Research Article

Multiresolution Expansion and Approximation Order of Generalized Tempered Distributions

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Received 22 October 2012; Accepted 25 November 2012

Academic Editor: Palle E. Jorgensen

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Let \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) be the generalized tempered distributions of \( e^{M(x)} \)-growth with restricted order \( r \in \mathbb{N}_0 \), where the function \( M(x) \) grows faster than any linear functions as \( |x| \to \infty \). We show the convergence of multiresolution expansions of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) in the test function space \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \). In addition, we show that the kernel of an integral operator \( K : \mathcal{H}^{r}_{M}(\mathbb{R}) \to \mathcal{H}^{r}_{M}(\mathbb{R}) \) provides approximation order in \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) in the context of shift-invariant spaces.

1. Introduction

Multiresolution analysis was shown to be very useful in extending the expansions in orthogonal wavelets from \( L^2(\mathbb{R}) \) to a certain class of tempered distributions. Some interactions between wavelets and tempered distributions have been presented by Walter’s work in [1–3]. Walter has found the analytic representation of tempered distributions of polynomial growth with restricted order, \( \mathcal{S}_r(\mathbb{R}) \), by wavelets [1] and the multiresolution expansions’ pointwise convergence of \( \mathcal{S}_r(\mathbb{R}) \) [3]. Pilipović and Teofanov have shown the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{S}_r(\mathbb{R}) \) and the convergence of multiresolution expansions of \( \mathcal{S}_r(\mathbb{R}) \) in the test function space \( \mathcal{S}_r(\mathbb{R}) \) of \( \mathcal{S}_r(\mathbb{R}) \). As an application, Pilipović and Teofanov have shown that the kernel of an integral operator \( K : \mathcal{S}_r(\mathbb{R}) \to \mathcal{S}_r(\mathbb{R}) \) provides approximation order in \( \mathcal{S}_r(\mathbb{R}) \) in the context of shift-invariant spaces [4]

In the meantime, the tempered distributions of polynomial growth were extended to tempered distributions of \( e^{x^r} \)-growth, \( \mathcal{H}^{r}_{1}(\mathbb{R}) \), in [5, 6] and \( e^{x^r} \)-growth, \( \mathcal{H}^{r}_{p}(\mathbb{R}) \), in [7, 8] or \( e^{M(x)} \)-growth, \( \mathcal{H}^{r}_{M}(\mathbb{R}) \), in [9, 10], where the function \( M(x) \) grows faster than any linear functions as \( |x| \to \infty \). We have considered the analytic representation of tempered distributions of \( e^{M(x)} \)-growth with restricted order, \( \mathcal{H}^{r}_{M}(\mathbb{R}) \), by wavelets [11]. Also, we have shown that the multiresolution expansions of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) converges pointwise to the value of the distribution where it exists [12].

In this paper, we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) and convergence of multiresolution expansions of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) in the test function space \( \mathcal{H}^{r}_{M}(\mathbb{R}) \) of \( \mathcal{H}^{r}_{M}(\mathbb{R}) \). In addition, we will show that the kernel of an integral operator \( K : \mathcal{H}^{r}_{M}(\mathbb{R}) \to \mathcal{H}^{r}_{M}(\mathbb{R}) \) provides approximation order in \( \mathcal{H}^{r}_{M}(\mathbb{R}) \). This is an extension of the works of Pilipović and Teofanov [4] in the context of generalized tempered distributions, \( \mathcal{H}^{r}_{M}(\mathbb{R}) \).

2. The Generalized Tempered Distribution Spaces \( \mathcal{H}^{r}_{M}(\mathbb{R}) \)

Throughout this paper, we will use \( C \) or \( C_i \) to denote the positive constants, which are independent parameters and may be different at each occurrence.

Let \( \mu(\xi) (0 \leq \xi \leq \infty) \) denote a continuous increasing function such that \( \mu(0) = 0 \) and \( \mu(\infty) = \infty \). For \( x \geq 0 \), we define

\[
M(x) = \int_{0}^{x} \mu(\xi) d\xi.
\] (1)

The function \( M(x) \) is an increasing, convex, and continuous function with \( M(0) = 0 \), \( M(\infty) = \infty \) and satisfies the
fundamental convexity inequality $M(x_1) + M(x_2) \leq M(x_1 + x_2)$. Further, we define $M(x)$ for negative $x$ by means of the equality $M(x) = M(-x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in $\mathbb{R}$, the function $M(x)$ will grow faster than any linear function as $|x| \to \infty$. Now we list some properties of $M(x)$ which will be frequently used later. Consider the following:

\[
M(x) + M(y) \leq M(x + y) \quad \forall x, y \geq 0,
\]

\[
M(x + y) \leq M(2x) + M(2y) \quad \forall x, y \geq 0.
\]

