Research Article

On Subspaces of an Almost $\varphi$-Lagrange Space

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We discuss the subspaces of an almost $\varphi$-Lagrange space (APL space in short). We obtain the induced nonlinear connection, coefficients of coupling, coefficients of induced tangent and induced normal connections, the Gauss-Weingarten formulae, and the Gauss-Codazzi equations for a subspace of an APL-space. Some consequences of the Gauss-Weingarten formulae have also been discussed.

1. Introduction

The credit for introducing the geometry of Lagrange spaces and their subspaces goes to the famous Romanian geometer Miron [1]. He developed the theory of subspaces of a Lagrange space together with Bejancu [2]. Miron and Anastasiei [3] and Sakaguchi [4] studied the subspaces of generalized Lagrange spaces (GL spaces in short). Antonelli and Hrimiuc [5, 6] introduced the concept of $\varphi$-Lagrangians and studied $\varphi$-Lagrange manifolds. Generalizing the notion of a $\varphi$-Lagrange manifold, the present authors recently studied the geometry of an almost $\varphi$-Lagrange space (APL space briefly) and obtained the fundamental entities related to such space [7]. This paper is devoted to the subspaces of an APL space.

Let $F^n = (M, F(x, y))$ be an $n$-dimensional Finsler space and $\varphi : \mathbb{R}^+ \to \mathbb{R}$ a smooth function. If the function $\varphi$ has the following properties:

(a) $\varphi'(t) \neq 0$,
(b) $\varphi'(t) + \varphi''(t) \neq 0$, for every $t \in \text{Im}(F^2),$

then the Lagrangian given by

$$L(x, y) = \varphi(F^2) + A_i(x)y^i + U(x), \quad (1.1)$$
where \( A_i(x) \) is a covector and \( U(x) \) is a smooth function, is a regular Lagrangian \([7]\). The space \( L^n = (M, L(x, y)) \) is a Lagrange space. The present authors \([7]\) called such space as an almost \( \varphi \)-Lagrange space (shortly APL space) associated to the Finsler space \( F^n \). An APL space reduces to a \( \varphi \)-Lagrange space if and only if \( A_i(x) = 0 \) and \( U(x) = 0 \). We take

\[
\begin{align*}
\mathbb{g}_{ij} &= \frac{1}{2} \partial_i \partial_j F^2, & a_{ij} &= \frac{1}{2} \partial_i \partial_j L, & \text{where } \partial_i \equiv \frac{\partial}{\partial y^i}.
\end{align*}
\]

(1.2)

We indicate all the geometrical objects related to \( F^n \) by putting a small circle "\( \circ \)" over them. Equations (1.2), in view of (1.1), provide the following expressions for \( a_{ij} \) and its inverse (cf. \([7]\)):

\[
\begin{align*}
a_{ij} &= \varphi' \left( g_{ij} + \frac{2q''}{\varphi} \circ y_i \circ y_j \right), & a^{ij} &= \frac{1}{\varphi} \left( g^{ij} - \frac{2q''}{\varphi^2} \circ y^i \circ y^j \right),
\end{align*}
\]

(1.3)

where \( g_{ij}, y^i = y_i \).

Let \( \bar{M} \) be a smooth manifold of dimension \( m, 1 < m < n \), immersed in \( M \) by immersion \( i : \bar{M} \rightarrow M \). The immersion \( i \) induces an immersion \( T_i : T\bar{M} \rightarrow TM \) making the following diagram commutative:

\[
\begin{array}{c}
T\bar{M} \xrightarrow{T_i} TM \\
\pi \downarrow \quad \downarrow \pi \\
\bar{M} \xrightarrow{i} M.
\end{array}
\]

(1.4)

Let \( (u^\alpha, v^\alpha) \) (throughout the paper, the Greek indices \( \alpha, \beta, \gamma, \ldots \) run from 1 to \( m \)) be local coordinates on \( T\bar{M} \). The restriction of the Lagrangian \( L \) on \( T\bar{M} \) is \( L(u, v) = L(x(u), y(u, v)) \). Let \( a_{\alpha\beta} = (1/2)(\partial^2 L/\partial u^\alpha \partial u^\beta) \). Then, we have (cf. \([8]\)) \( a_{\alpha\beta} = B^i_{\alpha} B^j_{\beta} a_{ij} \) where \( B^i_{\alpha}(u) = \partial x^i/\partial u^\alpha \) are the projection factors. The pair \( L^n = (\bar{M}, \bar{L}(u, v)) \) is also a Lagrange space, called the subspace of \( L^n \). For the natural bases \((\partial/\partial x^i, \partial/\partial y^j)\) on \( TM \) and \((\partial/\partial u^\alpha, \partial/\partial v^\alpha)\) on \( T\bar{M} \), we have \([8]\)

\[
\begin{align*}
\frac{\partial}{\partial u^a} &= B^i_{\mu} \frac{\partial}{\partial x^i} + B^i_{\nu} \frac{\partial}{\partial y^i}, & \frac{\partial}{\partial v^a} &= B^i_{\mu} \frac{\partial}{\partial y^i},
\end{align*}
\]

(1.5)

where \( B^i_{\alpha} = B^i_{\mu\alpha} v^\mu, \ B^i_{\nu} = \partial^2 x^i/\partial v^\nu \partial u^\alpha \).

