ON X-VALUED SEQUENCE SPACES

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ABSTRACT. Certain spaces of X-valued sequences are introduced and some of their properties are investigated. Köthe- Toeplitz duals of these spaces are examined.

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1. INTRODUCTION AND BACKGROUND.

Let c₀, c, l₁ and s respectively denote the spaces of null sequences, convergent sequences, bounded sequences and all sequences. Let X be a complex linear space with zero element θ and X = (X, ||.||) be a seminormed space. We may define c₀(X) the null X-valued sequences, c(X) the convergent X-valued sequences, l₁(X) the bounded X-valued sequences and s(X) the vector space of all X-valued sequences. If we take X = C the set of complex numbers these spaces reduce to the already familiar spaces c₀, c, l₁ and s respectively. These spaces of X-valued sequences have been studied by Maddox[2,3], Rath[5], Pehlivan[4] and others. We take X and Y to be complete seminormed spaces and (Aₙ) to be a sequence of linear operators from X into Y. We denote by B(X, Y) the space of bounded linear operators on X into Y. Throughout the paper S denotes the unit ball in X, that is S = \{x ∈ X : ||x|| ≤ 1\} is the closed unit sphere in X.

The α and β-duals of Köthe have been generalized by Robinson [6] who replaced scalar sequences by sequences of linear operators. Accordingly, we define α and β duals of a subspace \( E \) of s(X) by

\[ E^\alpha = \{(A_n) : \sum_n ||A_n x_n|| \text{ converges for all } x = (x_n) \in E\}, \]

\[ E^\beta = \{(A_n) : \sum_n A_n x_n \text{ converges in } Y \text{ for all } x = (x_n) \in E\}. \]

Clearly \( E^\alpha \subseteq E^\beta \) if Y is complete and the inclusion may be strict. \( X^* \) will denote the continuous dual of X, this is \( B(X, C) \).

2. MAIN RESULTS

Before proving the main results we give some definitions. We consider a set \( D \) of sequences \( d = (d_n) \) of non-negative real numbers with the following properties:

(i) For each positive integer \( n \) there exists \( d \in D \) with \( d_n > 0 \),

(ii) \( D \) is directed in the sense that for \( d, h \in D \) there exists \( u \in D \) such that \( u_n \leq d_n, h_n \) for all \( n \).

For \( d = (d_n) \in D \) and \( X \) a seminormed vector space, we define the following sequence spaces:

\[ L_\infty(X, d) = \{x = (x_n) : D_d(x) = \sup_n ||x_n||d_n < \infty, \quad x_n \in X \text{ for all } n, \quad d \in D\}, \]

\[ C_0(X, d) = \{x = (x_n) : \lim_n ||x_n||d'_n = 0, \quad x_n \in X \text{ for all } n, \quad d \in D\}. \]

PROPOSITION 2.1 \( C_0(X, d) \) is a closed subspace of \( L_\infty(X, d) \).
PROOF. Let $x \in \tilde{C}_0(X, d)$ and $d = (d_n) \in D$. Given $\epsilon > 0$ there exists $x' = (x'_n) \in C_0(X, d)$ such that $D_d(x - x') < \frac{1}{2}$. If $N$ is such that $d_n \|x_n\| < \frac{1}{2}$ for $n \geq N$, then for $n \geq N$ we have

$$d_n \|x_n\| = d_n \|x_n - x'_n + x'_n\| \leq d_n (\|x_n - x'_n\| + \|x'_n\|) < \epsilon$$

which proves that $x \in C_0(X, d)$.

PROPOSITION 2.2 If $X$ is complete then $C_0(X, d)$ and $L_\infty(X, d)$ are FK spaces.

PROOF. Let $X$ be a complete seminormed space. We show that $L_\infty(X, d)$ is complete. Let $x = (x'_n)$ be a Cauchy sequence in $L_\infty(X, d)$. Then $\|x'_n - x'_m\| \leq d^{-1}_n D_d(x'_n - x'_m)$ therefore $(x'_n)$ is Cauchy in $X$. Let $x_n = \lim x'_n$. Now we will show that $x = (x_n) \in L_\infty(X, d)$ and $x' \leq x$. In fact, let $h \in D$ and $\epsilon > 0$. Choose $N$ such that $D_h(x' - x) < \epsilon$ if $i, j \geq N$. It follows from this that, we have $\|x'_n - x_n\| h_n < \epsilon$ for all $n$ and $i \geq N$. Let $H = D_h(x_N)$. If $\|x_n\| \leq \|x_N\|$ then $\|x_n\| h_n \leq H$. If $\|x_n\| > \|x_N\|$ then

