Some Integral Formulas for the \((r + 1)\)th Mean Curvature of a Closed Hypersurface

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By using the operator \(L_r\), we define the notions of \(r\)th order and \(r\)th type of a Euclidean hypersurface. By the use of these notions, we are able to obtain some sharp estimates of the \((r + 1)\)th mean curvature for a closed hypersurface of the Euclidean space in terms of \(r\)th order.

1. Introduction

The study of submanifolds of finite type began in 1970, with Chen’s attempts to find the best possible estimate of the total mean curvature of a compact submanifold of the Euclidean space and to find a notion of “degree” for Euclidean submanifolds [1, 2].

In algebraic geometry varieties are the main objects to study. Since an algebraic variety is defined by using algebraic equations, one can define the degree of an algebraic variety by its algebraic structure, and it is well known that the concept of degree plays a fundamental role in algebraic geometry [3].

On the other hand, in differential geometry, the main objects to study are Riemannian (sub) manifolds. According to Nash’s immersion Theorem, every Riemannian manifold can be realized as a submanifold of the Euclidean space via an isometric immersion [4], but there is no notion of degree for submanifolds of the Euclidean space in general.

So inspired by algebraic geometry, in 1970, Chen defined the notions of order and type for submanifolds of the Euclidean space by the use of the Laplace operator. After that, Chen was able to obtain some sharp estimates of the total mean curvature for compact submanifolds of the Euclidean space in terms of their orders. Moreover, he could introduce submanifolds and maps of finite type [1, 2].

On one hand finite-type submanifolds provides a natural way to exploit the spectral theory to study the geometry of submanifolds and smooth maps, in particular the Gauss...
map. On the other hand, the techniques of the submanifold theory can be used in the study of spectral geometry via the study of finite-type submanifolds.

As is well known, the Laplace operator of a hypersurface $M$ immersed into $\mathbb{R}^{n+1}$ is an (intrinsic) second-order linear elliptic differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. From this point of view, the Laplace operator $\Delta$ can be seen as the first one of a sequence of $n$ operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where $L_r$ stands for the linearized operator of the first variation of the $(r+1)$th mean curvature arising from normal variations of the hypersurface (see [5]). These operators are given by $L_r(f) = tr(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where $P_r$ denotes the $r$th Newton transformation associated to the second fundamental form of the hypersurface, and $\nabla^2 f$ is the hessian of $f$ (see the next section for details).

In contrast to the operator $\Delta$, the operators $L_r$ are not elliptic in general, but they still share some nice properties with Laplacian of $M$; moreover, under appropriate natural geometric hypotheses on the hypersurface, they are elliptic [6]. Therefore, from this point of view, it seems natural and interesting to generalize the definition of finite-type hypersurface by replacing $\Delta$ by the operator $L_r$. Having this idea, for the first time in [7], the second author, inspired by a private communication with Ali, introduced such hypersurfaces and called them “$L_r$-finite type” hypersurfaces.

In this paper, by using the operator $L_r$, we define the notions of $r$th order and $r$th type of a Euclidean hypersurface. Then we are able to obtain some sharp estimates of the $(r+1)$th mean curvature for closed hypersurfaces of the Euclidean space in terms of their $r$th orders when $L_r$ is elliptic. The paper generalizes the results of [8, 9].

2. Preliminaries

In this section, we recall some prerequisites about Newton transformations $P_r$ and their associated second-order differential operators $L_r$ from [10].

Consider an orientable isometrically immersed hypersurface $x : M^n \rightarrow \mathbb{R}^{n+1}$ in the Euclidean space, with the Gauss map $N$. We denote by $\nabla^0$ and $\nabla$ the Levi-Civita connections on $\mathbb{R}^{n+1}$ and $M$, respectively. Then, the basic Gauss and Weingarten formulae of the hypersurface are written as

\[ \nabla_X^0 Y = \nabla_X Y + (SX, Y)N, \]
\[ SX = -\nabla_X^0 N \]  

for all tangent vector fields $X, Y \in \chi(M)$, where $S : \chi(M) \rightarrow \chi(M)$ is the shape operator of $M$ with respect to the Gauss map $N$. As is well known, $S$ defines a self-adjoint linear operator on tangent space $T_p M$, and its eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$ are called the principal curvatures of the hypersurface. Associated to the shape operator $S$, there are $n$ algebraic invariants given by

\[ s_r(p) = \sigma_r(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \leq r \leq n, \]  

where $\sigma_r : \mathbb{R}^n \rightarrow \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