(2)

Using the function $M(x)$, we define the space $\mathcal{H}_M(\mathbb{R})$ as the space of all functions $\varphi \in C^\infty(\mathbb{R})$ such that

\[
\left\| \varphi \right\|_{\mathcal{H}_M} = \sup_{x \in \mathbb{R}, \alpha \in \mathbb{N}} e^{M(x)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| < \infty, \quad k = 1, 2, \ldots
\]

(3)

The topology in $\mathcal{H}_M(\mathbb{R})$ is defined by the family of the seminorms $\left\| \cdot \right\|_{\mathcal{H}_M}$. Then $\mathcal{H}_M(\mathbb{R})$ become a Fréchet space and $\mathcal{D}(\mathbb{R}) \hookrightarrow \mathcal{H}_M(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R})$ are continuous and dense inclusions; here $\mathcal{D}(\mathbb{R})$ denotes the spaces of all $C^\infty(\mathbb{R})$ functions with compact supports, $\mathcal{S}(\mathbb{R})$ the spaces of polynomially decreasing functions (Schwartz functions), and $\mathcal{S}'(\mathbb{R})$ the space of all $C^\infty(\mathbb{R})$ functions. By $\mathcal{H}_M'(\mathbb{R})$, we mean the space of continuous linear functionals on $\mathcal{H}_M(\mathbb{R})$.

Definition 1. We say that the elements of $\mathcal{H}_M'(\mathbb{R})$ are generalized tempered distributions.

Clearly, when $M(x) = \log(1 + |x|)$, $\mathcal{H}_M'(\mathbb{R})$ are tempered distributions (Schwartz distributions), $\mathcal{S}(\mathbb{R})$. When $M(x) = |x|$, $\mathcal{H}_M'(\mathbb{R})$ are tempered distributions, are introduced and characterized by Yoshinaga [6] and Hasumi [5], independently. When $M(x) = |x|^p$, $p > 1$, $\mathcal{H}_M'(\mathbb{R})$ are tempered distributions, which are introduced and characterized by Szajdler and Zielensky [7, 8]. For details about $\mathcal{H}_M'(\mathbb{R})$, refer to [9, 10].

For a natural number $r$, we define by $\mathcal{H}_M'(\mathbb{R})$ the space of all $\varphi \in C^\infty(\mathbb{R})$ such that

\[
\left\| \varphi \right\|_{\mathcal{H}_M^r} = \sup_{x \in \mathbb{R}, \alpha \in \mathbb{N}^r} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| < \infty,
\]

\[
\lim_{|x| \to \infty} \sup_{\alpha \in \mathbb{N}^r} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| = 0.
\]

(4)

The topology of $\mathcal{H}_M^r(\mathbb{R})$ is defined by the family of $\left\| \cdot \right\|_{\mathcal{H}_M^r}$ and the dual of $\mathcal{H}_M^r(\mathbb{R})$ is denoted by $\mathcal{H}_M^{r'}(\mathbb{R})$. Clearly, $\mathcal{H}_M'(\mathbb{R})$ is the projective limit of $\mathcal{H}_M^r(\mathbb{R})$ when $r \to \infty$ and $\mathcal{H}_M'(\mathbb{R}) = \bigcup_{r \in \mathbb{N}} \mathcal{H}_M^r(\mathbb{R})$. Also, we have continuous and dense inclusion mapping as following:

\[
\mathcal{H}_M(\mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathcal{H}_M^{r+1}(\mathbb{R}) \hookrightarrow \mathcal{H}_M'(\mathbb{R}) \hookrightarrow \cdots
\]

(5)

Definition 2. We say that the elements of $\mathcal{H}_M'(\mathbb{R})$ are generalized tempered distributions of order $r$.

We define by $\mathcal{H}_M^r(\mathbb{R})$ the space of all $\psi \in C^\infty(\mathbb{R})$ such that

\[
\left\| \psi \right\|_{\mathcal{H}_M^r} = \sup_{x \in \mathbb{R}, \alpha \in \mathbb{N}^r} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \psi(x) \right| < \infty, \quad l = 1, 2, \ldots
\]

(6)

The topology of $\mathcal{H}_M^r(\mathbb{R})$ is defined by the family of $\left\| \cdot \right\|_{\mathcal{H}_M^r}$ and the dual of $\mathcal{H}_M^r(\mathbb{R})$ is denoted by $\mathcal{H}_M^{r'}(\mathbb{R})$. Obviously, $\mathcal{H}_M^r(\mathbb{R}) \subset \mathcal{H}_M'(\mathbb{R})$.

Now, we give a theorem that will be used later.