$$\|x_n\| = \|x_n - x_N\| + \|x_N\| h_n \leq \|x_n - x_N\| h_n + \|x_N\| h_n < \epsilon + H$$

which shows that $L_\infty(X, d)$ is complete. The completeness of $C_0(X, d)$ follows from the completeness of $L_\infty(X, d)$ and the Proposition 2.1.

THEOREM 2.3 $C_0(X, d) = L_\infty(X, d)$ if and only if for each $d = (d_n) \in D$ there exists $h = (h_n) \in D$ and a sequence $(u_n)$ of non-negative real numbers such that $u_n \to 0$ and $d_n \leq u_n h_n$ for all $n$.

PROOF. Let $x \in L_\infty(X, d)$. Given $d = (d_n) \in D$ there exist $h = (h_n) \in D$ and a sequence $(u_n)$ of non-negative real numbers such that $u_n \to 0$ and $d_n \leq u_n h_n$ for all $n$. Now, for $x \in L_\infty(X, d)$, we have

$$d_n \|x_n\| \leq u_n h_n \|x_n\| \leq u_n D_h(x)$$

This concludes the proof of the theorem with the Proposition 2.1.

LEMMA 2.4 In order for $C_0(X, d) \subset C_0(X, h)$ it is necessary and sufficient that $\lim inf_n \frac{d_n}{h_n} > 0$.

PROOF. Suppose that $\lim inf_n \frac{d_n}{h_n} = \alpha > 0$. Then since $d_n > \alpha h_n$ the inclusion $C_0(X, d) \subset C_0(X, h)$ is obvious. Now we suppose $\lim inf_n \frac{d_n}{h_n} = 0$. Then there exists a subsequence $(n(p))$ of $(n)$ such that $h_{n(p)} > pd_{n(p)}$ for $p = 1, 2, \ldots$. Define a sequence $x = (x_n)$ by putting $x_{n(p)} = vd_{n(p)}^{-1}p$ for $p = 1, 2, \ldots$ and $x_n = \theta$ otherwise where $v \in X$ and $\|v\| = 1$. Then we have $x = (x_n) \in C_0(X, d)$ but $x \notin C_0(X, h)$ since $\|h_{n(p)}x_{n(p)}\| = \|h_{n(p)}d_{n(p)}^{-1}p^{-1}v\| > 1$. The completes the proof of the theorem.

LEMMA 2.5 In order for $C_0(X, h) \subset C_0(X, d)$ it is necessary and sufficient that $\lim sup_n \frac{d_n}{h_n} < \alpha < \infty$.

PROOF. Suppose that $\lim sup_n \frac{d_n}{h_n} < \infty$. Then there is $K > 0$ such that $d_n < Kh_n$ for all large values of $n$. The inclusion $C_0(X, h) \subset C_0(X, d)$ is obvious. Now we suppose $\lim sup_n \frac{d_n}{h_n} = \infty$. Then there exists a subsequence $(n(p))$ of $(n)$ such that $d_{n(p)} > pd_{n(p)}$ for $p = 1, 2, \ldots$. Define a sequence $x = (x_n)$ by putting $x_{n(p)} = vh_{n(p)}^{-1}p^{-1}$ for $p = 1, 2, 3, \ldots$ and $x_n = \theta$ otherwise where $v \in X$ and $\|v\| = 1$. Then we have $x \in C_0(X, h)$ but $x \notin C_0(X, d)$ since $\|d_{n(p)}x_{n(p)}\| = \|d_{n(p)}h_{n(p)}^{-1}p^{-1}v\| > 1$. The concludes the proof of the lemma.

Combining Lemma 2.4 and 2.5, we have following theorem.

THEOREM 2.6 $C_0(X, h) = C_0(X, d)$ if and only if $0 \leq \lim inf_n \frac{d_n}{h_n} \leq \lim sup_n \frac{d_n}{h_n} < \infty$.

THEOREM 2.7 Let $\lim inf_n \frac{d_n}{h_n} > 0$. The identity mapping of $C_0(X, d)$ into $C_0(X, h)$ is continuous.