\[ \sigma_r(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} x_{i_1} \cdots x_{i_r}. \]
Observe that the characteristic polynomial of $S$ can be written in terms of the $s_r$, $s$ as

$$Q_S(t) = \det(tI - S) = \sum_{r=0}^{n} (-1)^r s_r t^{n-r}, \quad (2.4)$$

where $s_0 = 1$ by definition. The $r$th mean curvature $H_r$ of the hypersurface is then defined by

$$\binom{n}{k} H_r = s_r, \quad 0 \leq r \leq n. \quad (2.5)$$

In particular, when $r = 1$,

$$H_1 = \frac{1}{n} \sum_{i=1}^{n} \kappa_i = \frac{1}{n} \text{tr}(S) = H \quad (2.6)$$

is nothing but the mean curvature of $M$, which is the main extrinsic curvature of the hypersurface. On the other hand, $H_n = \kappa_1 \cdots \kappa_n$ is called the Gauss-Kronecker curvature of $M$. A hypersurface with zero $(r+1)$th mean curvature in $\mathbb{R}^{n+1}$ is called $r$-minimal.

The classical Newton transformations $P_r : \chi(M) \rightarrow \chi(M)$ are defined inductively by

$$P_0 = I, \quad P_r = s_r I - S \circ P_{r-1} = \binom{n}{r} H_r I - S \circ P_{r-1} \quad (2.7)$$

for $r = 1, \ldots, n$ where $I$ denotes the identity of $\chi(M)$. Equivalently, we have

$$P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} S^j = \sum_{j=0}^{r} (-1)^j \binom{n}{r-j} H_{r-j} S^j. \quad (2.8)$$

Note that, by the Cayley-Hamilton Theorem stating that any operator $T$ is annihilated by its characteristic polynomial, we have $P_n = 0$ from (2.4).

Each $P_r(p)$ is also a self-adjoint linear operator on the tangent space $T_p M$ which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively. Then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{1 \leq i_1 < \cdots < i_r \leq n, i_j \neq i} \kappa_{i_1}(p) \cdots \kappa_{i_r}(p), \quad (2.9)$$
for every \( r, 1 \leq i \leq n \). We have the following formulas for the Newton transformation \( P_r \) [5]:

\[
\text{tr}(P_r) = c_r H_r, \quad (2.10)
\]

\[
\text{tr}(S \circ P_r) = c_r H_{r+1}, \quad (2.11)
\]

\[
\text{tr}(S^2 \circ P_r) = \left( \frac{n}{r+1} \right) (nH_1 H_{r+1} - (n - r - 1) H_{r+2}), \quad (2.12)
\]

where

\[
c_r = (n - r) \binom{n}{r} = (r + 1) \binom{n}{r+1}. \quad (2.13)
\]

Associated to each Newton transformation \( P_r \), we consider the second-order linear differential operator \( L_r : C^\infty(M) \to C^\infty(M) \) given by

\[
L_r(f) = \text{tr}(P_r \circ \nabla^2 f). \quad (2.14)
\]

\( \nabla^2 f : \chi(M) \to \chi(M) \) denotes the self-adjoint linear operator metrically equivalent to the Hessian of \( f \) and is given by

\[
\left\langle \nabla^2 f(X), Y \right\rangle = \left\langle \nabla_X (\nabla f), Y \right\rangle, \quad \forall X, Y \in \chi(M). \quad (2.15)
\]

Let \( \{e_1, \ldots, e_n\} \) be a local orthonormal frame on \( M \) and observe that

\[
\text{div}(P_r(\nabla f)) = \sum_{i=1}^{n} \left\langle (\nabla_{e_i} P_r)(\nabla f), e_i \right\rangle + \sum_{i=1}^{n} \left\langle P_r(\nabla_{e_i} \nabla f), e_i \right\rangle
\]

\[
= (\text{div} P_r \nabla f) + L_r(f), \quad (2.16)
\]

where \( \text{div} \) denotes the divergence operator on \( M \).

Since \( \text{div} P_r = 0 \) (see [10]), as a consequence, from (2.16), one gets that

\[
L_r(f) = \text{div}(P_r(\nabla f)). \quad (2.17)
\]

3. The \( r \)th Order and the \( r \)th Type of a Hypersurface

As mentioned in the introduction, there is no notion of degree for submanifolds of the Euclidean space in general. However, Chen could use the induced Riemannian structure on a submanifold to introduce a pair of well-defined numbers \( p \) and \( q \) associated with a submanifold (see [1] for the precise definition). \( p \) is a natural number and \( q \geq p \) or \( +\infty \). The pair \( [p, q] \) is called the order of the submanifold \( M \); more precisely, \( p \) is the lower order, and \( q \) is the upper order of the submanifold. The submanifold is said to be of finite type if its upper order is finite, and it is of infinite type if its upper order is \( +\infty \) (see [1, 2] for details).
Consider an isometrically immersed closed orientable hypersurface \( x: M^n \to \mathbb{R}^{n+1} \) in the Euclidean space, with the Gauss map \( N \), and assume that, for a fixed \( r \), \( 1 \leq r \leq n-1 \), \( L_r \) is an elliptic differential operator on \( C^\infty(M) \), the ring of all smooth real functions on \( M \).

