Research Article

Generalizations of the Simpson-Like Type Inequalities for Co-Ordinated $s$-Convex Mappings in the Second Sense

Jaekeun Park

Department of Mathematics, Hanseo University, Chungnam-do, Seosan-si 356-706, Republic of Korea

Correspondence should be addressed to Jaekeun Park, jkpark@hanseo.ac.kr

Received 14 October 2011; Revised 6 December 2011; Accepted 21 December 2011

Academic Editor: Feng Qi

Copyright © 2012 Jaekeun Park. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A generalized identity for some partial differentiable mappings on a bidimensional interval is obtained, and, by using this result, the author establishes generalizations of Simpson-like type inequalities for coordinated $s$-convex mappings in the second sense.

1. Introduction

In recent years, a number of authors have considered error estimate inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of the Simpson-like type inequality which gives an error bound for the well-known Simpson rule.

**Theorem 1.1.** Let $f : \mathbb{I} \subset [0, \infty) \to \mathbb{R}$ be a four-time continuous differentiable mapping on $[a, b]$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in [a, b]} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:

$$
\left| \frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_{a}^{b} f(x)dx \right| \leq \frac{(b-a)^{4}}{2880} \left\| f^{(4)} \right\|_{\infty}.
$$

(1.1)

It is well known that the mapping $f$ is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on $(a, b)$, then we cannot apply the classical Simpson quadrature formula.
For recent results on Simpson type inequalities, you may see the papers [1–5].
In [2, 6–8], Dragomir et al. and Park considered among others the class of mappings which are $s$-convex on the coordinates.
In the sequel, in this paper let $\Delta = [a, b] \times [c, d]$ be a bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$.

**Definition 1.2.** A mapping $f : \Delta \to \mathbb{R}$ will be called $s$-convex in the second sense on $\Delta$ if the following inequality:

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w), \quad (1.2)$$

holds, for all $(x, y), (z, w) \in \Delta$, $\lambda \in [0, 1]$ and $s \in [0, 1]$.

Modification for convex and $s$-convex mapping on $\Delta$, which are also known as co-ordinated convex, $s$-convex mapping, and $s$-$r$-convex, respectively, were introduced by Dragomir, Sarikaya [5, 9, 10], and Park [4, 8, 11, 12].

**Definition 1.3.** A mapping $f : \Delta \to \mathbb{R}$ will be called coordinated $s$-convex in the second sense on $\Delta$ if the partial mappings

$$f_y : [a, b] \to \mathbb{R}, \quad f_y(u) = f(u, y),$$
$$f_x : [c, d] \to \mathbb{R}, \quad f_x(v) = f(x, v), \quad (1.3)$$

are $s$-convex in the second sense, for all $x \in [a, b], y \in [c, d]$, and $s \in [0, 1]$ [5, 9, 10].

A formal definition for coordinated $s$-convex mappings may be stated as follow [8].

**Definition 1.4.** A mapping $f : \Delta \to \mathbb{R}$ will be called coordinated $s$-convex in the second sense on $\Delta$ if the following inequality:

$$f(tx + (1 - t)z, ty + (1 - \lambda)w) \leq t^s\lambda^s f(x, y) + (1 - t)^s\lambda^s f(z, y) + t^s(1 - \lambda)^s f(x, w) + (1 - t)^s(1 - \lambda)^s f(z, w), \quad (1.4)$$

holds, for all $t, \lambda \in [0, 1], (x, y), (z, w) \in \Delta$, and $s \in [0, 1]$.

In [2], S.S. Dragomir established the following theorem.

**Theorem 1.5.** Let $f : \Delta \to \mathbb{R}$ be convex on the coordinates on $\Delta$. Then, one has the inequalities:

$$f\left(\frac{a + b}{2}, \frac{c + d}{2}\right) \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y)dydx \leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)]. \quad (1.5)$$

In [13], Hwang et al. gave a refinement of Hadamard’s inequality on the coordinates and they proved some inequalities for coordinated convex mappings.
In [1, 6, 14], Alomari and Darus proved inequalities for coordinated $s$-convex mappings.

In [15], Latif and Alomari defined coordinated $h$-convex mappings, established some inequalities for co-ordinated $h$-convex mappings and proved inequalities involving product of convex mappings on the coordinates.