PROOF. Let $\lim inf_n \frac{d_n}{h_n} > 0$. Then $C_0(X, d) \subset C_0(X, h)$. There exists $\alpha > 0$ such that $d_n > \alpha h_n$ for all $n$. Thus for $x \in C_0(X, d)$ we have $\alpha D_h(x) \leq D_d(x)$ hence the identity mapping is continuous.

3. GENERALIZED KÖTHE-TOEPLITZ DUALS

Now we determine Köthe-Toeplitz duals in the operator case for the sequence space $C_0(X, d)$. For the more interesting sequence spaces generalized Köthe-Toeplitz duals were determined by Maddox [3]. In the following theorems we suppose in general that $(A_n)$ is a sequence of linear operators $A_n$ mapping
a complete seminormed space $X$ into a complete seminormed space $Y$. Let $(A_n) = (A_1, A_2, \ldots)$ be a sequence in $B(X, Y)$. Then the group norm of $(A_n)$ is defined by

$$
\|(A_n)\| = \sup \left\| \sum_{n=1}^{k} A_n x_n \right\|
$$

where the supremum is taken over all $k \in \mathbb{N}$ and all $x_n \in S$. This argument was introduced by Robinson [6]. This concept was termed as group norm by Lorentz and Macphail [1]. We start with the proposition given by Maddox [3].

**Proposition [M] [3]** If $(A_n)$ is a sequence in $B(X, Y)$ and we write $R_k = (A_k, A_{k+1}, \ldots)$ then

$$
\| \sum_{n=k}^{k+p} A_n x_n \| \leq \| R_k \| \max\{\| x_n \| : k \leq n \leq k+p \},
$$

for any $x_n$ and all $k \in \mathbb{N}$, and all $p > 0$ integers.

**Theorem 3.1** Let $(d_n) \in D$. Then $(A_n) \in C_0^0(X, d)$ if and only if there exists an integer $k$ such that

(i) $A_n \in B(X, Y)$ for each $n \geq k$ and

(ii) $\sum_{n=k}^{p} \|(A_n)\| d_n^{-1} < \infty$.

**Proof.** For the sufficiency, let $x = (x_n) \in C_0(X, d)$ and (i), (ii) hold. Then there exists an integer $n_1$ such that $\| x_n \| d_n < 2\epsilon$ for all $n \geq n_1$ and there exists an integer $n_2 \geq k$ such that

$$
\sum_{n \geq n_2} \| A_n \| d_n^{-1} < \frac{\epsilon}{2}
$$

for a given $\epsilon > 0$. Put $H = \max(n_1, n_2)$ so that

$$
\sum_{n \geq H} \| A_n x_n \| = \sum_{n \geq H} \| A_n \| \| x_n \| \leq \sum_{n \geq H} \| A_n \| 2\epsilon d_n^{-1} < \epsilon,
$$

and therefore $(A_n) \in C_0^0(X, d)$.

Conversely, suppose that $(A_n) \in C_0^0(X, d)$. If (i) does not hold then there exists a strictly increasing sequence $(n_i)$ of natural numbers such that $A_{n_i}$ is not bounded for each $i$ and a sequence $(v_n)$ in $S$ such that $\| A_{n_i} v_n \| > d_{n_i}$. For each $i \geq 1$, define the sequence $x = (x_n)$ by putting $x_{n_i} = v_n d_{n_i}^{-1}$ for each $i \geq 1$ and $x = \theta$ otherwise. We have $x \in C_0(X, d)$ but $A_{n_i} x_{n_i} \| > 1$ for each $i \geq 1$ and so $\sum_{n_i} \| A_{n_i} x_{n_i} \|$ diverges, which gives a contradiction.

