Research Article

On Symmetric Left Bi-Derivations in BCI-Algebras

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The notion of symmetric left bi-derivation of a BCI-algebra X is introduced, and related properties are investigated. Some results on componentwise regular and d-regular symmetric left bi-derivations are obtained. Finally, characterizations of a p-semisimple BCI-algebra are explored, and it is proved that, in a p-semisimple BCI-algebra, F is a symmetric left bi-derivation if and only if it is a symmetric bi-derivation.

1. Introduction

BCI-algebras and BCI-algebras are two classes of nonclassical logic algebras which were introduced by Imai and Iséki in 1966 [1, 2]. They are algebraic formulation of BCK-system and BCI-system in combinatory logic. Later on, the notion of BCI-algebras has been extensively investigated by many researchers (see [3–6], and references therein). The notion of a BCI-algebra generalizes the notion of a BCK-algebra in the sense that every BCK-algebra is a BCI-algebra but not vice versa (see [7]). Hence, most of the algebras related to the t-norm-based logic such as MTL [8], BL, hoop, MV [9] (i.e. lattice implication algebra), and Boolean algebras are extensions of BCK-algebras (i.e. they are subclasses of BCK-algebras) which have a lot of applications in computer science (see [10]). This shows that BCK/BCI-algebras are considerably general structures.

Throughout our discussion, X will denote a BCI-algebra unless otherwise mentioned. In the year 2004, Jun and Xin [11] applied the notion of derivation in ring and near-ring theory to BCI-algebras, and as a result they introduced a new concept, called a (regular) derivation, in BCI-algebras. Using this concept as defined they investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a p-semisimple BCI-algebra. For a self-map d of a BCI-algebra, they defined a d-invariant ideal and gave conditions for an ideal to be d-invariant. According to Jun and Xin, a self map d : X → X is called a left-right derivation (briefly (l, r)-derivation) of X if d(x * y) = d(x) * y ∧ x * d(y) holds for all x, y ∈ X.

Similarly, a self map d : X → X is called a right-left derivation (briefly (r, l)-derivation) of X if d(x * y) = x * d(y) ∧ d(x) ∗ y holds for all x, y ∈ X. Moreover, if d is both (l, r)- and (r, l)-derivation, it is a derivation on X. After the work of Jun and Xin [11], many research articles have appeared on the derivations of BCI-algebras and a greater interest has been devoted to the study of derivations in BCI-algebras on various aspects (see [12–17]).

Inspired by the notions of σ-derivation [18], left derivation [19], and symmetric bi-derivations [20, 21] in rings and near-rings theory, many authors have applied these notions in a similar way to the theory of BCI-algebras (see [12, 13, 17]). For instance in 2005 [17], Zhan and Liu have given the notion of f-derivation of BCI-algebras as follows: a self map d : X → X is said to be a left-right f-derivation or (l, r)-f-derivation of X if it satisfies the identity d_{(x * y)} = d_{(x)} * f(y) ∧ f(x) * d_{(y)} for all x, y ∈ X.

Similarly, a self map d : X → X is said to be a right-left f-derivation or (r, l)-f-derivation of X if it satisfies the identity d_{(x * y)} = f(x) * d_{(y)} ∧ d_{(x)} * f(y) for all x, y ∈ X. Moreover, if d is both (l, r)- and (r, l)-f-derivation, it is said that d is an f-derivation, where f is an endomorphism. In the year 2007, Abujabal and Al-Shehri [12] defined and studied the notion of left derivation of BCI-algebras as follows: a self map D : X → X is said to be a left
derivation of $X$ if satisfying $D(x * y) = x * D(y) \land y * D(x)$ for all $x, y \in X$. Furthermore, in 2011 [13], Libera et al. have introduced the notion of symmetric bi-derivations in $BCI$-algebras. Following [13], a mapping $D(\cdot, \cdot) : X \times X \to X$ is said to be symmetric if $F(x, y) = F(y, x)$ holds for all pairs $x, y \in X$. A symmetric mapping $D(\cdot, \cdot) : X \times X \to X$ is called left-right symmetric bi-derivation (briefly $(l, r)$-symmetric bi-derivation) if it satisfies the identity $D(x * y, z) = D(x, z) * D(y, z) \land D(x, z) * y$ for all $x, y, z \in X$. $D$ is called right-left symmetric bi-derivation (briefly $(r, l)$-symmetric bi-derivation) if it satisfies the identity $D(x * y, z) = x * D(y, z) \land D(x, z) * y$ for all $x, y, z \in X$. Moreover, if $D$ is both a $(l, r)$- and a $(r, l)$-symmetric bi-derivation, it is said that $D$ is a symmetric bi-derivation on $X$.