**Theorem 3.** Let $\phi$ and sequence $\{\phi_n\}_{n \in \mathbb{N}}$ be given in $\mathcal{H}_M^{r+1}(\mathbb{R})$ such that $\{(d^\alpha/dx^\alpha)\phi_n\}_{n \in \mathbb{N}}$ converges uniformly to $(d^\alpha/dx^\alpha)\phi$ on every compact set $K \subset \mathbb{R}$ and for $\alpha = 0, 1, \ldots, r$. If $\{\phi_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_M^{r+1}(\mathbb{R})$, then the sequence $\{\phi_n\}_{n \in \mathbb{N}}$ converges to $\phi \in \mathcal{H}_M^{r+1}(\mathbb{R})$ in $\mathcal{H}_M'(\mathbb{R})$.

**Proof.** Let $\varepsilon > 0$ be given and let $\alpha \in \{0, 1, \ldots, r\}$. Then there exist $N$ such that

\[
\sup_{x \in K} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} (\phi_n - \phi)(x) \right| < \varepsilon, \quad n \geq N,
\]

(7)

for arbitrary $K \subset \mathbb{R}$. Also, since the sequence $\{\phi_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{H}_M^{r+1}(\mathbb{R})$, we can take a positive number $A > 0$ and a compact set $K$ such that $|x| > A$ when $x \notin K$ and

\[
\sup_{x \in K} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} (\phi_n - \phi)(x) \right| < Ce^{-M(A)} < \varepsilon.
\]

From (7) and (8), we have

\[
\lim_{n \to \infty} \sup_{x \in K} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} (\phi_n - \phi)(x) \right| = 0, \quad 0 \leq \alpha \leq r.
\]

(9)

\[\square\]

**3. Multiresolution Expansion of $\mathcal{H}_M'(\mathbb{R})$**

Definition 4. A multiresolution analysis (shortly MRA) consists of a sequence of closed subspaces $V_n, n \in \mathbb{Z}$, of $L^2(\mathbb{R})$ satisfying the following:

(i) $\{\psi(t - n)\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $V_0$,

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})$,

(iii) $f \in V_n \Longleftrightarrow f(2^j) \in V_{n-j}$,

(iv) $\bigcap_{n \in \mathbb{Z}} V_n = \{0\}$, $\bigcup_{n \in \mathbb{Z}} V_n = L^2(\mathbb{R})$.

The function $\psi$ whose existence is asserted in (i) is called a scaling function of the given MRA.
Definition 5. We say that a multiresolution analysis $V_n$, $n \in \mathbb{Z}$, is $(M, r)$-regular MRA of $L^2(\mathbb{R})$ if the scaling function $\psi$ is in $\mathcal{F}^r_M(\mathbb{R})$.

Example 6. It is impossible that the scaling function $\psi$ has exponential decay and $\psi \in C_0^\infty(\mathbb{R})$, with all derivatives bounded, unless $\psi = 0$. Refer to [13, Corollary 5.5.3]. So we will restrict our attention to $\mathcal{F}^r_M(\mathbb{R})$ or $\mathcal{F}^r_M(\mathbb{R})$. From the remark in [13] or, page 152 [2, Example 4, page 48], Battle-Lemarié's wavelets are in $\mathcal{F}^r_M(\mathbb{R})$ for some $r \in \mathbb{N}$ when $M(x) = |x|$, but not in $\mathcal{F}^r_M(\mathbb{R})$ even if they have exponential decay and smoothness. In [13], Daubechies shown that for an arbitrary nonnegative integer $r$, there exists an $(M, r)$-regular MRA of $L^2(\mathbb{R})$ such that the scaling function $\psi$ has compact supports.

Let $V_j$ be an $(M, r)$-regular MRA of $L^2(\mathbb{R})$ and let $\psi$ be a scaling function. The reproducing kernel of $V_0$ is given by

$$q_0(x, y) = \sum_{n \in \mathbb{Z}} \psi(x-n)\psi(y-n). \quad (10)$$

The series and its derivatives with respect to $x$ or $y$ of order $r \leq r$ converge uniformly on $\mathbb{R}$ because of the regularity of $\psi \in \mathcal{F}^r_M(\mathbb{R})$. The reproducing kernel of the projection operator onto $V_j$ is

$$q_j(x, y) = 2^j q_0(2^j x, 2^j y), \quad x, y \in \mathbb{R}, \quad (11)$$

and the projection of $f \in L^2(\mathbb{R})$ onto $V_j$ is given by

$$q_{j,f}(x) = \left\langle f(y), q_j(x, y) \right\rangle = \int f(y) q_j(x, y) dy, \quad x \in \mathbb{R}. \quad (12)$$

The sequence $\{q_j\}_{j \in \mathbb{Z}}$ given in (12), is called the multiresolution expansion of $f \in L^2(\mathbb{R})$.