For the bases \((dx^i, dy^j)\) and \((du^\alpha, dv^\alpha)\), we have

\[
dx^i = B^i_{\alpha} du^\alpha, \quad dy^j = B^j_{\mu} dv^\mu + B^j_{\nu} du^\alpha.
\]

(1.6)

Since \((B^i_{\alpha})\) are \( m \) linearly independent vector fields tangent to \( \bar{M} \), a vector field \( \xi^i(x, y) \) is normal to \( \bar{M} \) along \( T\bar{M} \) if on \( T\bar{M} \), we have

\[
a_{ij}(x(u), y(u, v))B^i_{\alpha} B^j_{\beta} = 0, \quad \forall \alpha = 1, 2, \ldots, m.
\]

(1.7)
There are, at least locally, \((n-m)\) unit vector fields \(B_a^i(u,v)\) \((a = m + 1, m + 2, \ldots, n)\) normal to \(\mathcal{M}\) and mutually orthonormal, that is,

\[
a_{ij}B_a^iB_b^j = 0, \quad a_{ij}B_a^iB_b^j = \delta_{ab}, \quad (a, b = m + 1, m + 2, \ldots, n). \tag{1.8}
\]

Thus, at every point \((u,v) \in T\mathcal{M}\), we have a moving frame \(\mathcal{R} = ((u,v), B_a^i(u,v), B_a^i(u,v))\).

Using (1.3) in the first expression of (1.8) and keeping \(\dot{y}_iB_a^i = 0\) (this fact is clear from \(g_{ij}\)) in view, we observe that \(B_a^i\)'s are normal to \(\mathcal{M}\) with respect to \(L^n\) if and only if they are so with respect to \(F^n\). The dual frame of \(\mathcal{R}\) is \(\mathcal{R}' = ((u,v), B_a^i(u,v), B_a^i(u,v))\) with the following duality conditions:

\[
B_a^iB_a^j = \delta_a^j, \quad B_a^iB_j^b = 0, \quad B_a^iB_j^b = \delta_a^b, \quad B_a^iB_j^b + B_a^bB_j^i = \delta_j^i. \tag{1.9}
\]

We will make use of the following results due to the present authors [7], during further discussion.

**Theorem 1.1** (cf. [7]). The canonical nonlinear connection of an APL space \(L^n\) has the local coefficients given by

\[
N_j^i = \dot{N}_j^i - V_j^i, \tag{1.10}
\]

where \(V_j^i = (1/2)F_j^i = S_j^r(2F_{rk}y^k + \partial_rU)\),

\[
S_j^r = \frac{1}{2q^r}C_{ij}^r \dot{q}^r + \frac{1}{2q^r}q'^r \dot{q}^r y_j^r + \frac{q''}{2q'} \left( \delta_j^r y_j^r + \delta_j^r y_j^r \right) + \frac{q'^2q'' - 2q'^3q'}{2q'^2(q' + 2F^2q'')} \dot{y}_j^r \dot{y}_j^r,
\]

\[
F_{rk}(x) = \frac{1}{2}(\partial_rA_k - \partial_kA_r), \quad F_j^i = a^i_k F_{kj}. \tag{1.11}
\]

**Theorem 2.1** (cf. [7]). The coefficients of the canonical metrical \(d\)-connection \(\Gamma(N)\) of an APL space \(L^n\) are given by

\[
C_{ijk}^i = C_{ijk}^i + \frac{q''}{q'} \left( \delta_j^i \delta_k^r + \delta_k^i \delta_j^r \right) + \frac{q'^2q'' - 2q'^3q'}{2q'^2(q' + 2F^2q'')} \dot{y}_j^r \dot{y}_j^r, \tag{1.12}
\]

\[
L_{ijk}^i = \frac{L_{ijk}}{2} + V'_k C_{ijk}^i + V'_j C_{ki}^i + V'_p a^p C_{rkj}. \tag{1.13}
\]

For basic notations related to a Finsler space, a Lagrange space, and their subspaces, we refer to the books [8, 9].
2. Induced Nonlinear Connection

Let \( \tilde{N} = (\tilde{N}_\beta^\alpha(u,v)) \) be a nonlinear connection for \( \tilde{L}^m = (\tilde{M}, \tilde{L}(u,v)) \). The adapted basis of \( T_{(u,v)}T\tilde{M} \) induced by \( \tilde{N} \) is \( (\delta/\delta u^a, \partial/\partial v^\alpha) \), where

\[
\delta_a = \partial_a - \tilde{N}_a^\beta \partial_\beta.
\]

(2.1)

The dual basis (cobasis) of the adapted basis \( (\delta_\alpha, \dot{\partial}^\alpha) \) is \( (du^\alpha, dv^\alpha + \tilde{N}_a^\beta du^\beta) \).