Motivated by the notion of symmetric bi-derivations [13] in the theory of $BCI$-algebras, in the present analysis, we introduced the notion of symmetric left bi-derivations on $BCI$-algebras and investigated related properties. Further, we obtained some results on componentwise regular and $d$-regular symmetric left bi-derivations. Finally, we characterize the notion of $p$-semisimple $BCI$-algebra $X$ by using the concept of symmetric left bi-derivation and show that, in a $p$-semisimple $BCI$-algebra $X$, $F$ is a symmetric left bi-derivation if and only if it is a symmetric bi-derivation.

2. Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set $X$ with a constant 0 and a binary operation $*$ is called a $BCI$-algebra if for all $x, y, z \in X$ the following conditions hold:

(I) $(x * y) * (x * z) * (z * y) = 0$,

(II) $x * (x * y) * y = 0$,

(III) $x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on $X$ by letting $x * y = 0$ if and only if $x \leq y$. Then $(X, \leq)$ is a partially ordered set. A $BCI$-algebra $X$ satisfying $0 \leq x$ for all $x \in X$ is called $BCK$-algebra.

A $BCI$-algebra $X$ has the following properties for all $x, y, z \in X$.

(a1) $x * 0 = x$.

(a2) $x * y * z = (x * z) * y$.

(a3) $y * z = xy * z$ and $z * y \leq z * x$.

(a4) $x * (y * z) \leq x * y$.

(a5) $x * (x * (x * y)) = x * y$.

(a6) $0 * (x * y) = (0 * x) * (0 * y)$.

(a7) $x * 0 = 0$ implies $x = 0$.

For a $BCI$-algebra $X$, denote by $X_+$ (resp., $G(X)$) the $BCK$-part (resp., the $BCI$-G part) of $X$; that is, $X_+$ is the set of all $x \in X$ such that $0 \leq x$ (resp., $G(X) := \{ x \in X \mid 0 * x = x \}$). Note that $G(X) \cap X_+ = \{ 0 \}$ (see [22]).

If $X_+ = \{ 0 \}$, then $X$ is called a $p$-semisimple $BCI$-algebra. In a $p$-semisimple $BCI$-algebra $X$, the following hold.

(a8) $(x * z) * (y * z) = x * y$.

(a9) $0 * (0 * x) = x$ for all $x \in X$.

(a10) $x * (0 * y) = y * (0 * x)$.

(a11) $x * 0 = 0$ implies $y = x$.

(a12) $x * a = x * b$ implies $a = b$.

(a13) $a * x = b * x$ implies $a = b$.

(a14) $a * (a * x) = x$.

(a15) $(x * y) * (w * z) = (x * w) * (y * z)$.

Let $X$ be a $p$-semisimple $BCI$-algebra. We define addition “$+$” as $x + y = x * (0 * y)$ for all $x, y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x + y = x * y$. Conversely, let $(X, +)$ be an abelian group with identity 0, and let $x * y = x + y$. Then $X$ is a $p$-semisimple $BCI$-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see (6)).

For a $BCI$-algebra $X$, we denote $x \land y = y * (y * x)$, in particular $0 * (0 * x) = a_0$, and $L_p(X) := \{ a \in X \mid x * a = 0 \Rightarrow x = a, \forall x \in X \}$. We call the elements of $L_p(X)$ the $p$-atoms of $X$. For any $a \in X$, let $V(a) := \{ x \in X \mid a * x = 0 \}$, which is called the branch of $X$ with respect to $a$. It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all $x, y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{ x \in X \mid a_0 = x \}$, which is the $p$-semisimple part of $X$, and $X$ is a $p$-semisimple $BCI$-algebra if and only if $L_p(X) = X$ (see [23, Proposition 3.2]). Note also that $a_0 \in L_p(X)$; that is, $0 * (0 * a_0) = a_0$, which implies that $a_0 * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subseteq L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. Let $D(\cdot, \cdot) : X \times X \to X$ be a symmetric mapping. Then for all $x \in X$, a mapping $d : X \to X$ defined by $d(x) = D(x, x)$ is called trace of $D$ [13]. For more details, refer to [3, 4, 6, 11, 22, 23].

3. Symmetric Left Bi-Derivations

The following definition introduces the notion of symmetric left bi-derivation for a $BCI$-algebra $X$.