It is well known that the eigenvalues of \( -L_r \) when it is elliptic (see [1], chapter 3, for properties of an elliptic operator) form a discrete infinite sequence

\[
0 = \lambda_{0}^{L_r} < \lambda_{1}^{L_r} < \lambda_{2}^{L_r} < \cdots \nearrow \infty.
\]  

Let \( V^L_k = \{ f \in C^\infty(M) : L_rf + \lambda_k f = 0 \} \) be the eigenspace of \( -L_r \) with eigenvalue \( \lambda_k^{L_r} \). Then \( V^L_k \) is finite dimensional. Define an inner product \( (, ) \) on \( C^\infty(M) \) by

\[
(f, h) = \int_M fhdM,
\]

where \( dM \) is the volume element of \( M \). Then \( \sum_{k=0}^{\infty} V^L_k \) is dense in \( C^\infty(M) \) (in \( L^2 \) sense). If we denote by \( \hat\oplus V^L_k \) the completion of \( \sum V^L_k \), we have

\[
C^\infty(M) = \hat\oplus V^L_k.
\]

For each function \( f \in C^\infty(M) \), let \( f_t \) denote the projection of \( f \) onto the subspace \( V^L_t \) \( (t = 0, 1, 2, \ldots) \). Then we have the following spectral decomposition:

\[
f = \sum_{t=0}^{\infty} f_t \quad \text{(in } L^2 \text{ sense)}.
\]

Because \( V^L_0 \) is 1-dimensional, for any nonconstant function \( f \in C^\infty(M) \), there is a positive integer \( p \geq 1 \) such that \( f_p \neq 0 \) and

\[
f - f_0 = \sum_{t=0}^{p} f_t,
\]

where \( f_0 \in V^L_0 \) is a constant. If there are infinitely many \( f_t \)'s which are nonzero, we put \( q = +\infty \); otherwise, there is an integer \( q, q \geq p \), such that \( f_q \neq 0 \) and

\[
f - f_0 = \sum_{t=p}^{q} f_t.
\]

Consider the set

\[
T^L_f = \{ t \in N : f_t \neq 0 \}.
\]

The smallest element of \( T^L_f \) is called the lower \( r \)th order of \( f \) and is denoted by \( l.o.L^r(f) \), and the supremum of \( T^L_f \) is called the upper \( r \)th order of \( f \) and is denoted by \( u.o.L^r(f) \). A
function \( f \) in \( C^\infty(M) \) is said to be of \( L_r \)-finite type if \( T^L_r f \) is a finite set, that is, if its spectral decomposition contains only finitely many nonzero terms. Otherwise, \( f \) is said to be of \( L_r \)-infinite type. \( f \) is said to be of \( L_r \)-\( k \) type if \( T^L_r f \) contains exactly \( k \) elements.

For an isometrically immersed closed hypersurface \( x : M^n \to \mathbb{R}^{n+1} \) in the Euclidean space \( \mathbb{R}^{n+1} \), we put

\[
x = (x_1, \ldots, x_{n+1}),
\]

where \( x_A \) is the \( A \)-th component of \( x \). For each \( x_A \), we have

\[
x_A - (x_A)_0 = \sum_{t \in p_A} (x_A)_t, \quad A = 1, \ldots, n + 1.
\]

For the isometric immersion \( x : M \to \mathbb{R}^{n+1} \), we put

\[
p = \inf_A \{ p_A \}, \quad q = \sup_A \{ q_A \}.
\]

It is easy to see that \( p \) and \( q \) are independent of the choice of the Euclidean coordinate system on \( \mathbb{R}^{n+1} \), and \( p \) is a positive integer and \( q \) is either \( +\infty \) or \( q \geq p \). Thus, \( p \) and \( q \) are well defined. Consequently, for each closed hypersurface \( M \) in \( \mathbb{R}^{n+1} \) (or, more precisely, for each isometric immersion \( x : M^n \to \mathbb{R}^{n+1} \)), we have a pair \([p, q]\) associated with \( M \). We call the pair \([p, q]\) the order of the hypersurface \( M \).

