Research Article
Fredholm Weighted Composition Operators on Dirichlet Space

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1. Introduction

Let $H$ be a Hilbert space of analytic functions on the unit disk $D$. For an analytic function $\psi$ on $D$, we can define the multiplication operator $M_\psi : f \rightarrow \psi f$, $f \in H$. For an analytic self-mapping $\psi$ of $D$, the composition operator $C_\psi$ defined on $H$ as $C_\psi f = f \circ \psi$, $f \in H$. These operators are two classes of important operators in the study of operator theory in function spaces [1–3]. Furthermore, for $\psi$ and $\phi$, we define the weighted composition operator $C_{\psi,\phi}$ on $H$ as

$$C_{\psi,\phi} : f \rightarrow \psi(f \circ \phi), \quad f \in H.$$ (1.1)

Recently, the boundedness, compactness, norm, and essential norm of weighted composition operators on various spaces of analytic functions have been studied intensively, see [4–9] and so on. In this paper, we characterize bounded Fredholm weighted composition operators on Dirichlet space of the unit disk.
Recall the Dirichlet space $D$ that consists of analytic function $f$ on $D$ with finite Dirichlet integral:

$$D(f) = \int_D |f'|^2 dA < \infty,$$

where $dA$ is the normalized Lebesgue area measure on $D$. It is well known that $D$ is the only mobius invariant Hilbert space up to an isomorphism [10]. Endow $\mathfrak{D}$ with norm

$$\|f\| = \left( |f(0)|^2 + D(f) \right)^{1/2}, \quad f \in \mathfrak{D}. \quad (1.3)$$

$\mathfrak{D}$ is a Hilbert space with inner product

$$\langle f, g \rangle = f(0)g(0) + \int_D f'(z)g'(z)dA(z), \quad f, g \in \mathfrak{D}. \quad (1.4)$$

Furthermore $\mathfrak{D}$ is a reproducing function space with reproducing kernel

$$K_\lambda(z) = 1 + \log \frac{1}{1-\lambda z}, \quad \lambda, z \in D. \quad (1.5)$$

Denote $\mathcal{M} = \{ \varphi : \varphi \text{ is analytic on } D, \varphi f \in \mathfrak{D} \text{ for } f \in \mathfrak{D} \}$. $\mathcal{M}$ is called the multiplier space of $\mathfrak{D}$. By the closed graph theorem, the multiplication operator $M_\varphi$ defined by $\varphi \in \mathcal{M}$ is bounded on $\mathfrak{D}$. For the characterization of the element in $\mathcal{M}$, see [11].

For analytic function $\varphi$ on $D$ and analytic self-mapping $\varphi$ of $D$, the weighted composition operator $C_{\varphi, \psi}$ on $\mathfrak{D}$ is not necessarily bounded. Even the composition operator $C_\varphi$ is not necessarily bounded on $\mathfrak{D}$, which is different from the cases in Hardy space and Bergman space. See [12] for more information about the properties of composition operators acting on the Dirichlet space.

The main result of the paper reads as the following.

**Theorem 1.1.** Let $\varphi$ and $\psi$ be analytic functions on $D$ with $\varphi(D) \subset D$. Then $C_{\varphi, \psi}$ is a bounded Fredholm operator on $\mathfrak{D}$ if and only if $\varphi \in \mathcal{M}$, bounded away from zero near the unit circle, and $\varphi$ is an automorphism of $D$.

If $\varphi(z) = 1$, then the result above gives the characterization of bounded Fredholm composition operator $C_\varphi$ on $\mathfrak{D}$, which was obtained in [12].

As corollaries, in the end of this paper one gives the characterization of bounded invertible and unitary weighted composition operator on $\mathfrak{D}$, respectively. Some idea of this paper is derived from [4, 13], which characterize normal and bounded invertible weighted composition operator on the Hardy space, respectively.

### 2. Proof of the Main Result

In the following, $\varphi$ and $\psi$ denote analytic functions on $D$ with $\varphi(D) \subset D$. It is easy to verify that $\varphi \in \mathfrak{D}$ if $C_{\varphi, \psi}$ is defined on $\mathfrak{D}$. 
Proposition 2.1. Let $C_{\psi,\varphi}$ be a bounded Fredholm operator on $\mathcal{D}$. Then $\psi$ has at most finite zeroes in $D$ and $\varphi$ is an inner function.