Definition 7. For a given $f \in \mathcal{F}^r_M(\mathbb{R})$, the sequence $\{q_j\}_{j \in \mathbb{Z}}$ defined by

$$\left\langle q_j, f, \phi \right\rangle = \left\langle f, q_j, \phi \right\rangle, \quad \phi \in \mathcal{F}^r_M(\mathbb{R}) \quad (13)$$

is called the multiresolution expansion of $f \in \mathcal{F}^r_M(\mathbb{R})$.

We deduce the following properties of the reproducing kernel $q_0$ with scaling function $\psi \in \mathcal{F}^r_M(\mathbb{R})$:

(a) $q_0(x, y) = q_0(y, x)$ and $q_0(x+k, y+k) = q_0(x, y)$ for all $k \in \mathbb{Z}$.

(b) For every $l \in \mathbb{N}$ and $0 \leq \alpha, \beta \leq r$, there exist $C_l > 0$ such that

$$\left| \frac{\partial^\alpha \partial^\beta}{\partial x^\alpha \partial y^\beta} q_0(x, y) \right| \leq \sum_j \left| \frac{\partial^\alpha}{\partial x^\alpha} \psi(x-j) \right| \left| \frac{\partial^\beta}{\partial y^\beta} \psi(y-j) \right| \leq C_l e^{-M((2l+1)(x-j))} e^{-M((2l+1)(y-j))} \leq C_l e^{-M(2l(x-j))} e^{-M(x-j)} \times e^{-M(2l(y-j))} e^{-M(y-j)} \leq C_l e^{-M(l(x-y))} \sum_j C_l e^{-M(x-j)} e^{-M(y-j)} \leq C_l e^{-M(l(x-y))}, \quad (14)$$

where we used the properties (2).

(c) $\int_0^\infty q_0(x, y) y^\alpha dy = x^\alpha, \quad y \in \mathbb{R}, \quad 0 \leq \alpha \leq r$.

Let $V_j$ be an $(M, r)$-regular MRA of $L^2(\mathbb{R})$. We fix a function $g \in \mathcal{D}(\mathbb{R})$ with $\int g(x) dx = 1$. We let $g_j$ denote the function $2^j g(2^j x)$ and let $G_j$ denote the operation of convolution by $g_j$. For each fixed $x$, we consider the function $\partial_x^\alpha q_0(x, y)$ of the variable $y$. From (c), we have

$$\int \partial_x^\alpha q_0(x, y) y^\beta dy = 0, \quad (15)$$

for $0 \leq \beta < \alpha$, whereas

$$\int \partial_x^\alpha q_0(x, y) y^\alpha dy = \alpha!. \quad (16)$$

Now, it follows from integration by parts that the kernel $g(x-y)$ of the operator $G$ shares these properties (15) and (16) with $q_0(x, y)$.

Let

$$R^\alpha(x, y) = \partial_x^\alpha q_0(x, y) - \partial_x^\alpha g(x-y). \quad (17)$$

From (b) and the fact that $g \in \mathcal{D}(\mathbb{R}) \subset \mathcal{F}^r_M(\mathbb{R})$, we have

$$|R^\alpha(x, y)| \leq q_0 e^{-M(k(x-y))}, \quad x, y \in \mathbb{R}, \quad k \in \mathbb{N}, \quad (18)$$

and these functions also satisfy

$$\int R^\alpha(x, y) dy = 0 \quad (19)$$

identically in $x$ for every $\alpha = 1, 2, \ldots, r$. They, for every $j \in \mathbb{Z}$ and $f \in \mathcal{C}^r(\mathbb{R})$ with at most $e^{M(x)}$-growth, define operator $R^\alpha_j$ by

$$R^\alpha_j f(x) = 2^j \int R^\alpha(2^j x, 2^j y) f(y) dy \quad (20)$$
which are such that
\[ q_j \frac{d^\alpha}{dx^\alpha} f(x) = G_j \frac{d^\alpha}{dy^\alpha} f(y) + R^\alpha_j \frac{d^\alpha}{dy^\alpha} f(y), \tag{21} \]
that is,
\[ \int q_j (x, y) \frac{d^\alpha}{dy^\alpha} f(y) \, dy \]
\[ = 2^j \int g(2^j (x - y)) \frac{d^\alpha}{dy^\alpha} f(y) \, dy \tag{22} \]
\[ + 2^j \int R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) \, dy. \]
From Theorem 1.1 in [14], we have
\[ \lim_{j \to \infty} G_j \frac{d^\alpha}{dy^\alpha} f(y) \, dy = \frac{d^\alpha}{dx^\alpha} f(x), \quad x \in \mathbb{R}, \quad \alpha \geq 0, \tag{23} \]
uniformly on compact sets. Now we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of $\mathcal{R}^\alpha_M(R)$.