In [3], Özdemir et al. gave the following theorems:

**Theorem 1.6.** Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta$. If $\partial^2 f / \partial t \partial \lambda$ is convex on the coordinates on $\Delta$, then the following inequality holds:

\[
\left| \frac{1}{9} \left[ f \left( \frac{a + c + d}{2} \right) + f \left( \frac{b + c + d}{2} \right) + 4f \left( \frac{a + b + c + d}{2} \right) \right] + f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + \frac{1}{36} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] \right| \\
\leq \left( \frac{5}{72} \right)^2 (b - a)(d - c) \\
\times \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right\},
\]

where

\[
A = \frac{1}{b - a} \int_a^b \left\{ f(x, c) + 4f(x, (c + d)/2) + f(x, d) \right\} dx \\
+ \frac{1}{d - c} \int_c^d \left\{ f(a, y) + 4f((a + b)/2, y) + f(b, y) \right\} dy.
\]

**Theorem 1.7.** Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta$. If $\partial^2 f / \partial t \partial \lambda$ is bounded, that is,

\[
\left\| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right\| \leq \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right| < \infty
\]

for all $(t, \lambda) \in [0, 1]^2$, then the following inequality holds:

\[
\left| \frac{1}{9} \left[ f \left( \frac{a + c + d}{2} \right) + f \left( \frac{b + c + d}{2} \right) + 4f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \right] + f \left( \frac{a + b}{2}, c \right) + f \left( \frac{a + b}{2}, d \right) + \frac{1}{36} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] \right|
\]
Lemma 2.1. To prove our main results, we need the following lemma.

2. Main Results

where $A$ is defined in Theorem 1.6.

In [3], Özdemir et al. proved a new equality and, by using this equality, established some inequalities on coordinated convex mappings.

In this paper the author give a generalized identity for some partial differentiable mappings on a bidimensional interval and, by using this result, establish a generalizations of Simpson-like type inequalities for coordinated $s$-convex mappings in the second sense.

2. Main Results

To prove our main results, we need the following lemma.

**Lemma 2.1.** Let $f : \Delta \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2$. If $(\partial^2 f / \partial t \partial \lambda) \in L^1(\Delta)$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $(1/r_1) \leq h_1 \leq (r_1 - 1/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following equality holds:

\[
\begin{align*}
I(f)(h_1, h_2, r_1, r_2) &= \frac{(r_1 - 2)(r_2 - 2)}{r_1 r_2} f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
&\quad + \frac{(r_1 - 2)}{r_1 r_2} \left\{ f(h_1 a + (1 - h_1)b, c) + f(h_1 a + (1 - h_1)b, d) \right\} \\
&\quad + \frac{(r_2 - 2)}{r_1 r_2} \left\{ f(a, h_2 c + (1 - h_2)d) + f(b, h_2 c + (1 - h_2)d) \right\} \\
&\quad + \frac{1}{r_1 r_2} \left\{ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right\} \\
&\quad - \frac{1}{r_2(b - a)} \int_a^b \left\{ f(x, c) + (r_2 - 2) f(x, h_2 c + (1 - h_2)d) + f(x, d) \right\} dx \\
&\quad - \frac{1}{r_1(d - c)} \int_c^d \left\{ f(a, y) + (r_1 - 2) f(h_1 a + (1 - h_1)b, y) + f(b, y) \right\} dy \\
&\quad + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \\
&= (b - a)(d - c) \int_0^1 p(h_1, r_1, t)q(h_2, r_2, \lambda) \\
&\quad \times \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt d\lambda,
\end{align*}
\]

(2.1)
where

\[ p(h_1, r_1, t) = \begin{cases} t - \frac{1}{r_1}, & t \in [0, h_1], \\ t - \frac{r_1 - 1}{r_1}, & t \in (h_1, 1), \end{cases} \]

\[ q(h_2, r_2, \lambda) = \begin{cases} \lambda - \frac{1}{r_2}, & \lambda \in [0, h_2], \\ \lambda - \frac{r_2 - 1}{r_2}, & \lambda \in (h_2, 1). \end{cases} \]

Proof. By the definitions of \( p(h_1, r_1, t) \) and \( q(h_2, r_2, \lambda) \), we can write

\[
I = \int_0^1 p(h_1, r_1, t) q(h_2, r_2, \lambda) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt d\lambda \\
= \int_0^1 q(h_2, r_2, \lambda) \left[ \int_0^1 p(h_1, r_1, t) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt \right] d\lambda \\
= \int_0^1 q(h_2, r_2, \lambda) \left[ \int_0^{h_1} \left( t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt \\
+ \int_{h_1}^1 \left( t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt \right] d\lambda \\
= \int_0^1 q(h_2, r_2, \lambda) [I_{11} + I_{12}] d\lambda,
\]

where

\[
I_{11} = \int_0^{h_1} \left( t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt,
\]

\[
I_{12} = \int_{h_1}^1 \left( t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt.
\]