Proof. If $C_{\psi,\varphi}$ is a bounded Fredholm operator, then there exist a bounded operator $T$ and a compact operator $S$ on $\mathcal{D}$ such that

$$T(C_{\psi,\varphi})^* = I + S, \quad (2.1)$$

where $I$ is the identity operator.

Since

$$\langle C_{\psi,\varphi}^* K_w, K_z \rangle = \langle C_{\psi,\varphi}^* K_z, K_w \rangle = \langle K_w, C_{\psi,\varphi} K_z \rangle = \langle K_w, \psi K_z \circ \varphi \rangle = \varphi(w) K_z(\varphi(w)), \quad (2.2)$$

we have

$$\|T\| \|\psi(w)\| \|K_{\varphi(w)}\| \geq \|T(C_{\psi,\varphi})^* K_w\| \geq \|k_w\| - \|Sk_w\| \geq 1 - \|Sk_w\|, \quad (2.3)$$

where $k_w = K_w/\|K_w\|$ is the normalization of reproducing kernel function $K_w$.

Since $S$ is compact and $k_w$ weakly converges to 0 as $|w| \to 1$, $\|Sk_w\| \to 0$ as $|w| \to 1$. It follows that there exists constant $r$, $0 < r < 1$, such that $\|Sk_w\| < 1/2$ for all $w$ with $r < |w| < 1$. Inequality (2.3) shows that

$$\|\psi(w)\| \|K_{\varphi(w)}\| \geq 1/2\|T\| \|K_{\varphi(w)}\|, \quad r < |w| < 1, \quad (2.4)$$

which implies that $\psi$ has no zeroes in $\{w \in D, \ r < |w| < 1\}$, and, hence, $\psi$ has at most finite zeroes in $\{w \in D, \ |w| \leq r\}$.

Since $k_w$ weakly converges to 0 as $|w| \to 1$, $\langle \psi, k_w \rangle \to 0$ as $|w| \to 1$, that is,

$$\frac{\psi(w)}{\|K_w\|} \to 0, \quad |w| \to 1. \quad (2.5)$$

It follows from (2.4) that $\|K_{\varphi(w)}\| = (1 + \log(1/(1 - |\varphi(w)|^2)))^{1/2} \to \infty$ and hence $|\varphi(w)| \to 1$ as $|w| \to 1$, that is, $\varphi$ is an inner function.

For the proof of the following lemma, we cite Carleson’s formula for the Dirichlet integral [14].
Lemma 2.2. Let $f \in \mathfrak{D}$, $f = BSF$ be the canonical factorization of $f$ as a function in the Hardy space, where $B = \prod_{j=1}^{\infty} (a_j/|a_j|)((a_j - z)/(1 - \bar{a}_j z))$, is a Blaschke product, $S$ is the singular part of $f$ and $F$ is the outer part of $f$. Then

$$D(f) = \int T \sum_{n=1}^{\infty} P_{a_n}(\xi) |f(\xi)|^2 \, \frac{d\xi}{2\pi} + \int \int \frac{2}{T |\xi - \eta|^2} |f(\xi)|^2 \, d\mu(\xi) \frac{|d\xi|}{2\pi}$$

$$+ \int \int \frac{e^{2u(\xi)} - e^{2u(\eta)}}{\xi - \eta} (u(\xi) - u(\eta)) \frac{|d\xi|}{2\pi} \frac{|d\eta|}{2\pi},$$

(2.6)

where $T$ is the unit circle, $u(\xi) = \log |f(\xi)|$, $P_{a}(\xi)$ is the Poisson kernel, and $\mu$ is the singular measure corresponding to $S$.

**Lemma 2.2.** Let $C_{\varphi,\psi}$ be a bounded operator on $\mathfrak{D}$, $\varphi = BF$ with $B$ a finite Blaschke product. Then $C_{\varphi,\psi}$ is bounded.

**Proof.** Let $M_B$ be the multiplication operator on $\mathfrak{D}$. Then $C_{\varphi,\psi} = M_B C_{\varphi,\psi}$. Since $B$ is a finite Blaschke product, by the Carleson’s formula, we have

$$D(\varphi(f \circ \psi)) = D(BF(f \circ \psi)) \geq D(F(f \circ \psi)), \quad f \in \mathfrak{D}. \quad (2.7)$$

Since $\|f\|^2 = |f(0)|^2 + D(f)$, $f \in \mathfrak{D}$, by the inequality above it is easy to verify that $C_{\varphi,\psi}$ is bounded on $\mathfrak{D}$ if $C_{\varphi,\psi}$ is bounded. \hfill \Box

**Lemma 2.3.** Let $F$ be an analytic function on $D$ with zero-free. If $C_{\varphi,\psi}$ is a bounded Fredholm operator on $\mathfrak{D}$, then $\varphi$ is univalent.