Now we suppose $(A_n) \in C_0^0(X, d)$ and $\sum_{n \geq k} \| A_n \| d_n^{-1} = \infty$. We choose $k = n_1 < n_2 < n_3 \ldots$ such that $\sum_{n=n_i}^{n_{i+1}} \| A_n \| d_n^{-1} > i$ for $i \in \mathbb{N}$. Moreover for each $n \geq k$ there exists a sequence $(v_n)$ in $S$ such that $2\| A_n v_n \| \geq \| A_n \|$. Define the sequence $x = (x_n)$ by putting $x_n = v_n d_n^{-1}$ for $n \leq n \leq n_{i+1} - 1$ for $i = 1, 2, \ldots$ and $x_n = \theta$ otherwise so that $x \in C_0(X, d)$ since

$$
\| x_n \| d_n = \frac{\| v_n \|}{i} \to 0 \text{ as } n \to \infty.
$$

Then we have

$$
\sum_n \| A_n x_n \| = \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \| A_n v_n d_n^{-1} i^{-1} \|
\geq \frac{1}{2} \sum_{i=1}^{\infty} \sum_{n=n_i}^{n_{i+1}-1} \| A_n \| d_n^{-1} i^{-1}
\geq \frac{1}{2} \sum_{i=1}^{\infty} i
$$

which contradicts our assumption that $\sum_n \| A_n x_n \| < \infty$. This completes the proof.

It is clear that the conditions of the theorem 3.1. are also necessary and sufficient for $(A_n) \in l_\infty^0(X, d)$ then we have $C_0^0(X, d) = l_\infty^0(X, d)$. 
COROLLARY 3.2 ([6], Theorem 1.) Let \( p_n = O(1) \). Then \( (A_n) \in C^0_0(X,p) \) if and only if there exists an integer \( k \) such that condition (i) of Theorem 3.1 holds and

(iii) there exists an integer \( N > 1 \) such that \( \sum_{n \geq k} ||A_n|| N^{-\frac{1}{n}} < \infty \).

COROLLARY 3.3 ([3], Proposition 3.4.) \( (A_n) \in C^0_0(X) \) if and only if there exists an integer \( k \) such that condition (i) of Theorem 3.1 holds and

(iv) \( \sum_{n=0}^{\infty} ||A_n|| < \infty \).

THEOREM 3.4 Let \( (d_n) \in D \). Then \( (A_n) \in C^0_0(X,d) \) if and only if there exists a strictly increasing sequence \( (m_i) \) of natural numbers such that \( \sum_{n=m_i}^{m_{i+1}-1} d_n A_n v_n || < i \) for \( i \in N \). Define the sequence \( x = (x_n) \) by putting \( x_n = v_n d_n^{-1} x_{n-1} \) for \( n \leq n \leq n+1 - 1, \ i = 1, 2, \ldots \) and \( x_n = 0 \) otherwise. We have \( x \in C_0(X,d) \) but for each \( i \geq 1 \)

\[
\sum_{n=n_i}^{n_i+1-1} \sum_{n=n_i}^{n_i+1-1} A_n v_n d_n^{-1} x_{n-1} = 1
\]

Therefore \( \sum_{k=n} A_n x_k \) diverges, which gives a contradiction. This proves the theorem.

COROLLARY 3.5 ([3], Proposition 3.1.) \( d_n \) for all \( n \), \( (A_n) \in C^0_0(X) \) if and only if condition (i) of Theorem 3.1 holds and \( ||R_k(d)|| < \infty \).

THEOREM 3.6 \( Y = X \) and \( f_n \in X^* \) for \( n \geq 1 \) then \( C^0_0(X,d) = C^0_0(X,d) \) where \( M_0(X^*,d) = \{ F = (f_n) : f_n \in X^*, \sum_{n=0}^{\infty} ||f_n|| d_n^{-1} < \infty \} \).

PROOF. We show that \( C^0_0(X,d) \subseteq M_0(X^*,d) \), which is sufficient to prove the theorem. We suppose \( F \not\in M_0(X^*,d) \) then there exists a strictly increasing sequence \( (n_i) \) of natural numbers such that \( \sum_{n=n_i}^{n_{i+1}-1} ||f_n|| d_n^{-1} > i \) for \( i \in N \). Define the sequence \( x = (x_n) \) by putting \( x_n = \text{sgn}(f_n(v_n)) d_n^{-1} x_{n-1} \) for \( n \leq n \leq n_i - 1, \ i = 1, 2, \ldots \) and \( x_n = 0 \) otherwise. Then \( x \in C_0(X,d) \) but \( \sum_{n=0}^{\infty} ||x_n|| d_n^{-1} x_{n-1} = 1 \) diverges and so \( F \not\in C^0_0(X,d) \). Thus \( C^0_0(X,d) \subseteq M_0(X^*,d) \) and the proof is complete.

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