Research Article

Combined Algebraic Properties of IP* and Central* Sets Near 0

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It is known that for an IP* set A in \( \mathbb{N} \) and a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) there exists a sum subsystem \( \langle y_n \rangle_{n=1}^{\infty} \) of \( \langle x_n \rangle_{n=1}^{\infty} \) such that \( FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A \). Similar types of results also have been proved for central* sets. In this present work we will extend the results for dense subsemigroups of \( ((0, \infty), +) \).

1. Introduction

One of the famous Ramsey theoretic results is Hindman’s Theorem.

**Theorem 1.1.** Given a finite coloring \( \mathbb{N} = \bigcup_{i=1}^{r} A_i \) there exists a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \) and \( i \in \{1, 2, \ldots, r\} \) such that

\[
FS(\langle x_n \rangle_{n=1}^{\infty}) = \left\{ \sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\} \subseteq A_i,
\]

where for any set \( X, \mathcal{P}_f(X) \) is the set of finite nonempty subsets of \( X \).

The original proof of this theorem was combinatorial in nature. But later using algebraic structure of \( \beta \mathbb{N} \) a very elegant proof of this theorem was established in [1, Corollary 5.10]. First we give a brief description of algebraic structure of \( \beta S_d \) for a discrete semigroup \( (S, \cdot) \).
We take the points of $\beta S_d$ to be the ultrafilters on $S$, identifying the principal ultrafilters with the points of $S$ and thus pretending that $S \subseteq \beta S_d$. Given $A \subseteq S$,

$$c\ell A = \overline{A} = \{ p \in \beta S_d : A \in p \}$$ (1.2)

is a basis for the closed sets of $\beta S_d$. The operation $\cdot$ on $S$ can be extended to the Stone-Čech compactification $\beta S_d$ of $S$ so that $(\beta S_d, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S_d$, the function $\rho_p : \beta S_d \to \beta S_d$ defined by $\rho_p(q) = q \cdot p$ is continuous) with a smallest ideal $\sum$ such that

$$\sum \subseteq \beta S_d.$$ (1.4)

A nonempty subset $I$ of a semigroup $T$ is called a left ideal if $TI \subseteq I$, a right ideal if $IT \subseteq I$, and a two-sided ideal (or simply an ideal) if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal.

Any compact Hausdorff right topological semigroup $T$ has a smallest two-sided ideal:

$$K(T) = \bigcup \{ L : L \text{ is a minimal left ideal of } T \}$$

$$= \bigcup \{ R : R \text{ is a minimal right ideal of } T \}.$$ (1.3)

Given a minimal left ideal $L$ and a minimal right ideal $R$, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a minimal idempotent. If $p$ and $q$ are idempotents in $T$ we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

Given $p, q \in \beta S$, and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{ x \in S : x^{-1}A \in q \} \subseteq p$, where $x^{-1}A = \{ y \in S : x \cdot y \in A \}$. See [1] for an elementary introduction to the algebra of $\beta S$ and for any unfamiliar details.

A set $A \subseteq \mathbb{N}$ is called an IP* set if it belongs to every idempotent in $\beta \mathbb{N}$. Given a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$, we let $FP((x_n)_{n=1}^{\infty})$ be the product analogue of Finite Sum. Given a sequence $(y_n)_{n=1}^{\infty}$ in $\mathbb{N}$, we say that $(y_n)_{n=1}^{\infty}$ is a sum subsystem of $(x_n)_{n=1}^{\infty}$ if there is a sequence $(H_n)_{n=1}^{\infty}$ of nonempty finite subsets of $\mathbb{N}$ such that $max H_n < min H_{n+1}$ and $y_n = \sum_{i \in H_n} x_i$ for each $n \in \mathbb{N}$.

Theorem 1.2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ and let $A$ be an IP* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that

$$FS((y_n)_{n=1}^{\infty}) \cup FP((y_n)_{n=1}^{\infty}) \subseteq A.$$ (1.4)

Proof. See [2, Theorem 2.6] or see [1, Corollary 16.21].

Definition 1.3. A subset $C \subseteq S$ is called central if and only if there is an idempotent $p \in K(\beta S)$ such that $C \in p$.