**Proof.** If $\varphi(a) = \varphi(b)$ for $a, b \in D$ with $a \neq b$, by a similar reasoning as [1, Lemma 3.26], there exist infinite sets $\{a_n\}$ and $\{b_n\}$ in $D$ which is disjoint such that $\varphi(a_n) = \varphi(b_n)$. Hence,

$$(C_{\varphi,\psi})^* \left( \frac{K_{a_n}}{F(a_n)} - \frac{K_{b_n}}{F(b_n)} \right) = 0, \quad (2.8)$$

which contradicts to that kernel of $(C_{\varphi,\psi})^*$ is finite dimensional. \hfill \Box

**Corollary 2.4.** If $C_{\varphi,\psi}$ is a bounded Fredholm operator on $\mathfrak{D}$, then $\varphi$ is an automorphism of $D$ and $\varphi \in \mathcal{M}$.

**Proof.** By Proposition 2.1, $\varphi$ has the factorization of $BF$ with $B$ a finite Blaschke product and $F$ zero free in $D$. By Lemma 2.2, $C_{\varphi,\psi}$ is a bounded operator on $\mathfrak{D}$. Since $C_{\varphi,\psi} = M_B C_{\varphi,\psi}$ and $M_B$ is a Fredholm operator, $C_{\varphi,\psi}$ is a Fredholm operator also. By Proposition 2.1 and Lemma 2.3, $\varphi$ is an univalent inner function, it follows from [1, Corollary 3.28] that $\varphi$ is an automorphism of $D$.

Since $C_{\varphi,\psi} C_{\varphi^{-1}} = M_{\varphi}$, $M_{\varphi}$ is a bounded multiplication operator on $\mathfrak{D}$, which implies that $\varphi \in \mathcal{M}$. \hfill \Box

The following lemmas is well-known. It is easy to verify by the fact $M_{\varphi}^* K_w = \overline{\varphi(w)} K_w$ also.
Lemma 2.5. Let \( \psi \in \mathcal{M} \). Then \( M_\psi \) is an invertible operator on \( D \) if and only if \( \psi \) is invertible in \( \mathcal{M} \).

Lemma 2.6. Let \( \psi \in \mathcal{M} \). Then \( M_\psi \) is a Fredholm operator on \( D \) if and only if \( \psi \) is bounded away from the unit circle.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. If \( C_{\psi,\varphi} \) is a bounded Fredholm operator on \( D \), by Corollary 2.4, \( \psi \in \mathcal{M} \) and \( \varphi \) is an automorphism of \( D \). Since \( C_\psi \) is invertible, \( M_\psi \) is a Fredholm operator. So \( \psi \) is bounded away from the unit circle follows from Lemma 2.6.

On the other hand, if \( \psi \in \mathcal{M} \) and bounded away from the unit circle, then \( M_\psi \) is a bounded Fredholm operator on \( D \). If \( \varphi \) is an automorphism of \( D \), then \( C_\varphi \) is invertible. Hence \( C_{\psi,\varphi} = M_\psi C_\varphi \) is a bounded Fredholm operator on \( D \).

As corollaries, in the following, we characterize bounded invertible and unitary weighted composition operators on \( D \).

Corollary 2.7. Let \( \psi \) and \( \varphi \) be analytic functions on \( D \) with \( \varphi(D) \subset D \). Then \( C_{\psi,\varphi} \) is a bounded invertible operator on \( D \) if and only if \( \psi \in \mathcal{M} \), invertible in \( \mathcal{M} \), and \( \varphi \) is an automorphism of \( D \).

Proof. Since a bounded invertible operator is a bounded Fredholm operator, the proof is similar to the proof of Theorem 1.1.

Corollary 2.8. Let \( \psi \) and \( \varphi \) be analytic functions on \( D \) with \( \varphi(D) \subset D \). \( C_{\psi,\varphi} \) is a bounded operator on \( D \). Then \( C_{\psi,\varphi} \) is a unitary operator if and only if \( \psi \) is a constant with \( |\psi| = 1 \) and \( \varphi \) is a rotation of \( D \).