Definition 1. A symmetric mapping $F(\cdot, \cdot) : X \times X \to X$ is called a symmetric left bi-derivation of $X$ if it satisfies the following identity:

$$\forall x, y, z \in X \quad (F(x * y, z) = (x * F(y, z)) \land (y * F(x, z))).$$

Example 2 (see [24]). Consider a $p$-semisimple $BCI$-algebra $X = \{ 0, 3, 4, 5 \}$ with the following Cayley table:

<table>
<thead>
<tr>
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<th>0</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tr>
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<td>3</td>
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<td>0</td>
</tr>
</tbody>
</table>
Define a mapping \( F(\cdot, \cdot) : X \times X \to X \) by

\[
\begin{align*}
F(0, 0) &= F(3, 3) = F(4, 4) = F(5, 5) = 0, \\
F(0, 3) &= F(3, 0) = 3, \\
F(0, 4) &= F(4, 0) = 4, \\
F(0, 5) &= F(5, 0) = 5, \\
F(3, 4) &= F(4, 3) = 5, \\
F(3, 5) &= F(5, 3) = 4, \\
F(4, 5) &= F(5, 4) = 3.
\end{align*}
\]

It is routine to verify that \( F \) is a symmetric left bi-derivation of \( X \).

**Theorem 3.** Let \( F(\cdot, \cdot) : X \times X \to X \) be a symmetric left bi-derivation of \( X \). Then

\[
\begin{align*}
(1) & \quad (\forall z \in X) \ (a \in G(X) \Rightarrow F(a, z) \in G(X)). \\
(2) & \quad (\forall z \in X) \ (a \in L_p(X) \Rightarrow F(a, z) \in L_p(X)). \\
(3) & \quad (\forall z \in X) \ (a \in L_p(X) \Rightarrow F(a, z) = 0 + F(0, z)). \\
(4) & \quad (\forall z \in X) \ (a \in L_p(X) \Rightarrow F(a, z) = a + F(0, z)).
\end{align*}
\]

**Proof.** (1) Let \( a \in G(X) \). Then \( 0 \ast a = a \), and so

\[
F(a, z) = F(0 \ast a, z) = (0 \ast F(a, z)) \land (a \ast F(0, z)) = 0 \ast F(a, z),
\]

since \( 0 \ast F(a, z) \in L_p(X) \). Hence \( F(a, z) \in G(X) \).

(2) For any \( a \in L_p(X) \) implies \( a = 0 \ast (0 \ast a) \) and so

\[
F(a, z) = F(0 \ast (0 \ast a), z) = (0 \ast F(0 \ast a, z)) \land ((0 \ast a) \ast F(0, z)) = (0 \ast a) \ast F(0, z) \land ((0 \ast a) \ast F(0, z)) = 0 \ast F(0, z) + (0 \ast a, z) \in L_p(X).
\]

(3) By (2), we have \( F(a, z) \in L_p(X) \). Then

\[
F(a, z) = 0 \ast (0 \ast F(a, z)) = 0 + F(a, z).
\]

(4) For any \( a \in L_p(X) \) and \( z \in X \), we have

\[
F(a, z) = F(a \ast 0, z) = (a \ast F(0, z)) \land (0 \ast F(a, z)) = (0 \ast F(a, z)) \land ((0 \ast F(a, z)) \land (a \ast F(0, z))) = a \ast F(0, z) = a \ast (0 \ast F(0, z)) = a + F(0, z).
\]

This completes the proof. \( \square \)

Using Theorem 3, we have the following corollary.

**Corollary 4.** Let \( F(\cdot, \cdot) : X \times X \to X \) be a symmetric left bi-derivation and \( d : X \to X \) be the trace of \( F \). Then

\[
\begin{align*}
(1) & \quad (\forall a \in G(X)) \ (d(a) \in G(X)). \\
(2) & \quad (\forall a \in L_p(X)) \ (d(a) \in L_p(X)).
\end{align*}
\]

**Theorem 5.** Let \( F \) be a symmetric left bi-derivation of \( X \). Then

\[
\begin{align*}
(1) & \quad (\forall z \in X) \ (a, b \in L_p(X) \Rightarrow F(a + b, z) = a + F(b, z)). \\
(2) & \quad (\forall z \in X) \ (a \in L_p(X) \Rightarrow F(a, z) = a \text{ if and only if } F(0, z) = 0). \\
(3) & \quad (\forall x, y, z \in X) \ (F(x \ast y, z) \leq x \ast F(y, z)). \\
(4) & \quad (\forall x, y, z \in X) \ (x \ast F(x, z) = y \ast F(y, z)).
\end{align*}
\]