The algebraic structure of the smallest ideal of $\beta S$ has played a significant role in Ramsey Theory. It is known that any central subset of $(\mathbb{N}, +)$ is guaranteed to have substantial additive structure. But Theorem 16.27 of [1] shows that central sets in $(\mathbb{N}, +)$ need not have any multiplicative structure at all. On the other hand, in [2] we see that sets which belong
to every minimal idempotent of \( \mathbb{N} \), called central\(^*\) sets, must have significant multiplicative structure. In fact central\(^*\) sets in any semigroup \((S, \cdot)\) are defined to be those sets which meet every central set.

**Theorem 1.4.** If \( A \) is a central\(^*\) set in \((\mathbb{N}, +)\) then it is central in \((\mathbb{N}, \cdot)\).

**Proof.** See [2, Theorem 2.4]. \( \square \)

In case of central\(^*\) sets a similar result has been proved in [3] for a restricted class of sequences called minimal sequences, where a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \) is said to be a minimal sequence if

\[
\bigcap_{m=1}^{\infty} FS(\langle x_n \rangle_{n=m}^{\infty}) \cap K(\beta \mathbb{N}) \neq \emptyset.
\]  

**Theorem 1.5.** Let \( \langle y_n \rangle_{n=1}^{\infty} \) be a minimal sequence and let \( A \) be a central\(^*\) set in \((\mathbb{N}, +)\). Then there exists a sum subsystem \( \langle x_n \rangle_{n=1}^{\infty} \) of \( \langle y_n \rangle_{n=1}^{\infty} \) such that

\[
FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A.
\]  

**Proof.** See [3, Theorem 2.4]. \( \square \)

A strongly negative answer to the partition analogue of Hindman’s theorem was presented in [4]. Given a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \), let us denote \( PS(\langle x_n \rangle_{n=1}^{\infty}) = \{ x_m + x_n : m, n \in \mathbb{N} \text{ and } m \neq n \} \) and \( PP(\langle x_n \rangle_{n=1}^{\infty}) = \{ x_m \cdot x_n : m, n \in \mathbb{N} \text{ and } m \neq n \} \).

**Theorem 1.6.** There exists a finite partition \( \mathcal{R} \) of \( \mathbb{N} \) with no one-to-one sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( \mathbb{N} \) such that \( PS(\langle x_n \rangle_{n=1}^{\infty}) \cup PP(\langle x_n \rangle_{n=1}^{\infty}) \) is contained in one cell of the partition \( \mathcal{R} \).

**Proof.** See [4, Theorem 2.11]. \( \square \)

A similar result in this direction in the case of dyadic rational numbers has been proved by V. Bergelson et al.

**Theorem 1.7.** There exists a finite partition \( \mathbb{D} \setminus \{0\} = \bigcup_{i=1}^{r} A_i \) such that there do not exist \( i \in \{1, 2, \ldots, r\} \) and a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) with

\[
FS(\langle x_n \rangle_{n=1}^{\infty}) \cup PP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i.
\]  

**Proof.** See [5, Theorem 5.9]. \( \square \)

In [5], the authors also presented the following conjecture and question.

**Conjecture 1.8.** There exists a finite partition \( \mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^{r} A_i \) such that there do not exists \( i \in \{1, 2, \ldots, r\} \) and a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) with

\[
FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i.
\]  

\( \square \)
In the following discussion, we will extend Theorem 1.2 for a dense subsemigroup of \( * \) and Central’ set near 0.

**Definition 1.9.** If \( S \) is a dense subsemigroup of \((0, \infty), +)\) one defines \( 0^+(S) = \{ p \in \beta S_d : (\text{for all } e > 0)((0, e) \in p)\} \).

It is proved in [6], that \( 0^+(S) \) is a compact right topological subsemigroup of \( (\beta S_d, +) \) which is disjoint from \( K(\beta S_d) \) and hence gives some new information which are not available from \( K(\beta S_d) \). Being compact right topological semigroup \( 0^+(S) \) contains minimal idempotents of \( 0^+(S) \). A subset \( A \) of \( S \) is said to be IP*-set near 0 if it belongs to every idempotent of \( 0^+(S) \) and a subset \( C \) of \( S \) is said to be central’ set near 0 if it belongs to every minimal idempotent of \( 0^+(S) \). In [7] the authors applied the algebraic structure of \( 0^+(S) \) on their investigation of image partition regularity near 0 of finite and infinite matrices. Article [8] used algebraic structure of \( 0^+(\mathbb{R}) \) to investigate image partition regularity of matrices with real entries from \( \mathbb{R} \).