Definition 2.1 (cf. [8]). A nonlinear connection \( \tilde{N} = (\tilde{N}_\beta^\alpha(u,v)) \) of \( L^m \) is said to be induced by the canonical nonlinear connection \( N \) if the following equation holds good:

\[
\delta v^\alpha = B_i^\alpha \delta y^i.
\]

(2.2)

The local coefficients of the induced nonlinear connection \( \tilde{N} = (\tilde{N}_\beta^\alpha(u,v)) \) for the subspace \( \tilde{L}^m = (\tilde{M}, \tilde{L}(u,v)) \) of a Lagrange space \( L^n = (M, L(x,y)) \) are given by (cf. [8])

\[
\tilde{N}_\beta^\alpha = B_i^\alpha \left( N_j^i B_j^\beta + B_0^i \right).
\]

(2.3)

\( N_j^i \) being the local coefficients of canonical nonlinear connection \( N \) of the Lagrange space \( L^n = (M, L(x,y)) \). Now using (1.10) in (2.3), we get

\[
\tilde{N}_\beta^\alpha = B_i^\alpha \left( \tilde{N}_j^i B_j^\beta + B_i^\beta \right) - B_i^\beta \nabla_j B_j^\beta.
\]

(2.4)

If we take \( \tilde{N}_\beta^\alpha = B_i^\alpha \left( \tilde{N}_j^i B_j^\beta + B_i^\beta \right) \), it follows from (2.4) that

\[
\tilde{N}_\beta^\alpha = \tilde{N}_\beta^\alpha - B_i^\beta \nabla_j B_j^\beta.
\]

(2.5)

Thus, we have the following.

Theorem 2.2. The local coefficients of the induced nonlinear connection \( \tilde{N} \) of the subspace \( L^m \) of an APL space \( L^n \) are given by (2.5).

In view of (2.5), (2.1) takes the following form, for the subspace \( L^m \) of an APL space \( L^n \):

\[
\delta_\beta = \tilde{\delta}_\beta + B_i^\alpha \nabla_j B_j^\beta \dot{\partial}_\beta,
\]

(2.6)

where \( \tilde{\delta}_\beta = \partial_\beta - \tilde{N}_\beta^\alpha \dot{\partial}_\alpha \).
We can put $\{dx^i, \delta y^i\}$ as(cf.\[8\])

\[
dx^i = B^i_a du^a, \quad \delta y^i = B^i_a \delta y^a + B^i_a H^a_i du^a,\tag{2.7}
\]

where

\[
H^a_i = B^a_i \left( N^j_i B^j_a + B^j_a \right).	ag{2.8}
\]

Using (1.10) in (2.8) and simplifying, we get

\[
H^a_i = B^a_i \left( N^j_i B^j_a + B^j_a \right) - B^a_i \delta y^a B^j_a.	ag{2.9}
\]

Taking $H^a_i = B^a_i (N^j_i B^j_a + B^j_a)$, in (2.9), it follows that

\[
H^a_i = H^a_i - B^a_i \delta y^a B^j_a.\tag{2.10}
\]

Now, $dx^i = B^i_a du^a$, $\delta y^i = B^i_a \delta y^a$ if and only if $H^a_i = 0$, that is, if and only if $H^a_i = B^a_i \delta y^a B^j_a$. Thus, we have the following.

**Theorem 2.3.** The adapted cobasis $(dx^i, \delta y^i)$ of the basis $(\partial/\partial x^i, \partial/\partial y^i)$ induced by the nonlinear connection $N$ of an APL space $L^n$ is of the form $dx^i = B^i_a du^a$, $\delta y^i = B^i_a \delta y^a$ if and only if $H^a_i = B^a_i \delta y^a B^j_a$.

**Definition 2.4** (cf.\[8\]). Let $D = D \Gamma(N)$ be the canonical metrical $d$-connection of $L^n$. An operator $\tilde{D}$ is said to be a coupling of $D$ with $\tilde{N}$ if

\[
\tilde{D}X^i = X^i_\alpha du^\alpha + X^i_\alpha \delta v^\alpha,\tag{2.11}
\]

where $X^i_\alpha = \delta_\alpha X^i + X^i L^j_{\alpha i}$, $X^i_\alpha = \delta_\alpha X^i + X^i C^i_{\alpha i}$.

The coefficients $(L^i_{jk \alpha}, C^i_{jk \alpha})$ of coupling $\tilde{D}$ of $D$ with $\tilde{N}$ are given by

\[
L^i_{jk \alpha} = L^i_{jk} B^k_a + C^i_{jk} B^k_a H^a_i,\tag{2.12}
\]

\[
C^i_{jk \alpha} = C^i_{jk} B^k_a.\tag{2.13}
\]
Using (1.12) and (1.13) in (2.12), we have
\[
\dot{L}_{ij\beta} = \left( L_{jk}^i + V_k^j C_{jr}^i + V_j^r C_{kr}^i + V_p^r a^\gamma C_{rkj}^i \right) B_{\beta}^k
\]
\[+ \left[ C_{jk}^i + \frac{q''}{q'} \left( \delta_i^j \dot{y}_k + \delta_i^k \dot{y}_j \right) + \frac{q''}{q' + 2F^2 q''} S_{jk} y^i \right. \]
\[\left. + \frac{2 \left( q'' q' - 2 q''^2 \right)}{q' \left( q' + 2 F^2 q'' \right)} \dot{y}_j \dot{y}_k \right] B_{\beta}^k. \tag{2.14}\]