By using the above notation, we have the following spectral decomposition of \( x \) in vector form:

\[
x = x_0 + \sum_{t=p}^{q} x_t.
\]

We define \( T^L_r(x) \) by

\[
T^L_r(x) = \{ t \in \mathbb{N} : x_t \neq 0 \}.
\]

The immersion \( x \) or the hypersurface \( M \) is said to be of \( L_r \)-\( k \) type if \( T^L_r(x) \) contains exactly \( k \) elements. Similarly, we can define the lower \( r \)th order and the upper \( r \)th order of the immersion.

The immersion \( x \) is said to be of \( L_r \) finite type if its upper \( r \)th order \( q \) is finite, and it is said to be of \( L_r \) infinite type if its upper order is \( +\infty \).

The following Lemma states that for an isometrically immersed closed orientable hypersurface \( x : M^n \to \mathbb{R}^{n+1} \), the constant vector \( x_0 \) in (3.11) is exactly the “center of mass” of \( M \) in \( \mathbb{R}^{n+1} \) (i.e., \( \int_M x \, dM / \text{vol}(M) \), where \( dM \) is a chosen volume form of \( M \)).

**Lemma 3.1.** Let \( x : M^n \to \mathbb{R}^{n+1} \) be an isometric immersion of a closed orientable hypersurface \( M \) into \( \mathbb{R}^{n+1} \). Assume that \( L_r \) is elliptic, for some \( 1 \leq r \leq n - 1 \). Then \( x_0 \) in (3.11) is the center of mass of \( M \) in \( \mathbb{R}^{n+1} \).
Proof. Consider the decomposition
\begin{equation}
  x = \sum_{t=0}^{\infty} x_t. \tag{3.13}
\end{equation}

We have \( L_r x_t + \lambda^L x_t = 0 \). If \( t \neq 0 \), then (2.17) and the Divergence Theorem imply that
\begin{equation}
  \int_M x_t dM = -\frac{1}{\lambda^L_t} L_r x_t dM = 0. \tag{3.14}
\end{equation}

Since \( x_0 \) is a constant vector in \( \mathbb{R}^{n+1} \), we obtain from (3.13) and (3.14) that
\begin{equation}
  x_0 = \int_M \frac{x dM}{\text{vol}(M)}. \tag{3.15}
\end{equation}

This shows that \( x_0 \) is the center of mass of \( M \). \( \square \)

On the set of all \( \mathbb{R}^{n+1} \)-valued functions on \( M \) which is a real vector space, we define an inner product on such space by
\begin{equation}
  (v, w) = \int_M (v, w) dM \tag{3.16}
\end{equation}

for any two \( \mathbb{R}^{n+1} \)-valued functions \( v, w \) on \( M \), where \( (v, w)(x) \) denotes the Euclidean inner product of \( v(x), w(x) \) for any \( x \in M \). Then we have the following lemma.

**Lemma 3.2.** For an isometric immersion \( x: M \rightarrow \mathbb{R}^{n+1} \) of a closed orientable hypersurface \( M \) into \( \mathbb{R}^{n+1} \) the components of the spectral decomposition (3.11) are mutually orthogonal, for example,
\begin{equation}
  (x_t, x_s) = 0, \quad t \neq s. \tag{3.17}
\end{equation}

**Proof.** Since \( L_r \) is self-adjoint with respect to the inner product (3.16), we have
\begin{equation}
  \lambda^L_t (x_t, x_s) = -(L_r x_t, x_s) = -(x_t, L_r x_s) = \lambda^L_s (x_t, x_s). \tag{3.18}
\end{equation}

Since \( \lambda^L_t \neq \lambda^L_s \), we obtain (3.17). \( \square \)

Before we give our main result and to facilitate the reader, we quote Theorem 3.3 from [11] and present Theorem 3.4 about \( L_r \)-finite-type Euclidean hypersurfaces.

**Theorem 3.3** (see [11]). *Let \( x: M^n \rightarrow \mathbb{R}^{n+1} \) be an orientable connected hypersurface immersed into the Euclidean space, and let \( L_r \) be the linearized operator of the \( (r+1) \)-th mean curvature of \( M \), for some \( r = 0, \ldots, n-1 \). Then, one has*
\begin{equation}
  L_r x + \lambda x = 0 \tag{3.19}
\end{equation}
for a real constant $\lambda$ if and only if either $\lambda = 0$ and $M$ is $r$-minimal in $\mathbb{R}^{n+1}$ (i.e., $H_{r+1} = 0$ on $M$), or $\lambda \neq 0$ and $M$ is an open piece of a round sphere $S^n(q) \subset \mathbb{R}^{n+1}$ of radius $q = (c_r/\|\lambda\|)^{1/(r+2)}$ centered at the origin of $\mathbb{R}^{n+1}$, where $c_r = (n-r)(\frac{n}{r})$.