### 2. IP* and Central’ Set Near 0

In the following discussion, we will extend Theorem 1.2 for a dense subsemigroup of \((0, \infty), +)\) in the appropriate context.

**Definition 2.1.** Let \( S \) be a dense subsemigroup of \((0, \infty), +)\). A subset \( A \) of \( S \) is said to be an IP set near 0 if there exists a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) such that \( \sum_{n=1}^{\infty} x_n \) converges and such that \( FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A \). One calls a subset \( D \) of \( S \) an IP* set near 0 if for every subset \( C \) of \( S \) which is IP set near 0, \( C \cap D \) is IP set near 0.

From [6, Theorem 3.1] it follows that for a dense subsemigroup \( S \) of \((0, \infty), +)\) a subset \( A \) of \( S \) is an IP set near 0 if and only if there exists some idempotent \( p \in 0^+(S) \) with \( A \subseteq p \). Further it can be easily observed that a subset \( D \) of \( S \) is an IP* set near 0 if and only if it belongs to every idempotent of \( 0^+(S) \).

Given \( c \in \mathbb{R} \setminus \{0\} \) and \( p \in \beta \mathbb{R}_d \setminus \{0\} \), the product \( c \cdot p \) is defined in \( (\beta \mathbb{R}_d, \cdot) \). One has \( A \subseteq \mathbb{R} \) is a member of \( c \cdot p \) if and only if \( c^{-1} A = \{ x \in \mathbb{R} : c \cdot x \in A \} \) is a member of \( p \).

**Lemma 2.2.** Let \( S \) be a dense subsemigroup of \((0, \infty), +)\) such that \( S \cap (0, 1) \) is a subsemigroup of \((0, 1), \cdot)\). If \( A \) is an IP set near 0 in \( S \) then \( sA \) is also an IP set near 0 for every \( s \in S \cap (0, 1) \). Further if \( A \) is an IP* set near 0 in \( (S, \cdot) \) then \( s^{-1} A \) is also an IP* set near 0 for every \( s \in S \cap (0, 1) \).

**Proof.** Since \( A \) is an IP set near 0 then by [6, Theorem 3.1] there exists a sequence \( \langle x_n \rangle_{n=1}^{\infty} \) in \( S \) with the property that \( \sum_{n=1}^{\infty} x_n \) converges and \( FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A \). This implies that \( \sum_{n=1}^{\infty} (s \cdot x_n) \) is also convergent and \( FS(\langle sx_n \rangle_{n=1}^{\infty}) \subseteq sA \). This proves that \( sA \) is also an IP* set near 0.

For the second let \( A \) be an IP* set near 0 and \( s \in S \cap (0, 1) \). To prove that \( s^{-1} A \) is an IP* set near 0 it is sufficient to show that if \( B \) is any IP set near 0 then \( B \cap s^{-1} A \neq \emptyset \). Since \( B \)
Theorem 2.3. Let $idempotent p$\, be an IP set near 0, $sB$ is also an IP set near 0 by the first part of the proof, so that $A \cap sB \neq \emptyset$. Choose $t \in sB \cap A$ and $k \in B$ such that $t = sk$. Therefore $k \in s^{-1}A$ so that $B \cap s^{-1}A \neq \emptyset$. \hfill \Box

Given $A \subseteq S$ and $s \in S$, $s^{-1}A = \{t \in S : st \in A\}$, and $-s + A = \{t \in S : s + t \in A\}$.