Proof. If \( C_{\psi,\varphi} \) is a unitary operator, then it must be an invertible operator. By Corollary 2.7, \( \varphi \) is an automorphism of \( D \) and \( \psi \) is invertible in \( \mathcal{M} \).

Let \( n \) be nonnegative integer, \( e_n(z) = z^n, z \in D \). A unitary is also an isometry, so we have

\[
\|\psi\| = \|C_{\psi,\varphi}e_0\| = \|e_0\| = 1, \quad (2.9)
\]

\[
\|\psi \varphi^n\| = \|C_{\psi,\varphi}e_n\| = \|e_n\| = \sqrt{n}, \quad n \geq 1. \quad (2.10)
\]

Let \( \alpha \in D \) such that \( \varphi(\alpha) = 0 \). Since \( \varphi \) is an automorphism of \( D \), \( \varphi^n \) is a finite Blaschke product with zero \( \alpha \) of order \( n \). By Carleson’s formula for Dirichlet integral, we have

\[
D(\psi \varphi^n) = n \int_T P_\alpha(\xi) |\psi(\xi)|^2 \frac{|d\xi|}{2\pi} + D(\psi). \quad (2.11)
\]

Hence,

\[
n = \|\psi \varphi^n\|^2 = |\psi(0) \varphi(0)^n|^2 + D(\psi \varphi^n)
\]

\[
= |\psi(0) \varphi(0)^n|^2 + n \int_T P_\alpha(\xi) |\psi(\xi)|^2 \frac{|d\xi|}{2\pi} + D(\psi), \quad n \geq 1. \quad (2.12)
\]
That is,

\[
1 = \frac{|\varphi(0)|^2}{n} + \int_T p_n(\xi) |\varphi(\xi)|^2 \frac{|d\xi|}{2\pi} + \frac{D(\varphi)}{n}, \quad n \geq 1.
\]

Let \( n \to \infty \), then \( 1 = \int_T p_n(\xi) |\varphi(\xi)|^2 \frac{|d\xi|}{2\pi} \).

By (2.12), we have \( D(\varphi) = 0 \) and \( |\varphi(0)|\varphi(0) = 0 \). By (2.9), we obtain \( \varphi \) is a constant with \( |\varphi| = 1 \), which implies that \( \varphi(0) = 0 \), that is, \( \varphi \) is a rotation of \( D \).

The sufficiency is easy to verify. \( \square \)

**Remark 2.9.** The key step in the proof of the main result is to analyze zeros of the symbol \( \varphi \) and univalency of \( \varphi \). The following result pointed out by the referee gives a simple characterization of the symbols \( \psi \) and \( \varphi \) for the bounded Fredholm operator \( C_{\psi,\varphi} \) on \( \mathfrak{D} \).

**Proposition 2.10.** Let \( \varphi \) and \( \psi \) be analytic functions on \( D \) with \( \varphi(D) \subset D \). \( C_{\psi,\varphi} \) is a bounded Fredholm operator on \( \mathfrak{D} \). Then \( \varphi \) has only finitely many zeros in \( D \) and \( \psi \) is univalent.

**Proof.** If \( \varphi(a) = 0 \) for \( a \in D \), then \( C_{\psi,\varphi}^* K_a = \psi(a)K_{\varphi(a)} = 0 \), which implies that \( K_a \) is in the kernel of \( C_{\psi,\varphi}^* \). Thus if \( \varphi \) had infinitely many zeros, the kernel of \( C_{\psi,\varphi}^* \) would be infinite dimensional and hence this operator would not be Fredholm.

If \( \varphi(a) = \varphi(b) \) for \( a, b \in D \) with \( a \neq b \), by a similar reasoning as [1, Lemma 3.26], there exist infinite sets \( \{a_n\} \) and \( \{b_n\} \) in \( D \) which is disjoint such that \( \varphi(a_n) \neq \varphi(b_n) \). Since \( \varphi \) has only finitely many zeros in \( D \), we can choose infinitely many \( a_n \) and \( b_n \) such that \( \varphi(a_n) \neq \varphi(b_n) \neq 0 \). Hence,

\[
(C_{\psi,\varphi})^* \left( \frac{K_{a_n}}{\varphi(a_n)} - \frac{K_{b_n}}{\varphi(b_n)} \right) = 0.
\]

Since \( C_{\psi,\varphi} \) is a Fredholm operator, \( \varphi \) must be univalent. \( \square \)

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**References**


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