**Theorem 8.** Let $f \in C^\alpha(R)$ such that the corresponding derivatives $(d^\alpha/dx^\alpha)f$ are bounded by an $e^{M(k,x)}$ when $|x| \to \infty$, for every $\alpha = 0, 1, \ldots$, and some $k_0 \in \mathbb{N}$. If $q_j f$, given by (12), be the projection of $f$ onto an $(M, r)$-regular MRA of $L^2(R)$, then the sequence $\{d^\alpha/dx^\alpha q_j f\}_{j \in \mathbb{Z}}$ converges uniformly on compact sets to $(d^\alpha/dx^\alpha)f$ as $j \to \infty$, for every $\alpha = 0, 1, \ldots, r$.

**Proof.** If $|y - x| \leq c$, we have
\[ \left| \frac{d^\alpha}{dx^\alpha} f(x) - \frac{d^\alpha}{dy^\alpha} f(y) \right| \leq e_M^\alpha (y - x), \tag{24} \]
where $e_M^\alpha(x)$ is a continuous function with $e^{M(k,x)}$ growth and $e_M^\alpha(0) = 0$. From (18), given a compact set $K$, we have
\[ 2^j \int R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) \, dy \]
\[ \leq 2^j \int R^\alpha (2^j x, 2^j y) \left( \frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dy^\alpha} f(y) \right) \, dy \leq 2^j \int q e^{-M(2^j(x-y))} e_M^\alpha (y - x) \, dy, \tag{25} \]
for large enough $j$ and $x \in K$. Since $k$ can be chosen arbitrary, we obtain by dominated convergence theorem,
\[ \lim_{j \to \infty} 2^j \int R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) \, dy \]
\[ \leq \lim_{j \to \infty} \int q e^{-M(2^j(x-y))} e_M^\alpha (y - x) \, dy = 0 \tag{26} \]
uniformly for $x \in K$. From (21) and (23), we have the conclusion.

We now ready to show the main theorem.

**Theorem 9.** Let $\phi \in \mathcal{R}^\alpha_M(R)$ and let $q_j \phi(x)$, given by (7), be a projection of $\phi$ onto an $(M, r)$-regular MRA of $L^2(R)$. If $\phi \in \mathcal{R}^\alpha_M(R)$, then the sequence $\{q_j \phi(x)\}$ converges to $\phi(x)$ in $\mathcal{R}^\alpha_M(R)$ as $j \to \infty$.

**Proof.** Let $g$ and $R^\alpha$ be given in (21) such that $g \in D(R)$ and $\int g(x) \, dx = 1$. From Theorems 3 and 8 and (21), it suffices to show that
\[ \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int q_0 \left( \frac{x}{h}, \frac{y}{h} \right) \phi(y) \, dy \right| \]
\[ = \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int g \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) \, dy \right| \]
\[ + \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) \, dy \tag{27} \]
is bounded for every $\alpha \in \{0, 1, \ldots, r\}$ and $h > 0$. Since $g$ has a compact support, then
\[ \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int g \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) \, dy \right| \]
\[ \leq \sup_{x \in \mathbb{R}} \frac{1}{h} \left| \int g \left( \frac{x}{h}, \frac{y}{h} \right) e^{M((r+1)\alpha)} x^{-M(2(r+1)\alpha)} \, dy \right| \tag{28} \]
\[ \leq \sup_{x \in \mathbb{R}} \frac{1}{h} \left| \int g \left( \frac{x}{h}, \frac{y}{h} \right) e^{M(2(r+1)(x-y))} \, dy \right| \leq C. \]
Hence we have only to show that
\[ K = \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) \, dy \right| \leq C, \quad x \in \mathbb{R}, \tag{29} \]
for every $\alpha \in \{0, 1, \ldots, r\}$ and $h > 0$. Let $S_1 = \{y : |x - y| \leq 1\}, S_2 = \{y : |x - y| > 1 \text{ and } (1/2)|x| \leq |y|\}$ and $S_3 = \{y : |x - y| > 1 \text{ and } (1/2)|x| > |y|\}$. Then, by (18), we have
\[ I = \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) \, dy \right| \]
\[ \leq q_1 \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left| \int e^{-M((x-y)/h)} e^{-M(2(r+1)y)} \, dy \right| \]
\[ = q_1 \sup_{x \in \mathbb{R}} e^{M((r+1)\alpha)} \frac{1}{h} \left( \int_{S_1} + \int_{S_2} + \int_{S_3} \right) e^{-M((x-y)/h)} e^{-M(2(r+1)y)} \, dy \tag{30} \]
\[ = q_1 \left( I_1 + I_2 + I_3 \right). \]
By a simple change of variable, we have
\[
I_1 = \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \int_S e^{-M((x-y)/h)} e^{-M((r+1)y)} dy \leq \sup_{x \in \mathbb{R}} \frac{1}{h} \int_S e^{-M((x-y)/h)} e^{-M((r+1)y)} dy \leq 2e^{M(2(r+1))} x \int_0^{1/h} e^{-M(u)} du \leq C_1.
\]