**Theorem 3.4.** There is no compact Euclidean hypersurface of $L_r$-2 type with constant $H_{r+1}$, when $L_r$ is elliptic.

**Proof.** If $M$ is of $L_r$-2 type, by using (3.11), the position vector field $x$ of $M$ in $\mathbb{R}^{n+1}$ has the following spectral decomposition:

$$x - x_0 = x_p + x_q, \quad L_rx_p + \lambda_px_p = 0, \quad L_rx_q + \lambda_qx_q = 0, \quad \text{for some } x_0 \in \mathbb{R}^{n+1}, \ p, q \in \mathbb{N},$$

so

$$L_r^2x = -(\lambda_p + \lambda_q)L_rx - \lambda_p\lambda_q(x - x_0). \quad (3.21)$$

From [10], we also have

$$L_rx = c_rH_{r+1}N, \quad (3.22)$$

$$L_r^2x = -c_r\left(\frac{n}{r + 1}\right)H_{r+1}(nH_1H_{r+1} - (n - r - 1)H_{r+2})N. \quad (3.23)$$

The formula (3.23) holds since $H_{r+1}$ is a nonzero constant, see [10]. Therefore, by using (3.21), (3.22), and (3.23), we obtain that

$$-c_r\left(\frac{n}{r + 1}\right)H_{r+1}(nH_1H_{r+1} - (n - r - 1)H_{r+2})N = c_r(\lambda_p + \lambda_q)H_{r+1}N - \lambda_p\lambda_q(x - x_0). \quad (3.24)$$

Since $\lambda_p\lambda_q \neq 0$, from (3.24), we have $x - x_0$ that is normal to $M$ at every point of $M$. So $(x - x_0, x - x_0)$ is a positive constant. In this case, $M$ is an open piece of $S^n$ centered at $x_0$, by Theorem 3.3, $M$ is of $L_r$-1 type, which is not. \qed

### 4. The $r$th Order and the $(r + 1)$th Mean Curvature

In this section, we will relate the notion of the $r$th order of a Euclidean hypersurface with the $(r + 1)$th mean curvature. In particular, we will obtain some sharp estimates of the $(r + 1)$th mean curvature for a closed hypersurface of the Euclidean space in terms of $r$th order of the hypersurface when $L_r$ is elliptic. In the following we will state several results from [11, 12] which guarantee the ellipticity of $L_r$.

A classical theorem of Hadamard [12] gives three equivalent conditions on a closed connected hypersurface $M^n$ immersed into the Euclidean space $\mathbb{R}^{n+1}$ which imply that $M$ is a convex hypersurface (i.e., $M$ is embedded in $\mathbb{R}^{n+1}$ and is the boundary of a convex body).
Theorem 4.1 (Hadamard Theorem, see [12]). Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed connected hypersurface immersed into the Euclidean space. The following assertions are equivalent.

(i) The second fundamental form is definite at every point of \( M \).

(ii) \( M \) is orientable, and its Gauss map is a diffeomorphism onto \( S^n \).

(iii) The Gauss-Kronecker curvature never vanishes on \( M \).

Moreover, any of the above conditions implies that \( M \) is a convex hypersurface.

Here we observe that the convexity of a hypersurface in \( \mathbb{R}^{n+1} \) is closely related to its Ricci curvature.

Theorem 4.2 (see [11]). Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed connected hypersurface immersed into the Euclidean space. The following assertion is equivalent to any of the assertions ((i)–(iii)) in Hadamard theorem, and therefore it also implies that \( M \) is a convex hypersurface

(iv) The Ricci curvature of \( M \) is positive everywhere on \( M \).

Corollary 4.3 (see [11]). Let \( x : M^n \to \mathbb{R}^{n+1} \) be a closed connected hypersurface isometrically immersed into the Euclidean space. If \( M \) has positive Ricci curvature, then each operator \( L_r \) on \( C^\infty (M) \) is elliptic, and each \( r \)-th mean curvatures of \( M \) is positive.

In [8] by using the concept of order, Chen obtained the following best possible lower bound of total mean curvature for a closed Euclidean hypersurface.