In view of (2.10) and \( \dot{y}_i B_{a}^i = 0 \), (2.14) becomes
\[
\dot{L}_{ij\beta} = \left( L_{jk}^i B_{\beta}^k + C_{jk} B_{a}^k H_{\beta}^a \right) + \left( V_k^j C_{jr}^i + V_j^r C_{kr}^i + V_p^r a^\gamma C_{rkj} - C_{jr} B_{k}^i B_{p}^k V_{V_{p}}^p \right) B_{\beta}^k
\]
\[+ \left( \frac{q''}{q'} \dot{y}_j \delta_i^k + \frac{q''}{q' + 2F^2 q''} S_{jk} y^i \right) B_{\beta}^k. \tag{2.15}\]

that is,
\[
L_{ij\beta} = L_{ij\beta} + \left( V_k^j C_{jr}^i + V_j^r C_{kr}^i + V_p^r a^\gamma C_{rkj} - C_{jr} B_{k}^i B_{p}^k V_{V_{p}}^p \right) B_{\beta}^k
\]
\[+ \left( \frac{q''}{q'} \dot{y}_j \delta_i^k + \frac{q''}{q' + 2F^2 q''} S_{jk} y^i \right) B_{\beta}^k. \tag{2.16}\]

where \( L_{ij\beta} = L_{ij\beta} + C_{jk} B_{a}^k H_{\beta}^a. \)

Using (1.12) in (2.13), we find that
\[
C_{ij\beta} = C_{jk} B_{a}^k + \left( \frac{q''}{q'} \left( \delta_i^j \dot{y}_k + \delta_i^k \dot{y}_j \right) + \frac{q''}{q' + 2F^2 q''} S_{jk} y^i \right. \]
\[\left. + \frac{2 \left( q'' q' - 2 q''^2 \right)}{q' \left( q' + 2 F^2 q'' \right)} \dot{y}_j \dot{y}_k \right] B_{\beta}^k. \tag{2.17}\]

that is,
\[
\tilde{C}_{ij\beta} = \tilde{C}_{ij\beta} + \left( \frac{q''}{q'} \left( \delta_i^j \dot{y}_k + \delta_i^k \dot{y}_j \right) + \frac{q''}{q' + 2F^2 q''} S_{jk} y^i \right. \]
\[\left. + \frac{2 \left( q'' q' - 2 q''^2 \right)}{q' \left( q' + 2 F^2 q'' \right)} \dot{y}_j \dot{y}_k \right] B_{\beta}^k. \tag{2.18}\]
where \( \circ_C^i_{jk} = C_{jk} B^k_{\beta} \). Thus, we have the following.

**Theorem 2.5.** The coefficients of coupling for the subspace \( L^m \) of an APL space \( L^n \) are given by (2.16) and (2.18).

**Definition 2.6** (cf. [8]). An operator \( D^T \) given by

\[
D^T X^\alpha = X^\alpha_{\beta} d\mu^\beta + X^\alpha_{\beta} \delta\sigma^\beta,
\]

where \( X^\alpha_{\beta} = \delta^\alpha_{\beta} X^\gamma + X^\gamma L^a_{\gamma\rho} \), \( X^\alpha_{\beta} = \delta^\alpha_{\beta} X^\gamma + X^\gamma C^a_{\gamma\rho} \) is called the induced tangent connection by \( D \). This defines an \( N \)-linear connection for \( L^m \).

The coefficients \( L^a_{\beta\gamma} \) of \( D^T \) are given by

\[
L^a_{\beta\gamma} = B^a_i \left( B^i_{\beta\gamma} B^i_{\beta\gamma} \right),
\]

and

\[
C^a_{\beta\gamma} = B^a_i B^i_{\beta\gamma}.
\]

Using (2.16) in (2.20), we get

\[
L^a_{\beta\gamma} = B^a_i B^i_{\beta\gamma} + B^a_i B^i_{\beta\gamma} \left[ \delta^i_{\beta\gamma} \left( V^r_{k} C_{jr} + V^r_{j} C_{kr} + V^r_{p} \alpha p C_{rj} - \delta^i_{\beta\gamma} B^b_p B^i_{p} V^p_{k} \right) B^k_{\gamma} \right.
\]

\[
+ \left( \frac{q^r_{\gamma}}{q^r_{\gamma}} \delta^i_{\beta\gamma} B^k_{\gamma} \right) B^a_{\gamma} H^a_{\gamma},
\]

(2.22)

that is,

\[
L^a_{\beta\gamma} = B^a_i \left( B^i_{\beta\gamma} + \delta^i_{\beta\gamma} \right) + B^a_i B^i_{\beta\gamma} \left[ \left( V^r_{k} C_{jr} + V^r_{j} C_{kr} + V^r_{p} \alpha p C_{rj} - \delta^i_{\beta\gamma} B^b_p B^i_{p} V^p_{k} \right) B^k_{\gamma}
\]

\[
+ \left( \frac{q^r_{\gamma}}{q^r_{\gamma}} \delta^i_{\beta\gamma} + \frac{q^r_{\gamma}}{q^r_{\gamma} + 2F^2 q^r_{\gamma} B_{ij} \gamma} B^a_{\gamma} \right) B^a_{\gamma} H^a_{\gamma},
\]