By integration by parts, we have

\[
I_{11} = \int_0^{h_1} \left( t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt \\
= \frac{1}{a - b} \left[ \left( h_1 - \frac{1}{r_1} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d) \\
+ \frac{1}{r_1} \frac{\partial f}{\partial \lambda} (b, \lambda c + (1 - \lambda)d) \\
- \int_0^{h_1} \frac{\partial f}{\partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) dt \right],
\]
\[
I_{12} = \int_{h_1}^1 \left( t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d)\,dt
\]
\[
= \frac{1}{a - b} \left[ \left( \frac{1}{r_1} \frac{\partial f}{\partial \lambda} (a, \lambda c + (1 - \lambda)d) \right. \\
- \left( h_1 - \frac{r_1 - 1}{r_1} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d) \\
- \int_{h_1}^1 \frac{\partial f}{\partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d)\,dt \right]. \\
\tag{2.6}
\]

By using the equalities (2.5) and (2.6) in (2.3), we have

\[
I = \left( \frac{1}{a - b} \right) \left\{ \left( \frac{r_1 - 2}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\,d\lambda \\
+ \left( \frac{1}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1 - \lambda)d)\,d\lambda \\
+ \left( \frac{1}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1 - \lambda)d)\,d\lambda \\
- \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d)\,d\lambda dt \right\}
\]
\[= \left( \frac{1}{a - b} \right) \left\{ \left( \frac{r_1 - 2}{r_1} \right) I_{21} + \left( \frac{1}{r_1} \right) I_{22} + \left( \frac{1}{r_1} \right) I_{23} - I_{24} \right\},
\]
where

\[
I_{21} = \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\,d\lambda,
\]
\[
I_{22} = \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1 - \lambda)d)\,d\lambda,
\]
\[
I_{23} = \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1 - \lambda)d)\,d\lambda,
\]
\[
I_{24} = \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d)\,d\lambda dt.
\] (2.8)

Note that

\[
(i) \int_0^{h_2} \left( \lambda - \frac{1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\,d\lambda
\]
\[= \frac{1}{c - d} \left\{ \left( h_2 - \frac{1}{r_2} \right) f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
+ \frac{1}{r_2} f(h_1 a + (1 - h_1)b, d) \\
- \int_0^{h_2} f(h_1 a + (1 - h_1)b, \lambda c + (1 - \lambda)d)\,d\lambda \right\},
\tag{2.9}
\]
By the similar way, we get the following:

\[
(ii) \int_{h_2}^{1} \left( \lambda - \frac{r_2 - 1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
= \frac{1}{c - d} \left\{ \left( \frac{1}{r_2} \right) f(h_1 a + (1 - h_1) b, c) \\
- \left( \frac{r_2 - 1}{r_2} \right) f(h_1 a + (1 - h_1) b, h_2 c + (1 - h_2) d) \\
- \int_{h_2}^{1} f(h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \right\}.
\] (2.10)

By the equalities (2.9) and (2.10), we have

\[
I_{21} = \int_{0}^{1} q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
= \int_{0}^{h_2} \left( \lambda - \frac{1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
+ \int_{h_2}^{1} \left( \lambda - \frac{r_2 - 1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(h_1 a + (1 - h_1) b, c) + \frac{1}{r_2} f(h_1 a + (1 - h_1) b, d) \\
+ \frac{r_2 - 2}{r_2} f(h_1 a + (1 - h_1) b, h_2 c + (1 - h_2) d) \\
- \int_{0}^{1} f(h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \right\}.
\] (2.11)

By the similar way, we get the following:

\[
I_{22} = \int_{0}^{1} q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1 - \lambda) d) d\lambda \\
= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(b, c) + \frac{1}{r_2} f(b, d) + \left( \frac{r_2 - 2}{r_2} \right) f(b, h_2 c + (1 - h_2) d) \\
- \int_{0}^{1} f(b, \lambda c + (1 - \lambda) d) d\lambda \right\},
\] (2.12)

\[
I_{23} = \int_{0}^{1} q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1 - \lambda) d) d\lambda \\
= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(a, c) + \frac{1}{r_2} f(a, d) + \left( \frac{r_2 - 2}{r_2} \right) f(a, h_2 c + (1 - h_2) d) \\
- \int_{0}^{1} f(a, \lambda c + (1 - \lambda) d) d\lambda \right\},
\] (2.13)
Remark 2.2. Lemma 2.1 is a generalization of the results which proved by Sarikaya, Set, Özdemir, and Dragomir [3, 5, 9, 10].

