MULTIPLICATION OPERATORS ON WEIGHTED SPACES IN
THE NON-LOCALLY CONVEX FRAMEWORK

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ABSTRACT. Let \( X \) be a completely regular Hausdorff space, \( E \) a topological vector space, \( V \) a Nachbin family of weights on \( X \), and \( CV_0(X, E) \) the weighted space of continuous \( E \)-valued functions on \( X \). Let \( \theta : X \rightarrow C \) be a mapping, \( f \in CV_0(X, E) \) and define \( M_\theta(f) = \theta f \) (pointwise). In case \( E \) is a topological algebra, \( \psi : X \rightarrow E \) is a mapping then define \( M_\psi(f) = \psi f \) (pointwise). The main purpose of this paper is to give necessary and sufficient conditions for \( M_\theta \) and \( M_\psi \) to be the multiplication operators on \( CV_0(X, E) \) where \( E \) is a general topological space (or a suitable topological algebra) which is not necessarily locally convex. These results generalize recent work of Singh and Manhas based on the assumption that \( E \) is locally convex.

KEY WORDS AND PHRASES: Nachbin family of weights, topological vector spaces, vector-valued continuous functions, weighted topology, multiplication operators, locally idempotent topological algebras.


1. INTRODUCTION

The fundamental work on weighted spaces of continuous scalar-valued functions has been done mainly by Nachbin [9,10] in the 1960's. Since then it has been studied extensively for a variety of problems such as weighted approximation, characterization of the dual space, approximation property, description of inductive limit and of tensor-product, etc for both scalar- and vector-valued functions (for instance see [1-5,8-14]). Recently Singh and Summers [13] have studied the notion of composition operators on \( CV_0(X, C) \). Later, Singh and Manhas [12] made an analogous study of multiplication operators on \( CV_0(X, E) \), assuming \( E \) to be a locally convex space or a locally \( m \)-convex algebra. The purpose of this paper is to generalize the results of Singh and Manhas [12] to the case when \( E \) is a general topological vector space which is not necessarily locally convex. Section 3 contains our main results while section 2 is devoted to some technical preliminaries required for the development of our results.

2. PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that \( X \) is a completely regular Hausdorff space and \( E \) is a non-trivial Hausdorff topological vector space. Let \( S^+(X) \) denote the set of
all non-negative upper-semicontinuous functions on $X$, and let $S^+_0(X)$ (respectively $S^-_0(X)$), be the subset of $S^+(X)$ consisting of those functions vanishing at infinity (respectively having compact support).

A Nachbin family on $X$ is a subset $V$ of $S_+^+(X)$ such that, given $u, v \in V$, there exist $w \in V$ and $t > 0$ so that $u, v \leq tw$ (pointwise), the elements of $V$ are called weights. Let $C(X, E)$ ($C_0(X, E)$) be the vector space of all continuous (and bounded) $E$-valued functions on $X$, and let $CV_0(X, E)$ ($CV_0(X, E)$) denote the subspace of $C(X, E)$ consisting of those $f$ such that $vf$ is bounded (vanishes at infinity) for each $v \in V$. When $E = C$ (or $R$), these spaces are denoted by $C(X)$, $C_0(X)$, $CV_0(X)$, and $CV_0(X)$. If $\phi \in C(X)$ and $a \in E$, then $\phi \odot a$ is a function in $C(X, E)$ defined by $(\phi \odot a)(x) = \phi(x)a(x) \in X$.

If $U$ and $V$ are two Nachbin families on $X$ and, for each $u \in U$, there is a $v \in V$ such that $u \leq v$, then we write $U \leq V$. If, for each $x \in X$, there is a $v \in V$ with $v(x) \neq 0$, we write $V \gg 0$. For any function $\theta : X \to C$, we let $V|\theta| = \{v|\theta| : v \in V\}$.

Given any Nachbin family $V$ on $X$, the weighted topology $w_\theta$ on $CV_0(X, E)$ is defined as the linear topology which has a base of neighborhoods of $0$ consisting of all sets of the form

$$N(u, G) = \{f \in CV_0(X, E) : (uf)(X) \subseteq G\},$$

where $v \in V$ and $G$ is a neighborhood of $0$ in $E$. $CV_0(X, E)$ endowed with $w_\theta$ is called a weighted space. We mention that if $V = S_0^+(X)$, then $CV_0(X, E) = CV_0(X, E) = CV_0(X)$ and $w_\theta = \beta$, the strict topology and write as $(C(X, E), \beta)$; if $V = S_0^-(X)$, then $CV_0(X, E) = CV_0(X, E) = CV_0(X)$ and $w_\theta = k$, the compact-open topology and we write as $(C(X, E), k)$. For more information on weighted spaces, we refer to [1-2,9-14] when $E$ is a scalar field or a locally convex space and to [1,3-8] in the general setting.