(2.23)

If we take \( L^a_{\beta\gamma} = B^a_i \left( B^i_{\beta\gamma} + \delta^i_{\beta\gamma} \right) \), the last expression gives

\[
L^a_{\beta\gamma} = \delta^a_{\beta\gamma} \left[ \left( V^r_{k} C_{jr} + V^r_{j} C_{kr} + V^r_{p} \alpha p C_{rj} - \delta^i_{\beta\gamma} B^b_p B^i_{p} V^p_{k} \right) B^k_{\gamma}
\]

\[
+ \left( \frac{q^r_{\gamma}}{q^r_{\gamma}} \delta^i_{\beta\gamma} + \frac{q^r_{\gamma}}{q^r_{\gamma} + 2F^2 q^r_{\gamma} B_{ij} \gamma} B^a_{\gamma} \right) B^a_{\gamma} H^a_{\gamma},
\]

(2.24)
Next, using (2.18) in (2.21), we obtain

\[
C^\alpha_{\beta\gamma} = B^a_i B^i_{\gamma} C_{\beta\gamma} + \left( \frac{\partial}{\partial \gamma} \left( \delta^i_j y_k + \delta^i_k y_j \right) \right) + \frac{q''}{q'} \frac{q''}{q'} B^k_i B^i_{\beta\gamma} + 2 \left( \frac{q'' q' - 2 q'''}{q''} \right) \frac{B^k_i B^i_{\beta\gamma} B^j_{\gamma'} y^j_j y_j} {q''}.
\]  

(2.25)

If we take \(C^\alpha_{\beta\gamma} = B^a_i B^i_{\gamma} C_{\beta\gamma}\), (2.25) becomes

\[
C^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} + \left( \frac{\partial}{\partial \gamma} \left( \delta^i_j y_k + \delta^i_k y_j \right) \right) + \frac{q''}{q'} \frac{q''}{q'} B^k_i B^i_{\beta\gamma} + 2 \left( \frac{q'' q' - 2 q'''}{q''} \right) \frac{B^k_i B^i_{\beta\gamma} B^j_{\gamma'} y^j_j y_j} {q''}.
\]  

(2.26)

Thus, we have the following.

**Theorem 2.7.** The coefficients of the induced tangent connection \(D^I\) for the subspace \(L^m\) of an APL space are given by (2.24) and (2.26).

**Remarks.** The torsion \(T^\alpha_{\beta\gamma} = L^\alpha_{\beta\gamma} - L^\alpha_{\beta\gamma}\) does not vanish, in general, while \(S^\alpha_{\beta\gamma} = C^\alpha_{\beta\gamma} - C^\alpha_{\beta\gamma} = 0\). These facts may be observed from (2.24) and (2.26).

**Definition 2.8 (cf. [8]).** An operator \(D^I\) given by

\[
D^I X^a = X^a_{\beta\gamma} \partial \phi^a + X^a_{\alpha} \delta \phi^a,
\]  

(2.27)

where \(X^a_{\beta\gamma} = \delta \phi^a X^a + X^b L_{\beta\gamma}^a\), \(X^a_{\alpha} = \partial \phi^a X^a + X^b C_{\beta\gamma}^a\), is called the induced normal connection by \(D\).

The coefficients \((L^a_{\beta\gamma}, C^a_{\beta\gamma})\) of \(D^I\) are given by

\[
L^a_{\beta\gamma} = B^a_i \left( \delta^i_j B^j_{\beta\gamma} + B^i_{\gamma} \right),
\]  

(2.28)

\[
C^a_{\beta\gamma} = B^a_i \left( \partial^i_j B^j_{\beta\gamma} + B^i_{\gamma} \right).
\]  

(2.29)

Using (2.6) and (2.16) in (2.28), we find

\[
L^a_{\beta\gamma} = B^a_i \left( \delta^i_j B^j_{\beta\gamma} + B^i_{\gamma} \right) + B^a_i B^i_{\gamma} B^i_{\gamma} \partial^i_j B^j_{\beta\gamma}
\]

\[
+ B^a_i B^i_{\gamma} \left[ \frac{\partial}{\partial \gamma} \left( \delta^i_j y_k + \delta^i_k y_j \right) \right] + \left( \frac{\partial}{\partial \gamma} \right) \left( \delta^i_j y_k + \delta^i_k y_j \right) \right) + \left( \frac{\partial}{\partial \gamma} \right) \left( \delta^i_j y_k + \delta^i_k y_j \right).
\]  

(2.30)
Taking $L_{by} = B_i^a \partial_{y}^{-i} B^a_b + B_i^a L_{by}$ and using $y_j B^i_b = 0$, (2.30) reduces to

\[
L_{by}^a = L_{by}^a + B_i^a B^b_k V^b_j \partial_{y}^{-i} B^b_k + \left( V^k_j C_{jr}^i + V_r^j C_{kr}^i - C_{jr}^i B^b_k B^b_p V^p_k \right) B^a_i B^a_j B^k_b,
\]

\[+ \frac{q''}{\gamma' + 2F^2q''} g_{jk} y^j B^a_i H^c_k B^a_l B^c_b.\]