Theorem 2.3. Let $S$ be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$. Also let $(x_n)_{n=1}^{\infty}$ be a sequence in $S$ such that $\sum_{n=1}^{\infty} x_n$ converges and let $A$ be a IP set near 0 in $S$. Then there exists a sum subsystem $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ such that
\begin{equation}
FS((y_n)_{n=1}^{\infty}) \cup FP((y_n)_{n=1}^{\infty}) \subseteq A.
\end{equation}

Proof. Since $\sum_{n=1}^{\infty} x_n$ converges, from [6, Theorem 3.1] it follows that we can find some idempotent $p \in 0^*(S)$ for which $FS((x_n)_{n=1}^{\infty}) \subseteq p$. In fact $T = \bigcap_{m=1}^{\infty} c_{x_{\beta_s}} FS((y_n)_{n=m}^{\infty}) \subseteq 0^*(S)$ and $p \in T$. Again, since $A$ is a IP set near 0 in $S$, by Lemma 2.2 for every $s \in S \cap (0, 1)$, $s^{-1}A \in p$. Let $A^* = \{s \in A : -s + A \in p\}$. Then by [1, Lemma 4.14] $A^* \in p$. We can choose $y_1 \in A^* \cap FS((x_n)_{n=1}^{\infty})$. Inductively let $m \in \mathbb{N}$ and $(y_i)_{i=1}^{m}$, $(H_i)_{i=1}^{m}$ in $P_f(\mathbb{N})$ be chosen with the following properties:
\begin{enumerate}
\item $i \in \{1, 2, \ldots, m-1\}$ max $H_i < \min H_{i+1};$
\item if $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_{m}} x_t \in A^*$ and $FP((y_i)_{i=1}^{m}) \subseteq A$.
\end{enumerate}

We observe that $\{\sum_{t \in H} x_t : H \in P_f(\mathbb{N}), \min H > \max H_m\} \in p$. Let $B = \{\sum_{t \in H} x_t : H \in P_f(\mathbb{N}), \min H > \max H_m\}$, let $E_1 = FS((y_i)_{i=1}^{m})$ and $E_2 = FP((y_i)_{i=1}^{m})$. Now consider
\begin{equation}
D = B \cap A^* \cap \bigcap_{s \in E_1} (-s + A^*) \cap \bigcap_{s \in E_2} (s^{-1}A^*).
\end{equation}

Then $D \in p$. Now choose $y_{m+1} \in D$ and $H_{m+1} \in P_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$. Putting $y_{m+1} = \sum_{t \in H_{m+1}} x_t$ shows that the induction can be continued and proves the theorem. \hfill \Box

If we turn our attention to central* sets then the above result holds for a restricted class of sequences which we call minimal sequence near 0.

Definition 2.4. Let $S$ be a dense subsemigroup of $((0, \infty), +)$. A sequence $(x_n)_{n=1}^{\infty}$ in $S$ is said to be a minimal sequence near 0 if
\begin{equation}
\bigcap_{m=1}^{\infty} FS((x_n)_{n=m}^{\infty}) \cap K(0^+(S)) \neq \emptyset.
\end{equation}

The notion of piecewise syndetic set near 0 was first introduced in [6].

Definition 2.5. For a dense subsemigroup $S$ of $((0, \infty), +)$, a subset $A$ of $S$ is piecewise syndetic near 0 if and only if $c_{x_{\beta_s}} A \cap K(0^+(S)) \neq \emptyset$.

The following theorem characterizes minimal sequences near 0 in terms of piecewise syndetic set near 0.
Theorem 2.6. Let $S$ be a dense subsemigroup of $((0, \infty), +)$. Then the following conditions are equivalent:

(a) $(x_n)_{n=1}^\infty$ is a minimal sequence near 0.

(b) $FS((x_n)_{n=1}^\infty)$ is piecewise syndetic near 0.

(c) There is an idempotent in $\bigcap_{m=1}^\infty FS((x_n)_{n=m}^\infty) \cap K(0^+(S)) \neq \emptyset$.

Proof. (a) $\Rightarrow$ (b) follows from (see [6, Theorem 3.5]).