Theorem 4.4 (see [8]). Let \( x : M \to \mathbb{R}^{n+1} \) be an orientable closed connected hypersurface isometrically immersed into the Euclidean space. Then, one has

\[
\int_M H^2 \, dM \geq \frac{\lambda_p}{n} \text{vol} (M),
\]

where \( p \) is the lower order of \( M \), and \( \text{vol} (M) \) denotes the \( n \)-dimensional volume of \( M \). Equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).

Now we establish the corresponding result for the operator \( L_r \) (since \( L_0 = \Delta \), taking \( r = 0 \), we recover Theorem 4.4).

Theorem 4.5. Let \( x : M \to \mathbb{R}^{n+1} \) be an orientable closed connected hypersurface isometrically immersed into the Euclidean space. Assume that \( L_r \) is elliptic, for some \( 1 \leq r \leq n-1 \). Then, one has

\[
\int_M H_r^2 \, dM \geq \frac{\lambda_r}{C_r} \int_M H_r \, dM,
\]
and equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \). In particular, if \( M \) is embedded in \( \mathbb{R}^{n+1} \), then

\[
\int_M H_{r,1}^2 \ dM \geq (n+1)^2 \left( \frac{\lambda_1^{t_r}}{c_r} \right)^2 \frac{\text{vol}(\Omega)^2}{\text{vol}(M)},
\]

where equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \). Here, \( \text{vol}(M) \) denotes the \( n \)-dimensional volume of \( M \), and \( \Omega \) is the compact domain in \( \mathbb{R}^{n+1} \) bounded by \( M \), and \( \text{vol}(\Omega) \) denotes its \((n+1)\)-dimensional volume of \( \Omega \).

**Proof.** We will follow the techniques introduced by Chen (Theorem 4.4) in our context, we generalize some properties of \( \Delta \) and \( H \), respectively, to \( L_r \) and \( H_{r,1} \). Since \( L_r x = c_r H_{r,1} N \) (see [10]), and

\[
x = x_0 + \sum_{t=p}^{q} x_t, \quad L_r x_t + \lambda_{t}^{L_r} x_t = 0,
\]

by using the inner product on the set of all \( \mathbb{R}^{n+1} \)-valued functions on \( M \), defined by (3.16), we have

\[
(c_r)^2 \int_M H_{r,1}^2 \ dM = (c_r)^2 (H_{r,1} N, H_{r,1} N) = (L_r x, L_r x) = \sum_{t=p}^{q} \left( \lambda_{t}^{L_r} \right)^2 ||x_t||^2.
\]

Since, by Lemma 3.1, \( x_0 \) is the center of mass of \( M \), we have the well-known Minkowski formula [13] as follows:

\[
\int (H_r + H_{r,1}(x - x_0, N)) \ dM = 0, \quad r = 0, \ldots, n-1,
\]

so we get that

\[
(c_r) \int_M H_r \ dM = -(c_r)(x - x_0, H_{r,1} N) = -(x - x_0, L_r x) = \sum_{t=p}^{q} \lambda_{t}^{L_r} ||x_t||^2.
\]

Thus, by (4.5) and (4.7), we find that

\[
(c_r)^2 \int_M H_{r,1}^2 \ dM - (c_r) \lambda_p^{L_r} \int_M H_r \ dM = \sum_{t=p+1}^{q} \lambda_{t}^{L_r} (\lambda_{t}^{L_r} - \lambda_p^{L_r}) ||x_t||^2 \geq 0.
\]

Therefore, we obtain (4.2). Moreover, equality holds if and only if

\[
L_r x + \lambda_{p}^{L_r} x = 0,
\]
Which, by Theorem 3.3, means that \( M \) is a round sphere. If \( M \) is embedded in \( \mathbb{R}^{n+1} \), by formula (15) of [11], we have

\[
\lambda^L_r \leq \frac{c_r}{(n+1)^2 \text{vol}(\Omega)^2} \int_M H_r\,dM,
\]

where \( \text{vol}(M) \) denotes the \( n \)-dimensional volume of \( M \), and \( \Omega \) is the compact domain in \( \mathbb{R}^{n+1} \) bounded by \( M \), and \( \text{vol}(\Omega) \) denotes its \((n+1)\)-dimensional volume of \( \Omega \). So (4.3) is obtained by (4.2) and (4.10) easily.

By applying Theorem 4.2, Corollary 4.3, and Theorem 4.5 to positively Ricci curved hypersurfaces in \( \mathbb{R}^{n+1} \), we have the following Corollary.