Since \((1/2)|x| < |y|\) and \((1/2)|y| \leq |x - y|\) on \(S_2\), then
\[
I_2 = \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \int_S e^{-M((x-y)/h)} e^{-M((r+1)y)} dy \leq \sup_{x \in \mathbb{R}} \frac{1}{h} \int_S e^{-M((x-y)/h)} e^{-M((r+1)x)} dy \leq \frac{1}{h} \int_S e^{-M((l/2h)y)} dy \leq C_2,
\]
for sufficiently large \(l\). Since \((1/2)|x| > |y|\) on \(S_3\), then
\[
I_3 = \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \frac{1}{h} \int_S e^{-M((x-y)/h)} e^{-M((r+1)y)} dy \leq \sup_{x \in \mathbb{R}} \frac{1}{h} \int_S e^{-M((l/2h)y)} dy \int_S e^{-M((r+1)y)} dy \leq C_3,
\]
for sufficiently large \(l\).

4. Approximation Order of \(\mathcal{H}_M^r(\mathbb{R})\)

A space of functions \(S\) is called shift invariant if it is invariant under all integer translates, that is,
\[
f \in S \iff f (\cdot + k) \in S \quad \forall k \in \mathbb{Z}.
\]

The principal shift-invariant subspaces \(S = S(\phi)\) are generated by the closure of the linear span of the shifts of \(\phi\). The stationary ladder of spaces \(\{S^h(\phi) : h > 0\}\) is given by
\[
S^h(\phi) = \left\{ f \left( \frac{\cdot}{h} \right) : f \in S \right\}.
\]

To rate the efficiency for approximation of such spaces, the concept of approximation order is widely used. We say that the scale of the space \(S^h(\phi)\) provides approximation order \(k\) in \(F\) if for every sufficiently smooth \(f\),
\[
\inf_{g \in S^h(\phi)} \|f - g\|_F \leq Ch^k, \quad h > 0,
\]
where \(C = C(f) > 0\). For further details about the theory on the approximation order provided by shift-invariant spaces, we refer to [15, 16]. We will focus our attention to the so-called approximation order of an integral operator.

Let \(K\) be an integral operator of the following form
\[
(Kf)(x) = \int K(x, y) f(y) dy, \quad x \in \mathbb{R}.
\]

We assume that \(K(x - k, y) = K(x, y + k)\), \(h \in \mathbb{Z}, x, y \in \mathbb{R}\). For \(h > 0\), we define
\[
K_h = \varphi_h K \varphi_{1/h},
\]
where \(\varphi\) is the scaling operator \(\varphi_h f = f(\cdot/h)\). We say that the integral operator \(K\) defined by (37) provides approximation order \(k\) in \(F\) if for every sufficiently smooth \(f\),
\[
\|K_h f - f\|_F \leq Ch^k, \quad h > 0,
\]
where \(C = C(f) > 0\). For further details about the theory on the approximation order provided by integral or kernel operator, we refer to [17, 18].

Definition 10 (see [4]). Let \(f \in \mathcal{H}_M^r(\mathbb{R})\). Let \(K(x, y)\), \(x, y \in \mathbb{R}\), be the kernel of an integral operator \(K : \mathcal{H}_M^r(\mathbb{R}) \to \mathcal{H}_M^r(\mathbb{R})\). \(K\) is given by \((Kf, \phi) = (f, K\phi)\). We say that the operator \(K\) provides approximation order \(k\) in \(\mathcal{H}_M^r(\mathbb{R})\) if
\[
\|K_h f - f\|_{\mathcal{H}_M^{r+k}} = \sup_{\|\phi\|_{\mathcal{H}_M^{r+k}}} \|\langle K_h f, \phi \rangle - \langle f, \phi \rangle\| \leq Ch^k,
\]
for \(h > 0\),

where the constant \(C = C(f) > 0\).

We will now show that the kernel of an integral operator \(K : \mathcal{H}_M^{r+k}(\mathbb{R}) \to \mathcal{H}_M^{r+k}(\mathbb{R})\) provides approximation order in \(\mathcal{H}_M^{r+k}(\mathbb{R})\).

