore for each \( r \in R \), either \( D(ara, *) = 0 \) or \( d(ara, *) = \mu_araI_{ara} \), where \( \mu_ara \neq 0 \).

**Lemma B.** If \( D(ara, *) \neq 0 \), then \( \mu_ara = \mu_a \).

**Proof.** \( D(ara, *) \neq 0 \) implies \( ara \neq 0 \). Suppose that \( D(a, *) = 0 \), then \( D(ara, x) = D(a, x)ra + aD(r, x)a + arD(a, x) = aD(r, x)a \), but \( D(ara, x) = \mu_araI_{ara}(x) = \mu_ara(arax - zarax) \), so that \( \mu_ara(arax - zarax) = aD(r, x)a \). Right-multiplying the last equation by \( a \), we have \( \mu_araaraxa = 0 \) for all \( x \in R \). It follows that \( ara = 0 \), a contradiction. Therefore \( D(a, *) = \mu_aI_{a} \), and consequently,
\[ D(ara, x) = \mu_aI_{a}(x)ra + aD(r, x)a + ar\mu_a(x); \]
and right-multiplying this equation by \( a \) yields
\[ D(ara, x)a = \mu_aaraxa \quad \text{for all} \quad x \in R. \]

Hence \( \mu_araaraxa = \mu_aaraxa \), immediately \( \mu_ara = \mu_a \).

**Lemma C.** If \( a^2 = 0 \), then \( a = 0 \).

**Proof.** Let \( S = \{ r \in R \mid D(ara, *) = \mu_araI_{ara}, \mu_ara \neq 0 \} \) and \( T = \{ r \in R \mid D(ara, *) = 0 \} \).

By Lemma A and B, \( R = S \cup T \) and \( S \) and \( T \) are additive subgroups of \( R \). We conclude that either \( S = R \) or \( T = R \).

Suppose that \( S = R \). Lemma A gives, either \( D(a, *) = 0 \) or \( D(a, *) = \mu_aI_{a} \). If \( D(a, *) = 0 \), then \( D(ara, x) = aD(r, x)a \), for all \( r, x \in R \), and \( D(ara, x)a = 0 \). It follows that \( \mu_aaraxa = 0 \). Since \( \mu_a = \mu_ara \neq 0 \), we have \( a = 0 \). If \( D(a, *) = \mu_aI_{a} \), then the equation
\[ D(ara, ya) = D(a, ya)ra + aD(r, ya)a + arD(a, ya) \]
gives \( \mu_aaraya = 2\mu_ayaaraxa + \mu_araaraya \). Hence we get \( aaraya = 0 \), and \( a = 0 \) again.

We suppose henceforth that \( T = R \). If \( D(a, *) = 0 \), then \( D(axa, yz) = aD(xa, yz) = 0 \), and \( ayD(xa, z) = 0 \). Thus \( D(xa, z) = D(x, za) = 0 \), and \( D(x, y)za = D(x, yz) = 0 \). Since \( D \neq 0 \), we then get \( a = 0 \). If \( D(a, *) = \mu_aI_{a} \), then, right-multiplying the equation \( D(axa, y) = 0 \) by \( a \), we obtain \( \mu_azaaxa = axD(a, y)a = 0 \), and \( a = 0 \) again. The proof of the theorem is complete.

In order to prove Vukman's conjecture, we need the following proposition.

**Proposition.** Let \( n \) be a positive integer; let \( R \) be a prime ring with char \( R = 0 \) or char \( R > n \), and let \( g \) be a derivation of \( R \) and \( f \) the trace of a symmetric bi-derivation \( D \). For \( i = 1, 2, \ldots, n \), let \( F_i(x, y, z) \) be a generalized polynomial such that \( F_i(kx, f(kz), g(kz)) = k^iF_i(x, f(x), g(x)) \) for all \( x \in R \) for \( k = 1, 2, \ldots, n \). Let \( a \in R \), and (a) the additive subgroup generated by \( a \) if for all \( x \in (a), \)
\[ F_n(x, f(x), g(x)) + F_{n-1}(x, f(x), g(x)) + \cdots + F(x, f(x), g(x)) \in Z(R), \quad (2.2) \]

then \( F_i(a, f(a), g(a)) \in Z(R) \) for \( i = 1, 2, \ldots, n \).

This proposition can be proved by replacing \( x \) by \( a, 2a, \ldots, na \) in (2.2) and applying a standard "Van der Monde argument."

**THEOREM 2.** Let \( n \) be a fixed positive integer and \( R \) be a prime ring with \( \text{char} \ R = 0 \) or \( \text{char} \ R > n + 2 \). Let \( f_{k+1}(x) = [f_k(x), x] \) for \( k > 1 \), and \( f_1(x) = f(x) \) the trace of a symmetric bi-derivaiton \( D \) of \( R \). If \( f_n(x) \in Z(R) \) for all \( x \in R \), then either \( D = 0 \) or \( R \) is commutative.