(2.31)

Next, using (2.18) in (2.29), we have

\[
C_{by}^a = B_i^a \left( \partial_{y}^{-i} B^a_b + B^i_b \partial_{y}^{-i} C_{by} \right) + \left[ \frac{q''}{\gamma'} \left( \delta_i^y y_k + \delta_k^y y_i \right) + \frac{q''}{\gamma' + 2F^2q''} g_{jk} y^j \right]
\]

\[+ \frac{2(q''\gamma' - 2q''\gamma'')}{\gamma' + 2F^2q''} y^i y_j y_k \]

\[
B^a_i B^a_j B^k_b.\]

(2.32)

Taking $C_{by} = B_i^a \partial_{y}^{-i} B^a_b + B^i_b \partial_{y}^{-i} C_{by}$ and using (1.9) and $y_j B^i_b = 0$, the last equation yields

\[
C_{by}^a = C_{by}^a + \frac{q''}{\gamma'} \delta_i^y y_k B^a_i + \frac{q''}{\gamma' + 2F^2q''} g_{jk} y^j B^a_i B^a_j B^k_b.\]

(2.33)

Thus, we have the following.

**Theorem 2.9.** The coefficients of induced normal connection $D^2$ for the subspace $L^m$ of an APL space $L^n$ are given by (2.31) and (2.33).

**Definition 2.10** (cf. [8]). The (mixed) derivative of a mixed d-tensor field $T_{j \beta \alpha \beta}^{i \alpha \alpha \alpha \alpha}$ is given by

\[
\nabla T_{j \beta \alpha \beta}^{i \alpha \alpha \alpha \alpha} = \left( \partial_{\beta} T_{j \beta \alpha \beta}^{i \alpha \alpha \alpha \alpha} + T_{k \beta \beta \beta \beta}^{i \alpha \alpha \alpha \alpha} + \ldots + T_{j \gamma \beta \beta \beta}^{i \alpha \alpha \alpha \alpha} \right) du^j
\]

\[+ \left( \partial_{\gamma} T_{j \beta \beta \gamma}^{i \alpha \alpha \alpha \alpha} + \ldots + T_{j \gamma \gamma \gamma}^{i \alpha \alpha \alpha \alpha} \right) \delta v^j.
\]

(2.34)

The connection 1-forms,

\[
\omega_{\alpha}^i = I_{\alpha}^i du^\alpha + C_{\alpha}^i \delta v^\alpha,
\]

\[
\omega_{\beta}^\alpha = I_{\beta}^\alpha du^\beta + C_{\beta}^\alpha \delta v^\beta,
\]

\[
\omega_{\gamma}^a = I_{\gamma}^a du^\gamma + C_{\gamma}^a \delta v^\gamma.
\]

(2.35, 2.36, 2.37)
are called the connection 1-forms of $\nabla$. We have the following structure equations of $\nabla$.

**Theorem 2.11** (cf. [8]). The structure equations of $\nabla$ are as follows:

\[
\begin{align*}
    d(du^a) - du^\theta \wedge \omega^a_{\theta} &= -\Omega^a, \\
    d(du^\theta) - du^\phi \wedge \omega^\theta_{\phi} &= -\bar{\Omega}^\theta, \\
    d\omega^i_j - \omega^h_j \wedge \omega^i_{h} &= -\bar{\Omega}^i_j, \\
    d\omega^a_{\phi} - \omega^b_{\phi} \wedge \omega^a_{b} &= -\Omega^a_{\phi}, \\
    d\omega^a_{\theta} - \omega^b_{\theta} \wedge \omega^a_{b} &= -\Omega^a_{\theta},
\end{align*}
\]

where the 2-forms of torsions $\Omega^a, \bar{\Omega}^\theta$ are given by

\[
\begin{align*}
    \Omega^a &= \frac{1}{2} T^a_{\beta \gamma} du^\beta \wedge du^\gamma + C^a_{\beta \gamma} du^\beta \wedge \delta v^\gamma, \\
    \Omega^\theta &= \frac{1}{2} R^\theta_{\beta \gamma} du^\beta \wedge du^\gamma + P^\theta_{\beta \gamma} du^\beta \wedge \delta v^\gamma,
\end{align*}
\]

with $P^a_{\beta \gamma} = \partial_{\gamma} N^a_{\beta} - L^a_{\beta \gamma}$, and the 2-forms of curvature $\bar{\Omega}^i_j, \Omega^a_{\phi}$ and $\Omega^a_{\theta}$, are given by

\[
\begin{align*}
    \bar{\Omega}^i_j &= \frac{1}{2} \bar{R}^i_{j \alpha \beta} du^\alpha \wedge du^\beta + \bar{P}^i_{j \alpha \beta} du^\alpha \wedge \delta v^\beta + \frac{1}{2} \bar{S}^i_{j \alpha \beta} \delta v^\alpha \wedge \delta v^\beta, \\
    \Omega^a_{\phi} &= \frac{1}{2} R^a_{\beta \gamma} du^\beta \wedge du^\gamma + P^a_{\beta \gamma} du^\beta \wedge \delta v^\gamma + \frac{1}{2} S^a_{\beta \gamma} \delta v^\beta \wedge \delta v^\gamma, \\
    \Omega^a_{\theta} &= \frac{1}{2} R^a_{\beta \gamma} du^\beta \wedge du^\gamma + P^a_{\beta \gamma} du^\beta \wedge \delta v^\gamma + \frac{1}{2} S^a_{\beta \gamma} \delta v^\beta \wedge \delta v^\gamma.
\end{align*}
\]

We will use the following notations in Section 4:

(a) $\bar{\Omega}_{ij} = \bar{\Omega}^h_i a_{hj}$, (b) $\Omega_{ab} = \Omega^d_a a_{db}$, (c) $\Omega_{ab} = \Omega^c_b \delta_{ac}$.