**Corollary 4.6.** Let \( x : M \to \mathbb{R}^{n+1} \) be a closed connected hypersurface of the Euclidean space with positive Ricci curvature, and let \( \Omega \) be the convex body in \( \mathbb{R}^{n+1} \) bounded by \( M \). Then for every \( r = 1, \ldots, n-1 \), it follows that

\[
\int_M H^{2r+1}_r\,dM \geq \frac{\lambda^L_r}{c_r} \int_M H_r\,dM,
\]

where \( q \) is the upper order of \( M \), and \( \text{vol}(M) \) denotes the \( n \)-dimensional volume of \( M \). Equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).

By using the concept of order Chen in [9], we obtained the following best possible upper bound of total mean curvature for closed Euclidean hypersurface.

**Theorem 4.7 (see [9]).** Let \( x : M \to \mathbb{R}^{n+1} \) be an orientable closed connected isometrically immersed hypersurface into the Euclidean space. Then, one has

\[
\int_M H^2\,dM \leq \frac{\lambda^L_q}{n} \text{vol}(M),
\]

where \( q \) is the upper order of \( M \), and \( \text{vol}(M) \) denotes the \( n \)-dimensional volume of \( M \). Equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).

Now we establish the corresponding result for the operator \( L_r \) (since \( L_0 = \Delta \), taking \( r = 0 \), we recover Theorem 4.7).

**Theorem 4.8.** Let \( x : M \to \mathbb{R}^{n+1} \) be an orientable closed connected isometrically immersed hypersurface of the Euclidean space. Assume that \( L_r \) is elliptic on \( M \), for some \( 1 \leq r \leq n-1 \). Then, one has

\[
\int_M H^{2r+1}_r\,dM \leq \frac{\lambda^L_q}{c_r} \int_M H_r\,dM,
\]

and equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).
Proof. We will follow the techniques introduced by Chen (Theorem 4.7) in our context, we generalize some properties of $\Delta$ and $H$, respectively, to $L_r$ and $H_{r+1}$. Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface in the Euclidean space with Gauss map $N$. From [10], we have

$$L_r(H_{r+1}N) = -\binom{n}{r+1}H_{r+1} \nabla H_{r+1} - 2(S \circ P_r)(\nabla H_{r+1}) + \left((L_rH_{r+1})N - H_{r+1}tr(S^2 \circ P_r)N\right).$$  \hfill (4.14)

Formula (4.14) implies that

$$\langle L_r(H_{r+1}N), H_{r+1}N \rangle = H_{r+1}L_r H_{r+1} - H_{r+1}^2 tr(S^2 \circ P_r).$$  \hfill (4.15)

Furthermore, from (4.4), (4.5), and (4.7), we have

$$c_r^2 \int_M H_{r+1}^2 \ dM = \sum_{t=p}^q \left(\frac{1}{\lambda_i^r}\right)^2 \|x_i\|^2,$$

$$c_r^2 \int_M \langle L_r(H_{r+1}N), H_{r+1}N \rangle \ dM = -\sum_{t=p}^q \left(\frac{1}{\lambda_i^r}\right)^3 \|x_i\|^2,$$  \hfill (4.16)

$$c_r \int_M H_r \ dM = \sum_{t=p}^q \lambda_i^r \|x_i\|^2.$$

Assume that $q \leq \infty$. We put

$$\Lambda = -c_r^2 \int_M \langle L_r(H_{r+1}N), H_{r+1}N \rangle \ dM - c_r^2 \left(\frac{1}{\lambda_p^r} + \frac{1}{\lambda_q^r}\right) \int_M H_{r+1}^2 \ dM$$

$$+ c_r \lambda_p^r \lambda_q^r \int_M H_r \ dM.$$  \hfill (4.17)

Then we have

$$\Lambda = \sum_{t=p+1}^q \left(\lambda_i^L - \lambda_p^L\right)\left(\lambda_i^L - \lambda_q^L\right) \|x_i\|^2 \leq 0,$$  \hfill (4.18)

where equality holds if and only if $M$ is either of $L_r$-1 type or of $L_r$-2 type.

Combining (4.15), (4.17), and (4.18), we find that

$$-c_r^2 \int_M H_{r+1}L_r H_{r+1} \ dM + c_r^2 \int_M H_{r+1}^2 tr(S^2 \circ P_r) \ dM - c_r^2 \left(\frac{1}{\lambda_p^L} + \frac{1}{\lambda_q^L}\right) \int_M H_{r+1}^2 \ dM$$

$$+ c_r \lambda_p^L \lambda_q^L \int_M H_r \ dM \leq 0.$$  \hfill (4.19)
By Proposition 3.1 of [14], we have the following equation:

\[- \int_M H_{r+1} L_r H_{r+1} dM = \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1} \rangle dM. \tag{4.20}\]