**PROOF.** Linearizing \( f_n(x) \in Z(R) \), we obtain

\[
[[\ldots[[f(x) + f(y) + 2D(x,y), x - y], x + y], x + y] \in Z(R);\]

and using the Proposition, we get

\[
[[\ldots[[f(x), y], x], x] + \cdots + [[\ldots[[f(x), x], y], \ldots x], x + y], x + y] \in Z(R),
\]

equivalently,

\[
(-1)^{n-2}I^{-2}_x([f_1(x), x^2]) + (-1)^{n-3}I^{-3}_x([f_2(x), y]) + \cdots + [f_{n-1}(x), y] + 2(-1)^{n-1}I^{-1}_x(D(x,y)) \in Z(R). \quad (2.3)
\]

Noting that

\[
(-1)^{n-2}I^{-2}_x([f_1(x), x^2]) = (1)^n([f_2(x), x^2]) = \ldots = [f_{n-1}(x), x^2] = (-1)^{n-1}I^{-1}_x(D(x, x^2)) = 2f_n(x),
\]
and replacing \( y \) by \( x^2 \) in (2.3), we then get \( 2(n + 1)f_n(x) \in Z(R) \). Since \( f_n(x) \in z(R) \), it follows that \( f_n(x) = 0 \).

The linearization of \( f_n(x) = 0 \) gives

\[
(-1)^{n-2}I^{-2}_x([f_1(x), y]) + (-1)^{n-3}I^{-3}_x([f_2(x), y]) + \cdots + [f_{n-1}(x), y] + 2(-1)^{n-1}I^{-1}_x(D(x,y)) = 0. \quad (2.4)
\]

Since \( I^{-k}_x([f_{k-1}(x), xy]) = xI^{-1}_x([f_{k-1}(x), y]) + I^{-k}_x(f_k(x)) \) for \( k = 2, 3, \ldots, n \), and \( I^{-1}_x(D(x, xy)) = xI^{-1}_x(D(x,y)) + I^{-1}_x(f_1(x)) \). Substituting \( xy \) for \( y \) in (2.4), we have

\[
(-1)^{n-2}I^{-2}_x(f_2(x) + \cdots + (-1)^{n-1}I^{-1}_x(f_n(x)y) + 2(-1)^{n-1}I^{-1}_x(D(x,y)) = 0.
\]

Taking \( y = f_{n-2}(x) \), applying \( I^k_x(ab) = \sum_{j=0}^{k} \binom{k}{j} I^{k-j}(a)I^j_x(b) \) and noting \( I^k_x(f_i(x)) = 0 \) for \( i > n \), we then conclude that

\[
2(-1)^{n-1}
\]

But \( (-1)^{k}I^{-1}_x(f_{n-k}(x))I_x(f_{n-2}(x)) = (f_{n-1}(x))^2 \), so \( (n + 2)(n - 1)(f_{n-1}(x))^2 = 0 \), and by the hypotheses on the characteristic, we get \( (f_{n-1}(x))^2 = 0 \). Suppose that \( D \neq 0 \). By Theorem 1, \( f_{n-1}(x) = 0 \), and by induction, \( f_2(x) = [f(x), x] = 0 \) Using Vukman [3, Theorem 1], \( R \) is commutative, we complete the proof of Theorem 2.

**THEOREM 3.** Let \( n > 1 \) be an integer and \( R \) be a prime ring with \( \text{char} \ R = 0 \) or \( \text{char} \ R > n + 1 \), and let \( f(x) \) be the trace of a symmetric bi-derivaiton \( D \) of \( R \). Suppose that \([x^2, f(x)] \in Z(R) \) for all \( x \in R \). In this case either \( D = 0 \) or \( R \) is commutative.
PROOF. Using the condition \([x^n, f(x)] \in Z(R)\), we get \([x^{2n}, f(x^2)] \in Z(R)\), and
\[
[x^{2n}, f(x)] x^2 + x^2 [x^{2n}, f(x)] + 2x [x^{2n}, f(x)] x \in Z(R).
\] (2.5)
Noting that \([x^{2n}, f(x)] = 2[x^n, f(x)] x^n\), we now have from (2.5) that \(8[x^n, f(x)] x^{n+2} \in Z(R)\) Thus either \([x^n, f(x)] = 0\) or \(x^{n+2} \in Z(R)\).

But linearizing \([x^n, f(x)] \in Z(R)\) and applying the Proposition gives
\[
[x^{n-1} y + x^{n-2} y x + \ldots + y x^{n-1}, f(x)] + 2[x^n, D(x, y)] \in Z(R)
\]
for all \(x, y \in R\), and taking \(y = x^3\), yields
\[
n [x^{n+2}, f(x)] + 6[x^n, f(x)] x^2 \in Z(R).
\]
Suppose that \([x^n, f(x)] \neq 0\), then \(x^{n+2} \in Z(R)\) and \([x^n, f(x)] x^2 \in Z(R)\), hence \(x^2 \in Z(R)\) Now this condition, together with \(x^{n+2} \in Z(R)\), implies either \(x^2 = 0\) or \(x^n \in Z(R)\), so that in each event, \([x^n, f(x)] = 0\)

Linearizing \([x^n, f(x)] = 0\) and using the Proposition, we have
\[
[x^{n-1} y + x^{n-2} y x + \ldots + y x^{n-1}, f(x)] + 2[x^n, D(x, y)] = 0
\]
Replacing \(y\) by \(x^2\) yields \(n [x^{n+1}, f(x)] = 0\), hence \([x, f(x)] x^n = 0\) If \(D \neq 0\), then by Theorem 1, \([x, f(x)] = 0\), and by Vukman [3, Theorem 1], \(R\) is commutative This completes the proof

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REFERENCES