**Proof.** (1) Let \( a, b \in L_p(X) \). Then

\[
F(a + b, z) = (a \ast F(0, z)) \land ((0 \ast F(0, z)) \ast (a \ast F(0, z))) = a \ast F(0, z) \land ((0 \ast F(0, z)) \ast (a \ast F(0, z))) = a \ast F(0, z) = a + F(0, z).
\]

Thus, we can write \( F(0, z) = x \ast F(x, z) = y \ast F(y, z) \) for any \( y \in X \). This completes the proof. \( \square \)

**Definition 6.** A symmetric left bi-derivation \( F(\cdot, \cdot) : X \times X \to X \) of a BCI-algebra \( X \) is said to be componentwise regular if \( F(0, z) = 0 \) for all \( z \in X \). In particular, \( F \) is called \( d \)-regular if \( F(0, 0) = d(0) = 0 \).

**Theorem 7.** Let \( F \) be a symmetric left bi-derivation of BCI-algebra \( X \). Then \( X \) is a BCK-algebra if and only if \( F \) is componentwise regular symmetric left bi-derivation.
Proof. Suppose $X$ is a $BCK$-algebra. Then for any $x, z \in X$, we have

$$F(0, z) = F(0 \ast x, z)$$

$$= (0 \ast F(x, z)) \land (x \ast F(0, z))$$

$$= 0 \land (x \ast F(0, z)) = 0.$$  \hfill (11)

Hence $F$ is componentwise regular.

Conversely, let $F$ be a componentwise regular symmetric left bi-derivation. Let for any $a \in L_p(X)$ be such that $a \neq 0$. Then

$$F(a \ast 0, 0) = F(a, 0) = 0.$$  \hfill (12)

But it is clear that

$$a \ast F(0, 0) \land 0 \ast F(a, 0) = a \land 0 = 0 \ast (0 \ast a)$$

$$= a \neq 0,$$  \hfill (13)

which is not possible as $F$ is a componentwise regular symmetric left bi-derivation. Thus $0$ is the unique $p$-atom. Assume that for some $m \in X$, we have $0 \ast m \neq 0$, then $a_{0 \ast m} = 0 \ast (0 \ast (0 \ast m)) = 0$, so $0 \ast m \in L_p(X)$, which is a contradiction. Henceforth, for all $m \in X$, we have $0 \ast m = 0$ implying thereby, $X$ is a $BCK$-algebra.

This completes the proof. \hfill \square

Theorem 8. Let $F$ be a componentwise regular symmetric left bi-derivation of a $BCI$-algebra $X$. Then

(1) Both $x$ and $F(x, z)$ belong to the same branch for all $x, z \in X$.

(2) $(\forall x, y, z \in X) (F(x, z) \leq x)$.

(3) $(\forall x, y, z \in X) (F(x, z) \ast y \leq x \ast F(y, z))$.

Proof. (1) For any $x, z \in X$, we get

$$0 = F(0, z) = F(a_x \ast x, z)$$

$$= (a_x \ast F(x, z)) \land (x \ast F(a_x, z))$$

$$= (x \ast F(a_x, z)) \ast ((x \ast F(a_x, z)) \ast (a_x \ast F(x, z)))$$

$$= a_x \ast F(x, z),$$  \hfill (14)

since $a_x \ast F(x, z) \in L_p(X)$. Hence $a_x \leq F(x, z)$, and so $F(x, z) \in V(a_x)$. Obviously, $x \in V(a_x)$.

(2) Since $F$ is componentwise regular, $F(0, z) = 0$. Then

$$F(x, z) = F(x \ast 0, z)$$

$$= (x \ast F(0, z)) \land (0 \ast F(x, z))$$

$$= (x \ast 0) \land (0 \ast F(x, z))$$

$$= (0 \ast F(x, z)) \ast ((0 \ast F(x, z)) \ast x)$$

$$\leq x.$$  \hfill (15)

(3) Since $F(x, z) \leq x$ for all $x, z \in X$ by (2), using (a3) we obtain

$$F(x, z) \ast y \leq x \ast y \leq x \ast F(y, z).$$  \hfill (16)

This completes the proof. \hfill \square

Next, we prove some results in a $p$-semisimple $BCI$-algebra.