**3. The Gauss-Weingarten Formulae**

The Gauss-Weingarten formulae for the subspace $L^m = (M, L(u, v))$ of a Lagrange space $L^n$ are given by (cf. [8])

\[
\nabla B^i_a = B^i_a \Pi^a_{\theta}, \quad \nabla B^i_a = -B^i_a \Pi^{	heta}_{\theta},
\]

\[
\begin{align*}
    \text{(3.1)}
\end{align*}
\]
where

\[ \Pi^a_a = H^a_a du^\beta + K^a_a \delta_\beta^\gamma, \]

\[ \Pi^a_b = g^a_{\cdots b} \delta_{ab} \Gamma^b, \]

(a) \( H^a_{ap} = B^a_a \left( \delta_\beta^a B^a_a + B^a_a \tilde{L}^a_{a \beta} \right) \),

(b) \( K^a_{a \beta} = B^a_i B^a_{j \beta} \).

Using (2.6) and (2.16) in (3.3)(a), we have

\[ \begin{align*}
H^a_{ap} &= B^a_a \left( \delta_\beta^a B^a_a + B^a_a \tilde{L}^a_{a \beta} \right) + B^a_i B^a_j V^p_j B^a_p B^a_y \\
&\quad + \left( V^i_k C^j_{jr} + V^i_j C^k_{kr} + V^i_j a^p C_{r} + C^i_{jr} B^a_p B^a_l V^p \right) B^a_i B^a_j B^a_k \\
&\quad + \left( \frac{q^\gamma}{q^\gamma} y^j \delta^j_k + \frac{q^\gamma}{q^\gamma + 2F^2 q^\gamma \tilde{S}^j_k y^j} \right) B^a_b H^a_b B^a_a. \end{align*} \]

If we take \( H^a_{ap} = B^a_a \left( \delta_\beta^a B^a_a + B^a_a \tilde{L}^a_{a \beta} \right) \), the last expression provides

\[ \begin{align*}
H^a_{ap} &= H^a_{ap} + B^a_i B^a_j V^p_j B^a_p B^a_y \\
&\quad + \left( V^i_k C^j_{jr} + V^i_j C^k_{kr} + V^i_j a^p C_{r} + C^i_{jr} B^a_p B^a_l V^p \right) B^a_i B^a_j B^a_k \\
&\quad + \left( \frac{q^\gamma}{q^\gamma} y^j \delta^j_k + \frac{q^\gamma}{q^\gamma + 2F^2 q^\gamma \tilde{S}^j_k y^j} \right) B^a_b H^a_b B^a_a. \end{align*} \]

Next, using (2.18) in (3.3)(b) and keeping (1.9) in view, we find

\[ \begin{align*}
K^a_{a \beta} &= K^a_{a \beta} + \left( \frac{q^\gamma}{q^\gamma + 2F^2 q^\gamma \tilde{S}^j_k y^j} + \frac{2 \left( q^{\gamma^2} - q^{\gamma^2} \right)}{q^\gamma \left( q^\gamma + 2F^2 q^\gamma \right)} y^j \delta^j_k \right) B^a_i B^a_a B^a_b, \end{align*} \]

where \( K^a_{a \beta} = B^a_i B^a_a \tilde{C}_{a \beta} \). Thus, we have the following.

**Theorem 3.1.** The following Gauss-Weingarten formulae for the subspace \( \tilde{L}^m \) of an APL space hold:

\[ \nabla B^i_a = B^i_a \Pi^a_a, \quad \nabla B^i_a = -B^i_a \Pi^a_b. \]
where

\[ \Pi^a_{\alpha} = H^a_{\alpha \beta} du^\beta + K^a_{\alpha \beta} \partial v^\beta, \quad \Pi^b_{\beta} = g^{\beta \gamma} \delta_{ab} \Pi^b_{\gamma}, \]

\[ H^a_{\alpha \beta} = H^a_{\alpha \beta \gamma} + B^a_{\gamma} V^j_i B^i_j B^i_{\alpha \gamma}, \quad \Pi^a_{\beta} = g^{\alpha \gamma} B^a_{\gamma} \]

\[ + \left( \frac{\eta''}{\eta'} \eta_j i - \frac{\eta''}{\eta' + 2 \eta' \phi} \right) \Pi^a_{\beta}, \quad K^a_{\alpha \beta} = K^a_{\alpha \beta} \]

\[ \left( \frac{\eta''}{\eta'} \eta_j i - \frac{\eta''}{\eta' + 2 \eta' \phi} \right) B^i_{\alpha} B^i_{\beta}. \]

Remark 3.2. \( H^a_{\alpha \beta} \) and \( K^a_{\alpha \beta} \), given, respectively, by (3.5) and (3.6), are called the second fundamental \( d \)-tensor fields of immersion \( i \).