Let $\theta : X \to C$ and $\psi : X \to E$ be two mappings, and let $L(X, E)$ be the vector space of all functions from $X$ into $E$. The scalar multiplication on $E$ and, in case $E$ is an algebra, multiplication on $E$ give rise to two linear mappings $M_\theta$ and $M_\psi$ from $CV_0(X, E)$ into $L(X, E)$ defined by $M_\theta(f) = \theta f$ and $M_\psi(f) = \psi f$, where the product of functions is defined pointwise. If $M_\theta$ and $M_\psi$ map $CV_0(X, E)$ into itself and are continuous, they are called multiplication operators on $CV_0(X, E)$ induced by $\theta$ and $\psi$, respectively.

A neighborhood $G$ of $0$ in $E$ is called shrinkable if $rG \subseteq \text{int } G$ for $0 \leq r < 1$. By ([6], Theorems 4 and 5), every Hausdorff topological vector space has a base of shrinkable neighborhoods of $0$ and also the Minkowski functional $\rho_G$ of any such neighborhood $G$ is continuous.

Now let $E$ be a topological algebra with jointly continuous multiplication and having $W$, a base of neighborhoods of $0$. Then, given any $G \subseteq W$, there exists an $H \subseteq W$ such that $H^2 \subseteq G$. (Here $H^2 = \{ab : a, b \in H\}$.) A subset $G \subseteq W$ is called idempotent (or multiplicative) if $G^2 \subseteq G$. Following Zelazko ([16], p. 31), $E$ is said to be a locally idempotent algebra if it has a base of neighborhoods of $0$ consisting of idempotent sets. It is easily seen that if $G \subseteq W$ is idempotent, then $\rho_G$ is submultiplicative: $\rho_G(ab) \leq \rho_G(a)\rho_G(b)$ for all $a, b \in E$; further, if $E$ has an identity $e$, $\rho_G(e) \geq 1$. The notion of locally idempotent algebras is a strict generalization of the notion of locally $m$-convex algebras introduced by Michael [7] (see also [15, p. 348]).

3. CHARACTERIZATION OF MULTIPLICATION OPERATORS

In this section, we give necessary and sufficient conditions for $M_\theta$ and $M_\psi$ to be the multiplication operators on the weighted space $CV_0(X, E)$. (These results hold also for the space $CV_0(X, E)$ with slight modification in the proofs and are therefore omitted.) To avoid trivial cases we assume that the Nachbin family $V$ on $X$ satisfies the following conditions

(*) $V \gg 0$,

(**) corresponding to each $x \in X$, there exists an $h_x \in CV_0(X)$ such that $h_x(x) \neq 0$ (This holds in particular, when each $v$ in $V$ vanishes at infinity or $X$ is locally compact.)

THEOREM 3.1. For a mapping $\theta : X \to C$, the following are equivalent:
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(a) \( \theta \) is continuous and \( V|\theta| \leq V \);
(b) \( M_\theta \) is a multiplication operator on \( CV_0(X, E) \).

PROOF. Let \( W \) be a base of closed, balanced, and shrinkable neighborhoods of 0 in \( E \).

(a) \( \Rightarrow \) (b). We first show that \( M_\theta \) maps \( CV_0(X, E) \) into itself. Let \( f \in CV_0(X, E) \), and let \( v \in V \) and \( G \in W \). Choose \( u \in V \) such that \( v|\theta| \leq u \). There exists a compact set \( K \subseteq X \) such that \( u(x)f(x) \in G \) for all \( x \in X \setminus K \). Then, since \( G \) is balanced,

\[
v(x)M_\theta(f)(x) = v(x)\theta(x)f(x) \in G
\]

for all \( x \in X \setminus K \). Hence \( vM_\theta(f) \) vanishes at infinity; further, since \( \theta \) is continuous, \( M_\theta(f) \in CV_0(X, E) \). To prove the continuity of \( M_\theta \), let \( \{f_\alpha\} \) be a net in \( CV_0(X, E) \) with \( f_\alpha \to 0 \). Let \( v, G \) and \( u \) be chosen as above. Choose an index \( \alpha_0 \) such that \( \theta f_\alpha \in N(v, G) \) for all \( \alpha \geq \alpha_0 \). Then it follows that \( \theta f_\alpha \to 0 \) in \( CV_0(X, E) \). So \( M_\theta \) is continuous at 0 and hence, by linearity, it is continuous on \( CV_0(X, E) \).