Theorem 2.3. Let \( f : \Delta \rightarrow \mathbb{R}^2 \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \). If \( \nabla^2 f / \partial \lambda \partial \lambda \) is in \( L_1(\Delta) \) and is a coordinated \( s \)-convex mapping in the second sense on \( \Delta \), then, for \( r_1, r_2 \geq 2 \) and \( h_1, h_2 \in (0, 1) \) with \( (1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1) \), and \( (1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2) \) the following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \left| I(f) (h_1, h_2, r_1, r_2) \right| \\
\leq \mu_1(r, s) \left\{ \mu_2(r, s) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, c) \right| + \nu_2(r, s) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, d) \right| \right\} \\
+ \nu_1(r, s) \left\{ \mu_2(r, s) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, c) \right| + \nu_2(r, s) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, d) \right| \right\},
\]

(2.15)

where

\[
\mu_1(r, s) = M(r_1, s) + N(h_1, s), \\
\mu_2(r, s) = M(r_2, s) + N(h_2, s), \\
\nu_1(r, s) = M(r_1, s) + N(1 - h_1, s), \\
\nu_2(r, s) = M(r_2, s) - N(1 - h_2, s)
\]

(2.16)

for

\[
M(r, s) = \frac{2 + 2(r - 1)s^2 + r^{s+1}(s - r + 2)}{(s + 1)(s + 2)r^{s+2}}
\]

(2.17)

and

\[
N(h, s) = \frac{h^{s+1}((2h - 1)s + 2(h - 1))}{(s + 1)(s + 2)}.
\]
Proof. From Lemma 2.1 and by the coordinated $s$-convexity in the second sense of $\frac{\partial^2 f}{\partial t \partial \lambda}$, we can write

\begin{equation}
\frac{1}{(b - a)(d - c)}|I(f)(h_1, h_2, r_1, r_2)|
\leq \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| dt d\lambda
\times \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right| dt d\lambda
\leq \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right|
\times \left\{ t^\lambda \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + t^\lambda (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|
+ (1 - t)^\lambda \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + (1 - t)^\lambda (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right\} dt d\lambda
\end{equation}

\begin{equation}
= \left\{ \int_0^1 \left| p(h_1, r_1, t) \right| t^s dt \right\} \left\{ \int_0^1 \left| q(h_2, r_2, \lambda) \right| \lambda^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|
+ \int_0^1 \left| q(h_2, r_2, \lambda) \right| (1 - \lambda)^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \right\}
+ \left\{ \int_0^1 \left| p(h_1, r_1, t) \right| (1 - t)^s dt \right\} \left\{ \int_0^1 \left| q(h_2, r_2, \lambda) \right| \lambda^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|
+ \int_0^1 \left| q(h_2, r_2, \lambda) \right| (1 - \lambda)^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right\}.
\end{equation}

Note that

\begin{enumerate}
\item[(i)] \( \int_0^1 \left| p(h_1, r_1, t) \right| t^s dt = \mu_1(r, s) \),
\item[(ii)] \( \int_0^1 \left| p(h_1, r_1, t) \right| (1 - t)^s dt = \nu_1(r, s) \),
\item[(iii)] \( \int_0^1 \left| q(h_2, r_2, \lambda) \right| \lambda^s d\lambda = \mu_2(r, s) \),
\item[(iv)] \( \int_0^1 \left| q(h_2, r_2, \lambda) \right| (1 - \lambda)^s d\lambda = \nu_2(r, s) \).
\end{enumerate}

By (2.18) and (2.19), we get the inequality (2.15) by the simple calculations.
Remark 2.4. In Theorem 2.3,

(i) if we choose \( h_1 = h_2 = 1/2, r_1 = r_2 = 6, \) and \( s = 1 \) in (2.15), then we get

\[
\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left( \frac{5}{72} \right)^2 M(b - a)(d - c), \tag{2.20}
\]

(ii) if we choose \( h_1 = h_2 = 1/2, r_1 = r_2 = 2, \) and \( s = 1 \) in (2.15), then we get

\[
\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left( \frac{1}{8} \right)^2 M(b - a)(d - c), \tag{2.21}
\]

where

\[
M = \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|, \tag{2.22}
\]

which implies that Theorem 2.3 is a generalization of Theorem 1.6.