The following consequences of Theorem 3.1 are straightforward.

Corollary 3.3. In a subspace \( \tilde{L}^m \) of an APL space, we have the following:

\[ (a) \ \nabla a_{\alpha \beta} = 0, \]
\[ (b) \ \nabla B^i_{\alpha} = 0, \]

if and only if

\[ \hat{H}^a_{\alpha \beta} = - \left( B^a_{\gamma} V^j_i B^i_j B^i_{\alpha \gamma} + \left( V^r_{\gamma} C^i_j + V^j_i C^r_{\gamma} + V^r_i a^r_{\gamma} C^r_{kj} - C^r_{ij} B^r_{\gamma} V^r_i \right) B^i_{\alpha} B^i_{\beta} \right) \]

\[ + \left( \frac{\eta''}{\eta'} \eta_j i - \frac{\eta''}{\eta' + 2 \eta' \phi} \right) B^i_{\alpha} B^i_{\beta}, \]

\[ \hat{K}^a_{\alpha \beta} = - \left( \frac{\eta''}{\eta'} \eta_j i - \frac{\eta''}{\eta' + 2 \eta' \phi} \gamma^i j k \right) B^i_{\alpha} B^i_{\beta}. \]

4. The Gauss-Codazzi Equations

The Gauss-Codazzi Equations for the subspace \( \tilde{L}^m \) of a Lagrange space \( L^n \) are given by (cf. [8])

\[ B^i_{\alpha} B^j_{\beta} \Omega_{ij} - \Omega_{a \beta} = \Pi^a_{\beta \alpha} \wedge \Pi^a_{\beta}, \]

\[ B^i_{\alpha} B^j_{\beta} \Omega_{ij} - \Omega_{a \beta} = \Pi^a_{\beta \alpha} \wedge \Pi^a_{\beta}, \]

\[ -B^i_{\alpha} B^j_{\beta} \Omega_{ij} = \delta_{ab} \left( d \Pi^b_{\alpha} + \Pi^b_{\beta} \wedge \omega_{\alpha} - \Pi^a_{\beta} \wedge \omega_{\beta} \right), \]
where

\[ (a) \quad \Pi_{aa} = g_{ab} \Pi_{a}^{b}, \quad (b) \quad \Pi_{ab} = \delta_{bc} \Pi_{b}^{c}. \] (4.4)

Using (1.3) in (2.41)(a), we find that

\[ \tilde{\Omega}_{ij} = \varphi' \tilde{\Omega}_{i}^{h} g_{ij} + 2 \varphi'' \tilde{\Omega}_{i}^{h} \tilde{y}_{i}^{h} \tilde{y}_{j}^{h}. \] (4.5)

Applying \( a_{ij} = B_{i}^{j} B_{j}^{k} a_{ij} \) in (2.41)(b), we have \( \Omega_{ab} = B_{i}^{j} B_{j}^{k} \Omega_{i}^{k} a_{ij} \), which in view of (1.3) becomes

\[ \Omega_{ab} = \varphi' g_{ij} B_{i}^{j} B_{j}^{k} \Omega_{i}^{k} + 2 \varphi'' \tilde{y}_{i}^{k} \tilde{y}_{j}^{k} B_{i}^{j} B_{j}^{k} \Omega_{i}^{k}, \] (4.6)

that is,

\[ \Omega_{ab} = \varphi' g_{ij} \Omega_{i}^{k} + 2 \varphi'' \tilde{y}_{i}^{k} \tilde{y}_{j}^{k} B_{i}^{j} B_{j}^{k} \Omega_{i}^{k}. \] (4.7)

For the subspace \( L^{m} \) of an APL space, (4.4)(a) is of the form \( \Pi_{aa} = a_{ab} \Pi_{a}^{b} \), which in view of \( a_{ab} = B_{i}^{j} B_{j}^{k} a_{ij} \) and (1.3) becomes \( \Pi_{aa} = \varphi' B_{i}^{a} B_{j}^{a} a_{ij} + 2 \varphi'' \tilde{y}_{i}^{k} \tilde{y}_{j}^{k} B_{i}^{a} B_{j}^{a} \Pi_{i}^{a} \), that is,

\[ \Pi_{aa} = \varphi' g_{ab} \Pi_{a}^{b} + 2 \varphi'' \tilde{y}_{i}^{k} \tilde{y}_{j}^{k} B_{i}^{a} B_{j}^{a} \Pi_{i}^{a}. \] (4.8)

Thus, we have the following.

**Theorem 4.1.** The Gauss-Codazzi equations for a Lagrange subspace \( L^{m} \) of an APL space are given by (4.1)–(4.3) with \( \Pi_{aa}, \Pi_{ab}, \tilde{\Omega}_{ij}, \Omega_{ab}, \) and \( \omega_{c}^{b} \), respectively, given by (4.8), (4.4)(b), (4.5), (4.7), and (2.37).

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**References**


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