(b) \( \Rightarrow \) (a). We first show that \( \theta \) is continuous. Let \( \{x_\alpha\} \) be a net in \( X \) with \( x_\alpha \to x \). By assumption (**), there exists an \( h \in CV_0(X) \) such that \( h(x) \neq 0 \). Since \( M_\theta \) is a self-map on \( CV_0(X, E) \), it follows that the function \( \theta h \) from \( X \) into \( C \) is continuous. Hence \( \theta(x_\alpha)h(x_\alpha) \to \theta(x)h(x) \) and consequently \( \theta(x_\alpha) \to \theta(x) \). We next show that \( V|\theta| \leq V \). Let \( v \in V \). By continuity of \( M_\theta \), given \( G \in W \), there exist \( u \in V \) and \( H \in W \) such that

\[
M_\theta(N(u, H)) \subseteq (v, G).
\]

Without loss of generality we may assume that \( G \cup H \) is a proper subset of \( E \). Choose \( a \in X \setminus (G \cup H) \), and put \( t = \rho_H(a)/\rho_G(a) \). We claim that \( v|\theta| \leq 2tu \). Fix \( x_0 \in X \). We shall consider two cases \( u(x_0) \neq 0 \) and \( u(x_0) = 0 \).

Suppose that \( u(x_0) \neq 0 \), and let \( \epsilon = u(x_0) \). Then \( D = \{x \in X : u(x) < 2\epsilon\} \) is an open neighborhood of \( x_0 \). Using the complete regularity of \( X \) and the assumption (**), there is an \( h \in CV_0(X, E) \) with \( 0 \leq h \leq 1 \), \( h(x_0) = 1 \), and \( h(X \setminus D) = 0 \). Define \( f = (h \otimes a)/2\epsilon \rho_H(a) \). Since \( \rho_H \) is homogeneous, for any \( x \in X \),

\[
\rho_H(u(x)f(x)) = u(x)h(x)/2\epsilon < 1,
\]

by considering the cases \( x \in D \) and \( x \in X \setminus D \). Since \( H = \{b \in E : \rho_H(b) \leq 1\} \), we have \( f \in N(u, H) \). Hence, by (1), \( \theta f \in N(v, G) \). This implies that, for any \( x \in X \),

\[
\rho_G(\theta(x)v(x)h(x)a/2\epsilon \rho_H(a)) \leq 1,
\]

or \( v(x)h(x)|\theta(x)| \leq 2\epsilon u(x_0) \). In particular, \( v(x_0)|\theta(x_0)| \leq 2tu(x_0) \).

Now suppose that \( u(x_0) = 0 \) but \( v(x_0)|\theta(x)| > 0 \). Put \( \epsilon = v(x_0)|\theta(x_0)|/2t \) Let \( D = \{x \in X : u(x) < \epsilon\} \), and choose an \( h \in CV_0(X) \) as above. Define \( g = (h \otimes a)/\epsilon \rho_H(a) \). We easily have \( g \in N(u, H) \) and hence \( \theta g \in N(v, G) \). From this we obtain

\[
v(x_0)|\theta(x_0)| \leq \epsilon e = v(x_0)|\theta(x_0)|/2t,
\]

which is impossible unless \( v(x_0)|\theta(x_0)| = 0 \). This completes the proof.

We next consider the case of the operator \( M_\varphi \).

THEOREM 3.2. Let \( E \) be a Hausdorff locally idempotent algebra with identity \( e \) and \( W \) a base of neighborhoods of 0. Then, for a mapping \( \psi : X \to E \), the following are equivalent:

(a) \( \psi \) is continuous and \( V\rho_G \circ \psi \leq V \) for every \( G \in W \).
(b) \( M_\psi \) is a multiplication operator on \( CV_0(X, E) \).

PROOF. We may assume that \( W \) consists of closed, balanced, shrinkable, and idempotent sets.
(a) ⇒ (b) We first show that $M_\psi$ maps $CV_0(X, E)$ into itself. Let $f \in CV_0(X, E)$, and let $v \in V$ and $G \in W$. Choose $u \in V$ such that $\rho_G \circ \psi \leq u$. There exists a compact set $K \subseteq X$ such that $u(x)f(x) \in G$ for all $x \in X \setminus K$. Since $\rho_G$ is submultiplicative, for any $x \in X \setminus K$, we have

$$\rho_G(v(x)\psi(x)f(x)) \leq v(x)\rho_G(\psi(x))\rho_G(f(x)) \leq u(x)\rho_G(f(x)) \leq 1;$$

hence $M_\psi(f) \in CV_0(X, E)$. Using again the submultiplicativity of $\rho_G$, the continuity of $M_\psi$ follows in the same way as in the proof of Theorem 1.