To prove that (b) implies (a) let us consider that $FS((x_n)_{n=1}^\infty)$ be a piecewise syndetic near 0. Then there exists a minimal left ideal $L$ of $0^+(S)$ such that $L \cap FS((x_n)_{n=1}^\infty) \neq \emptyset$. We choose $q \in L \cap FS((x_n)_{n=1}^\infty)$. By [6, Theorem 3.1], $\bigcap_{m=1}^\infty cL_{\beta S_n}FS((x_n)_{n=m}^\infty)$ is a subsemigroup of $0^+(S)$, so it suffices to show that for each $m \in \mathbb{N}$, $L \cap FS((x_n)_{n=m}^\infty) \neq \emptyset$. In fact minimal left ideals being closed, we can conclude that $L \cap \bigcap_{m=1}^\infty FS((x_n)_{n=m}^\infty) \neq \emptyset$ and so $L \cap \bigcap_{m=1}^\infty FS((x_n)_{n=m}^\infty)$ is a compact right topological semigroup so that it contains idempotents. To this end, let $m \in \mathbb{N}$ with $m > 1$. Then $FS((x_n)_{n=1}^\infty) = FS((x_n)_{n=m}^\infty) \cup FS((x_n)_{n=m}^{m-1}) \cup \{t + FS((x_n)_{n=m}^\infty) : t \in FS((x_n)_{n=m}^{m-1})\}$. So we must have one of the following:

(i) $FS((x_n)_{n=m}^\infty) \in q$,

(ii) $FS((x_n)_{n=m}^{m-1}) \in q$.

(iii) $t + FS((x_n)_{n=m}^\infty) \in q$ for some $t \in FS((x_n)_{n=1}^{m-1})$.

Clearly (ii) does not hold, because in that case $q$ becomes a member of $S$ while it is a member of minimal left ideal. If (iii) holds then we have $t + FS((x_n)_{n=m}^\infty) \in q$ for some $t \in FS((x_n)_{n=1}^{m-1})$. Since $q \in 0^+(S)$, we have $(0, t) \cap S \in q$. But $(0, t) \cap (t + FS((x_n)_{n=m}^\infty)) = \emptyset$, a contradiction. Hence (i) must hold so that $q \in L \cap FS((x_n)_{n=m}^\infty)$.

(a) $\iff$ (c) is obvious. \hfill $\Box$

Let us recall following lemma for our purpose.

Lemma 2.7. Let $S$ be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $y \in S \cap (0, 1)$ and each $x \in S$, $x/y \in S$ and $yx \in S$. If $A \subseteq S$ and $y^{-1}A$ is a central set near 0, then $A$ is also a central set near 0.

Proof. See [6, Lemma 4.8]. \hfill $\Box$

Lemma 2.8. Let $S$ be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $s \in S \cap (0, 1)$ and each $t \in S$, $t/s \in S$ and $st \in S$. If $A$ is central set near 0 in $S$ then $sA$ is also central set near 0.

Proof. Since $s^{-1}(sA) = A$ and $A$ is central set near 0 then by Lemma 2.7, $sA$ is central set near 0. \hfill $\Box$

Lemma 2.9. Let $S$ be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $s \in S \cap (0, 1)$ and each $t \in S$, $t/s \in S$ and $st \in S$. If $A$ is a central set near 0 in $S$ then $s^{-1}A$ is also central set near 0.

Proof. Let $A$ be a central set near 0 and $s \in S \cap (0, 1)$. To prove that $s^{-1}A$ is a central set near 0 it is sufficient to show that for any central set near 0 $C$, $C \cap s^{-1}A \neq \emptyset$. Since $C$ is central set
near 0, \( sC \) is also central set near 0 so that \( A \cap sC \neq \emptyset \). Choose \( t \in sC \cap A \) and \( k \in C \) such that \( t = sk \). Therefore \( k \in s^{-1}A \) so that \( C \cap s^{-1}A \neq \emptyset \).

We end this paper by following generalization of Theorem 2.3, whose proof is also straightforward generalization of Theorem 2.3 and hence omitted.

**Theorem 2.10.** Let \( S \) be a dense subsemigroup of \( ((0, \infty), +) \) such that \( S \cap (0,1) \) is a subsemigroup of \( ((0,1), \cdot) \) and assume that for each \( s \in S \cap (0,1) \) and each \( t \in S \), \( t/s \in S \) and \( st \in S \). Also let \( \langle x_n \rangle_{n=1}^{\infty} \) be a minimal sequence near 0 and let \( A \) be a central set near 0 in \( S \). Then there exists a sum subsystem \( \langle y_n \rangle_{n=1}^{\infty} \) of \( \langle x_n \rangle_{n=1}^{\infty} \) such that

\[
FS(\langle y_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A. \tag{2.4}
\]

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