Theorem 11. Let \(\phi \in \mathcal{H}_M^{r+k}(\mathbb{R})\) with compact support such that the integer shifts of \(\phi\) form an orthogonal basis of \(S(\phi)\) with respect to the inner product in \(L^2(\mathbb{R})\). Assume that \(\phi(x) = \sum_{n \in \mathbb{N}} c_n \phi(2x - k)\) for some sequence \(\{c_n\}_{n \in \mathbb{N}}\). Let
\[
K(x, y) = \sum_{l \in \mathbb{Z}} \phi(x - i) \varphi(y - i), \quad x, y \in \mathbb{R}
\]
be the kernel of the integral operator given by (37). Then \(K\) provides approximation order \(k\) in \(\mathcal{H}_M^{r+k}(\mathbb{R})\).

Proof. Firstly, we will show that
\[
J = \|K_h \phi - \phi\|_{\mathcal{H}_M^r} \leq C \|\phi\|_{\mathcal{H}_M^{r+k}} h^k,
\]
for \(h > 0\),

where \(C = C(f) > 0\). For further details about the theory on the approximation order provided by shift-invariant spaces,
where \( \phi \in \mathcal{H}_M^{r+k}, \) \( k \in \mathbb{N}. \) If we accept the result (42) for a moment, it follows that for \( f \in \mathcal{H}_M^r(\mathbb{R}) \subset \mathcal{H}_M^{r+k}(\mathbb{R}), \) we have

\[
|\langle K_h f - f, \phi \rangle| = |\langle f, K_h \phi - \phi \rangle| \\
\leq \|f\|_{\mathcal{H}_M^r} \|K_h \phi - \phi\|_{\mathcal{H}_M^{r+k}} \leq C h^k \|\phi\|_{\mathcal{H}_M^{r+k}}. \tag{43}
\]

hence

\[
\|K_h f - f\|_{\mathcal{H}_M^{r+k}} = \sup_{\|\phi\|_{\mathcal{H}_M^{r+k}}} |\langle K_h f, \phi \rangle - \langle f, \phi \rangle| \leq C h^k, \tag{44}
\]

which implies the conclusion.

Since \( \{S_j^2(\phi) : j \in \mathbb{Z}\} \) satisfy the conditions of \((M, r)\) regular MRA of \( L^2(\mathbb{R}) \) with \( S(\phi) = V_0, \) we can apply (21) to the operator \( K, \) that is,

\[
\frac{d^a}{dx^a} K_h f(x) = \frac{1}{h} \int g \left( \frac{x - y}{h} \right) \frac{d^a}{dy^a} f(y) dy \\
+ \frac{1}{h} \int R^a \left( \frac{x - y}{h} \right) \frac{d^a}{dy^a} f(y) dy, \tag{45}
\]

where \( g \) and \( R^a \) are given in (21). For \( 0 \leq \alpha \leq r, \)

\[
J = \sup_{x \in \mathbb{R}} e^{M(rx)} \left| \frac{d^a}{dx^a} \left( \langle K_h (x, y), \phi(y) \rangle - \phi(x) \right) \right| \\
\leq \sup_{x \in \mathbb{R}} e^{M(rx)} \left| \frac{1}{h} \int g \left( \frac{x - y}{h} \right) \right| \\
\times \left| \frac{d^a}{dy^a} \phi(y) - \frac{d^a}{dy^a} \phi(y) \right|_{y=x} dy \\
+ \sup_{x \in \mathbb{R}} e^{M(rx)} \left| \frac{1}{h} \int R^a \left( \frac{x - y}{h} \right) \right| \\
\times \left| \frac{d^a}{dy^a} \phi(y) - \frac{d^a}{dy^a} \phi(y) \right|_{y=x} dy \\
= J_1 + J_2. \tag{46}
\]

In order to estimate \( J_1, \) we consider \( g \in \mathcal{D}(\mathbb{R}) \) with \( \int g(x) dx = 1 \) and \( \int g(x) x^a dx = 0, 0 < |\alpha| < \max\{r, k-1\}. \)

Let \( c \) be a constant such that \( \sup g \subset [-c, c]. \) If we assume \( h \in (0, 1), \) the smoothness of \( \phi \in \mathcal{H}_M^{r+k}(\mathbb{R}) \subset C^{r+k}(\mathbb{R}) \) implies