Theorem 2.5. Let \( f : \Delta \to \mathbb{R}^2 \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \). If \( \partial^2 f / \partial t \partial \lambda \) is bounded, that is,

\[
\left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} = \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right| < \infty \tag{2.23}
\]

for all \( (t, \lambda) \in [0,1]^2 \), then, for \( r_1, r_2 \geq 2 \) and \( h_1, h_2 \in (0,1) \) with \( 1/r_1 \leq h_1 \leq (r_1 - 1)/r_1 \) and \( 1/r_2 \leq h_2 \leq (r_2 - 1)/r_2 \), the following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2, r_1, r_2) \right| \leq \left\{ \frac{1}{2} - h_1 + h_1^2 + \frac{2 - r_1}{r_1^2} \right\} \left\{ \frac{1}{2} - h_2 + h_2^2 + \frac{2 - r_2}{r_2^2} \right\} \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty}. \tag{2.24}
\]

Proof. From Lemma 2.1, using the property of modulus and the boundedness of \( \partial^2 f / \partial t \partial s \), we get

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2, r_1, r_2) \right| \leq \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} \int_0^1 \int_0^1 p(h_1, r_1, t) q(h_2, r_2, \lambda) |dt dl|. \tag{2.25}
\]
By the simple calculations, we have

$$(i) \int_{0}^{1} |p(h_1, r_1, t)| dt = \frac{1}{2} - h_1 + h_1^2 + \frac{(2 - r_1)}{r_1^2}, \quad (2.26)$$

$$(ii) \int_{0}^{1} |q(h_2, r_2, \lambda)| dt = \frac{1}{2} - h_2 + h_2^2 + \frac{(2 - r_2)}{r_2^2}. \quad (2.27)$$

By using the inequality (2.25) and the equalities (2.26)-(2.27), the assertion (2.24) holds.

**Remark 2.6.** In Theorem 2.5,

(i) if we choose $h_1 = h_2 = \frac{1}{2}$ and $r_1 = r_2 = 6$, then we get

$$\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left( \frac{5}{36} \right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_\infty (b - a)(d - c), \quad (2.28)$$

(ii) if we choose $h_1 = h_2 = 1/2$ and $r_1 = r_2 = 2$, then we get

$$\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left( \frac{1}{4} \right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_\infty (b - a)(d - c), \quad (2.29)$$

which implies that Theorem 2.5 is a generalization of Theorem 1.7.

The following theorem is a generalization of Theorem 1.6.

**Theorem 2.7.** Let $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $|\partial^2 f / \partial t \partial \lambda|^q(q > 1)$ is in $L_1(\Delta)$ and $f$ is a coordinated s-convex mapping in the second sense on $\Delta$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following inequality holds:

$$\frac{1}{(b - a)(d - c)} |I(f)(h_1, h_2, r_1, r_2)| \leq \mu_1^{1/p} v_3^{1/p} \times \left\{ \left[ \frac{\partial^2 f / \partial t \partial \lambda(a, c)}{s + 1} \right]^q + \left[ \frac{\partial^2 f / \partial t \partial \lambda(a, d)}{s + 1} \right]^q + \left[ \frac{\partial^2 f / \partial t \partial \lambda(b, c)}{s + 1} \right]^q + \left[ \frac{\partial^2 f / \partial t \partial \lambda(b, d)}{s + 1} \right]^q \right\}^{1/p}, \quad (2.30)$$

where

\[
\mu_3 = \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p + 1)},
\]

\[
v_3 = \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p + 1)}.
\]

**Proof.** From Lemma 2.1, we can write

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2, r_1, r_2) \right|
\]

\[
\leq \left\{ \int_{0}^{1} \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \right. 
\times \left. \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right| dtd\lambda \right\}^{1/p}
\]

\[
\leq \left\{ \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial c} (h_1, h_2, r_1, r_2, \lambda) \right|^p dtd\lambda \right\}^{1/p}
\]

Hence, by the inequality (2.32) and the coordinated s-convexity in the second sense of \( |\partial^2 f / \partial t \partial \lambda|^q \), it follows that