(b) ⇒ (a). Let \{x_\alpha\} be a net in $X$ such that $x_\alpha \to x \in X$. Choose an $h \in CV_0(X)$ with $h(x) \neq 0$. Since $M_\psi$ is a self-map on $CV_0(X, E)$, it follows that the function $\psi(h \otimes a)$ from $X$ into $E$ is continuous. Hence $h(x_\alpha)\psi(x_\alpha) \to h(x)\psi(x)$ and consequently $\psi(x_\alpha) \to \psi(x)$. This proves the continuity of $\psi$. Next, let $v \in V$ and $G \in W$. There exist $u \in V$ and $H \in W$ such that

$$M_\psi(N(u, H)) \subseteq N(v, G). \quad (2)$$

Without loss of generality, we may assume that $H$ is a proper subset of $E$. We claim that $v\rho_G \circ \psi \leq 2\rho_H(e)u$.

Fix $x_0 \in X$. First assume that $u(x_0) \neq 0$, and let $\epsilon = u(x_0)$. Then $D = \{x \in X : u(x) < 2\epsilon\}$ is an open neighborhood of $x_0$, so there exists an $h \in CV_0(X)$ such that $0 \leq h \leq 1, h(x_0) = 1$, and $h(X \setminus D) = 0$. Define $f = (h \otimes e)/2\rho_H(e)$. Then, for any $x \in X$,

$$\rho_H(u(x)f(x)) = \rho_H(u(x)h(x)e)/2\rho_H(e) \leq 1;$$

that is, $f \in N(u, H)$. Hence, by (2), $\psi f \in N(v, G)$. This implies that, for any $x \in X$,

$$v(x)h(x)\rho_G(\psi(x)) \leq 2\epsilon\rho_H(e).$$

In particular, $v(x_0)\rho_G(\psi(x_0)) \leq 2\epsilon\rho_H(e)u(x_0)$. Next suppose that $u(x_0) = 0$, but $v(x_0)\rho_G(\psi(x_0)) > 0$. Put $\epsilon = v(x_0)\rho_G(\psi(x_0))/2\rho_H(e)$. Let $D = \{x \in X : u(x) < \epsilon\}$, and choose an $h \in CV_0(X)$ as above.

Define $g = (h \otimes e)/\epsilon\rho_H(e)$. Then $g \in N(u, H)$, so by (2), $\psi g \in N(v, G)$. From this we obtain

$$v(x_0)\rho_G(\psi(x_0)) \leq \rho_H(e)\epsilon = v(x_0)\rho_G(\psi(x_0))/2,$$

which is impossible unless $v(x_0)\rho_G(\psi(x_0)) = 0$. This completes the proof.

Finally, we apply the above results to the cases: $V = S^+(X)$ and $V = S^0(X)$ and obtain the following.

**THEOREM 3.3.**

(i) If $0_\theta : X \to C$ is a continuous mapping, then $M_\theta$ is a multiplication operator on $(C(X, E), k)$.

(ii) If $E$ is a Hausdorff locally idempotent algebra with identity $e$ and $\psi : X \to E$ a continuous mapping, then $M_\psi$ is a multiplication operator on $(C(X, E), k)$.

**PROOF.** (i) In view of Theorem 1, we only need to verify that $V|\theta| \leq V$, where $V = S^+_\theta(X)$. Let $v \in V$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X \setminus K$. Let $s = \sup\{|\theta(x)| : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st_\chi_K$. Then $u \in V$ and clearly $v(x)|\theta(x)| \leq u(x)$ for all $x \in X$.

(ii) Let $W$ be a base of neighborhoods of 0 in $E$ consisting of closed, balanced, shrinkable, and idempotent sets. In view of Theorem 2, we only need to verify that $V\rho_G \circ \psi \leq V$ for every $G \in W$, where $V = S^+_\theta(X)$. Let $v \in V$ and $G \in W$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X \setminus K$. Let $s = \sup\{\rho_G(\psi(x)) : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st_\chi_K$. Then $u \in V$ and clearly $v(x)\rho_G(\psi(x)) \leq u(x)$ for all $x \in X$. This completes the proof of the theorem.

**REMARK.** The above result need not hold for the space $(C_0(X, E), \beta)$. To see this, consider $X = R^+$, $E = C$, and $V = S^0_\theta(X)$. Let $\theta = \psi : X \to C$ be a mapping given by $\theta(x) = x^2 (x \in X)$, and let $v \in V$ be given by $v(x) = \frac{1}{x}(x \in X)$. Then $v(x)|\theta(x)| = x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v|\theta| \nleq u$ for every $u \in V$. Hence $V|\theta| \nleq V$ does not hold and so, by Theorem 1, $M_\theta$
is not a multiplication operator on \((C_b(X), \beta)\). The same is also true for the space \((C_b(X), u)\), where \(u\) is the uniform topology, since \(\beta \leq u\). However, if \(\theta\) and \(\psi\) are bounded continuous functions, then it is easily seen that \(M_{\theta}\) and \(M_{\psi}\) are always multiplication operators on \(CV_0(E)\) for any Nachbin family \(V\).

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**REFERENCES**