\[
J_1 = \sup_{x \in \mathbb{R}} e^{M(rx)} \left| \int_{|x-y| \leq c} g(x - y) \right| \\
\times \left| \left( \frac{d^a}{dy^a} \phi(y) \right)_{y=hx} - \left( \frac{d^a}{dy^a} \phi(y) \right)_{y=hx} \right| dy \\
= \sup_{x \in \mathbb{R}} e^{M(rx)} \left| \int_{|x-y| \leq c} g(x - y) \right| \\
\times \left| \left( \left( y - x \right) \times \frac{d^{a+1}}{dy^{a+1}} \phi(y) \right)_{y=hx} + \cdots + \left( \left( y - x \right)^{k-1} \frac{d^{a+k-1}}{dy^{a+k-1}} \phi(y) \right)_{y=hx} \right. \\
\left. + \left( \frac{y-x}{k!} \right) \times \frac{d^{a+k}}{dy^{a+k}} \phi(y) \right|_{y=\xi(y)} dy \\
\leq \sup_{x \in \mathbb{R}} e^{M(rx)} C \frac{1}{k!} \sup_{x \in [\xi(hx), \xi(hx + h\xi)]} \left| \frac{d^{a+k}}{dy^{a+k}} \phi(y) \right| \\
= C \frac{1}{k!} h^k \sup_{x \in [\xi(hx), \xi(hx + h\xi)]} \left| \frac{d^{a+k}}{dy^{a+k}} \phi(t) \right| \\
= C \frac{1}{k!} h^k e^{-M(r\theta)} \sup_{x \in \mathbb{R}} \left| \frac{d^{a+k}}{dy^{a+k}} \phi(t) \right| \\
\leq \sup_{x \in \mathbb{R}} e^{M(rx)} e^{-M(r\theta)} \sup_{x \in \mathbb{R}} \left| \frac{d^{a+k}}{dy^{a+k}} \phi(t) \right| \\
= C_1 \|\phi\|_{\mathcal{H}_M^{r+k}}, \tag{47}
\]

where \( \xi(y) = hx + \theta h(y-x) \in [hx-hc, hx+hc] \) for some \( \theta \in (0, 1) \) and \( C_1 = C_1(k/\theta) h^k \sup_{x \in \mathbb{R}} e^{M(rx)} e^{-M(r\theta \alpha)} < \infty. \) To show the finiteness of \( C_1 \) in the last statement, we use

\[
\sup_{x \in \mathbb{R}} e^{M(rx)} e^{-M(r\theta \alpha)} \\
\leq \sup_{|x| \leq c} e^{M(r\theta \alpha)} \\
< \infty.
\]
\[ J_J = \sup_{|s| \leq c} e^{M(|s|)} e^{-M(|s| - \phi)} \]
\[ \leq \sup_{|s| \leq c} e^{M(|s|)} e^{-M(|s| - \phi)} + \sup_{|s| > c} e^{M(|s|)} e^{-M(|s| - \phi)} \]
\[ \leq e^{M(|s|)} + e^{-M(|s|)}. \] (48)

We will estimate \( J_J \) by using the following facts. Since \( \phi \) has a compact support, there exists \( M > 0 \) such that \( K(x, y) = 0 \) for \( |x - y| > M \). Also, by the choice of \( g \) and property (c) of the reproducing kernel \( q_0 \), we have

\[ \int R^a (x, y) y' dy = \frac{d^a}{dy^a} \int K(x, y) y' dy \]
\[ - \frac{d^a}{dy^a} \int g(x - y) y' dy \]
\[ = \frac{d^a}{dy^a} x^s - \frac{d^a}{dy^a} \int g(t) x^s dt \]
\[ = \frac{d^a}{dy^a} x^s - \frac{d^a}{dy^a} x^s = 0, \quad 0 \leq s \leq r + k - 1. \] (49)

Hence

\[ J_J = \sup_{h \in R} e^{M(r/k)} \left| \int R^a (x, y) \right. \]
\[ \times \left( \frac{d^a}{dy^a} \phi(y) \right)_{y = hy} - \frac{d^a}{dy^a} \phi(y)_{y = hy} dy \]
\[ = \sup_{h \in R} e^{M(r/k)} \left| \int R^a (x, y) \right. \]
\[ \times \left( (y - x) \times h \frac{d^{a+1}}{dy^{a+1}} \phi(y) \right)_{y = hy} \]
\[ + \cdots + \left( (y - x)^{k-1} \right)_{(k - 1)!} \times h^{k-1} \frac{d^a}{dy^{a-1}} \phi(y)_{y = hy} \]
\[ + \left( (y - x)^k \right)_{k!} \times h^{k} \frac{d^{a+k}}{dy^{a+k}} \phi(y)_{y = hy} dy \]
\[ = \frac{h^k}{k!} \sup_{h \in R} e^{M(r/k)} \left| \int_{|x - y| \leq M} R^a (x, y) (y - x)^k \right. \]
\[ \times \frac{d^{a+k}}{dy^{a+k}} \phi(y)_{y = hy} dy \]
\[ \leq \frac{M^k h^k}{k!} \sup_{h \in R, \xi \in \{y < hy, c \}} e^{M(r/k)} \left| \frac{d^{a+k}}{dy^{a+k}} \phi(y)_{y = hy} \right| \]
\[ \leq C_2 \phi \left\| \phi \right\|_{2^{a+k}}. \] (50)

References


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