Since \(L_r\) is elliptic, it follows from (2.16) that the Newton transformation \(P_r\) is positive definite, so (4.20) implies that \(- \int_M H_{r+1} L_r H_{r+1} dM \geq 0\). On the other hand, by using (2.12), we have

\[H_{r+1}^2 tr \left( S^2 \circ P_r \right) = \frac{H_{r+1}^4}{H_r} + \frac{n \binom{n}{r+1} H_1 H_{r+1}^3 H_r - (n-r-1) \binom{n}{r+1} H_{r+2} H_{r+1}^2 H_r - H_{r+1}^4}{H_r}. \tag{4.21}\]

We suppose that \(B = \{n \binom{n}{r+1} H_1 H_{r+1}^3 H_r - (n-r-1) \binom{n}{r+1} H_{r+2} H_{r+1}^2 H_r - H_{r+1}^4 \}/H_r\), and we show that \(B\) is positive.

For every \(1 \leq j \leq n\), one has the following inequalities (see, for instance, [15, Theorems 51 and 144]):

\[H_{j-1} H_{j+1} \leq H_j^2. \tag{4.22}\]

Since each \(H_j > 0\) for \(j = 1, \ldots, r\), this is equivalent to

\[\frac{H_{r+1}}{H_r} \leq \frac{H_r}{H_{r-1}} \leq \cdots \frac{H_2}{H_1} \leq H_1. \tag{4.23}\]

And these inequalities imply that

\[H_1 H_r \geq H_{r+1}. \tag{4.24}\]

So by using (4.22) and (4.24), we get that \(B\) is positive.

Combining (4.19), (4.20), and (4.21) and Schwartz’s inequality, we get that

\[
0 \geq c_r^2 \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1} \rangle dM + c_r^3 \int_M \frac{H_{r+1}^4}{H_r} dM + c_r^3 \int_M B \, dM
- c_r^2 \left( \lambda^L_p + \lambda^L_q \right) \int_M H_{r+1}^2 dM + c_r \lambda^L_p \lambda^L_q \int_M H_r \, dM
\geq c_r^2 \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1} \rangle dM + c_r^3 \left( \int_M H_{r+1}^2 dM \right)^2 \int_M H_r \, dM
+ c_r^3 \int_M B \, dM - c_r^2 \left( \lambda^L_p + \lambda^L_q \right) \int_M H_{r+1}^2 dM + c_r \lambda^L_p \lambda^L_q \int_M H_r \, dM. \tag{4.25}\]
Hence, we obtain that
\[
0 \geq c_r \int_M H_r \, dM \int_M \langle P_r \nabla H_{r+1}, \nabla H_{r+1} \rangle \, dM + \int_M H_r \, dM \int_M B \, dM \\
+ \left( c_r \int_M H_{r+1}^2 \, dM - \lambda_p^{L_r} \int_M H_r \, dM \right) \left( c_r \int_M H_{r+1}^2 \, dM - \lambda_q^{L_r} \int_M H_r \, dM \right)
\]
(4.26)

By inequalities (4.2) and (4.26), we obtain (4.13). If in (4.13) the equality holds, then all the inequalities in (4.17) through (4.26) have to be equalities. Thus, we find that \( M \) is either of \( L_r^{-1} \) type or of \( L_r^{-2} \) type, and \( H_{r+1} \) is constant. So, by Theorems 3.3 and 3.4, \( M \) is a round sphere. \( \square \)

By applying Theorem 4.2, Corollary 4.3, and Theorem 4.8 to positively Ricci curved hypersurfaces in \( \mathbb{R}^{n+1} \), we have the following Corollary.

**Corollary 4.9.** Let \( x : M \to \mathbb{R}^{n+1} \) be a closed connected hypersurface of the Euclidean space with positive Ricci curvature. Then for every \( r = 1, \ldots, n-1 \) the following inequality holds:
\[
\int_M H_{r+1}^2 \, dM \leq \frac{\lambda_p^{L_r}}{c_r} \int_M H_r \, dM,
\]
(4.27)

where equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).

An immediate consequence of Corollaries 4.6 and 4.9 is the following.

**Corollary 4.10.** Let \( x : M \to \mathbb{R}^{n+1} \) be a closed connected hypersurface of the Euclidean space \( \mathbb{R}^{n+1} \) with positive Ricci curvature. If \( H_{r+1} \) is constant, then
\[
\lambda_p^{L_r} \leq \frac{c_r H_{r+1}^2}{H_r} \leq \lambda_q^{L_r},
\]
(4.28)

and equality holds if and only if \( M \) is a round sphere in \( \mathbb{R}^{n+1} \).

**References**