Theorem 9. Let $F$ be a symmetric left bi-derivation of a $p$-semisimple $BCI$-algebra $X$; one has the following assertions.

(1) $(\forall x, y, z \in X) (F(x \ast y, z) = x \ast F(y, z))$.

(2) $(\forall x, y, z \in X) (F(x, z) \ast y = F(y, z) \ast y)$.

(3) $(\forall x, y, z \in X) (F(x, z) \ast y = y \ast F(y, z))$.

Proof. (1) Let $X$ be a $p$-semisimple $BCI$-algebra. Then for any $x, y, z \in X$, we have

$$F(x \ast y, z) = (x \ast F(y, z)) \land (y \ast F(x, z)) = x \ast F(y, z).$$  \hfill (17)

(2) Let $x, y, z \in X$. Using (I), we have

$$(x \ast y) \ast (x \ast F(y, z)) \leq F(y, z) \ast y,$$

$$(y \ast x) \ast (y \ast F(x, z)) \leq F(x, z) \ast x.$$  \hfill (18)

These above inequalities can be rewritten as

$$((x \ast y) \ast (x \ast F(y, z))) \ast (F(y, z) \ast y) = 0,$$

$$((y \ast x) \ast (y \ast F(x, z))) \ast (F(x, z) \ast x) = 0.$$  \hfill (19)

Consequently, we get

$$((x \ast y) \ast (x \ast F(y, z))) \ast (F(y, z) \ast y)$$

$$= ((y \ast x) \ast (y \ast F(x, z))) \ast (F(x, z) \ast x)$$  \hfill (20)

Also, using Theorem 5(4) and (I), we obtain

$$(x \ast y) \ast F(x \ast y, z) = (y \ast x) \ast F(y \ast x, z)$$

$$\implies (x \ast y) \ast (x \ast F(y, z)) = (y \ast x) \ast (y \ast F(x, z)).$$  \hfill (21)

Since $X$ is a $p$-semisimple $BCI$-algebra, hence, by using (21) and (a12), the above (20) yields $F(x, z) \ast x = F(y, z) \ast y$.

(3) We have $F(0, z) = x \ast F(x, z)$ by Theorem 5(4). Further, on letting $x = 0$, we get that $F(0, z) \ast 0 = F(y, z) \ast y$ implies $F(0, z) = F(y, z) \ast y$. Henceforth $F(y, z) \ast y = x \ast F(x, z)$, which amounts to say that $F(x, z) \ast x = y \ast F(y, z)$.

This completes the proof. \hfill \square

Theorem 10. Let $X$ be a $p$-semisimple $BCI$-algebra. Then $F$ is a symmetric left bi-derivation if and only if it is a symmetric bi-derivation on $X$. 

\hfill \square
Proof. Suppose that $F$ is a symmetric left bi-derivation on $X$. First, we show that $F$ is a $(r,l)$-symmetric bi-derivation on $X$. Let $x, y, z \in X$. Using Theorem 9(1) and (a14), we have

\[ F(x \ast y, z) = x \ast F(y, z) = (F(x, z) \ast y) \ast ((F(x, z) \ast y) \ast (x \ast F(y, z))) = (x \ast F(y, z)) \land (F(x, z) \ast y). \]

Hence $F$ is a $(r,l)$-symmetric bi-derivation on $X$.

Again, we show that $F$ is a $(l,r)$-symmetric bi-derivation on $X$. Let $x, y, z \in X$. Using Theorem 9(1), (3) and (a15), we have

\[ F(x \ast y, z) = (x \ast F(y, z)) \land (F(x, z) \ast y) \ast ((x \ast F(y, z)) \ast (F(x, z) \ast y)) = (x \ast F(y, z)) \land (x \ast F(y, z)) \land (x \ast F(y, z)). \]

Conversely, suppose that $F$ is a symmetric bi-derivation of $X$. As $F$ is a $(r,l)$-symmetric bi-derivation on $X$, then for any $x, y, z \in X$ and using (a14), we have

\[ F(x \ast y, z) = (x \ast F(y, z)) \land (F(x, z) \ast y) = (F(x, z) \ast y) \ast ((F(x, z) \ast y) \ast (x \ast F(y, z))) = x \ast F(y, z) = (y \ast F(x, z)) \ast ((y \ast F(x, z)) \ast (x \ast F(y, z))) = (x \ast F(y, z)) \land (y \ast F(x, z)). \]

Hence $F$ is a symmetric left bi-derivation. This completes the proof.

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