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2, r_1, r_2) \right|
\]

\[
\leq \left\{ \int_{0}^{1} \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right|^p dtd\lambda \right\}^{1/p}
\]

\[
\times \left\{ \frac{|\partial^2 f / \partial t \partial \lambda(a, c)|^q + |\partial^2 f / \partial t \partial \lambda(a, d)|^q + |\partial^2 f / \partial t \partial \lambda(b, c)|^q + |\partial^2 f / \partial t \partial \lambda(b, d)|^q}{(s + 1)^2} \right\}^{1/q}.
\]

Note that

\[
(i) \int_{0}^{1} \left| p(h_1, r_1, t) \right|^p dt = \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p + 1)},
\]

\[
(ii) \int_{0}^{1} \left| q(h_2, r_2, \lambda) \right|^p d\lambda = \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p + 1)}.
\]

By the inequality (2.33) and the equalities (2.34) and (2.35), the assertion (2.30) holds. \(\square\)
Remark 2.8. In Theorem 2.7,

(i) if we choose \(h_1 = h_2 = 1/2, r_1 = r_2 = 6, \) and \(s = 1,\) then we get

\[
\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left\{ \frac{2(1 + 2^{p+1})}{6^{p+1}(p + 1)} \right\}^{2/p} (b - a)(d - c)M_{q}^{1/q},
\]

(2.36)

(ii) if we choose \(h_1 = h_2 = 1/2, r_1 = r_2 = 2, \) and \(s = 1,\) then we get

\[
\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left\{ \frac{1}{2^{p}(p + 1)} \right\}^{2/p} (b - a)(d - c)M_{q}^{1/q},
\]

(2.37)

(iii) if we choose \(h_1 = h_2 = 1/2, r_1 = r_2 = 6, s = 1, \) and \(q = 1,\) then we get

\[
\left| I(f) \left( \frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left( \frac{5}{36} \right)^{2} (b - a)(d - c)M_{1},
\]

(2.38)

where

\[
M_{q} = \frac{\left| \partial^{2}f/\partial t\partial \lambda(a,c) \right|^{q} + \left| \partial^{2}f/\partial t\partial \lambda(a,d) \right|^{q} + \left| \partial^{2}f/\partial t\partial \lambda(b,c) \right|^{q} + \left| \partial^{2}f/\partial t\partial \lambda(b,d) \right|^{q}}{4}.
\]

(2.39)

Theorem 2.9. Let \(f : \Delta \rightarrow \mathbb{R}^2\) be a partial differentiable mapping on \(\Delta = [a,b] \times [c,d] \subset \mathbb{R}^2.\) If \(\left| \partial^{2}f/\partial t\partial \lambda \right|^{q}(q \geq 1)\) is in \(L_1(\Delta)\) and is a coordinated \(s\)-convex mapping in the second sense on \(\Delta,\) then, for \(r_1, r_2 \geq 2\) and \(h_1, h_2 \in (0,1)\) with \((1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)\) and \((1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2),\) the following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2, r_1, r_2) \right| \\
\leq \left\{ \left( \frac{1}{2} - h_1 + h_1^2 + \frac{2 - r_1}{r_1^2} \right) \left( \frac{1}{2} - h_2 + h_2^2 + \frac{2 - r_2}{r_2^2} \right) \right\}^{1/(1 - q)} \\
\times \left\{ \mu_1 \left( \mu_2 \left| \frac{\partial^{2}f}{\partial t\partial \lambda}(a,c) \right|^{q} + \nu_2 \left| \frac{\partial^{2}f}{\partial t\partial \lambda}(a,d) \right|^{q} \right) \\
+ \nu_1 \left( \mu_2 \left| \frac{\partial^{2}f}{\partial t\partial \lambda}(b,c) \right|^{q} + \nu_2 \left| \frac{\partial^{2}f}{\partial t\partial \lambda}(b,d) \right|^{q} \right) \right\}^{1/q},
\]

(2.40)

where \(\mu_i\) and \(\nu_i\) \((i = 1, 2)\) are as given in Theorem 2.3.
Proof. From Lemma 2.1, we can write

\[ \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \]

\[ \leq \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \times \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right| \, dtd\lambda \]

\[ \leq \left\{ \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \, dtd\lambda \right\}^{1-(1/q)} \]

\[ \times \left\{ \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \times \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^{q} \, dtd\lambda \right\}^{1/q}. \] (2.41)

By the simple calculations, we have

\[ (i) \int_0^1 \left| p(h_1, r_1, t) \right| \, dt = \frac{1}{2} - h_1 + h_1^2 + \frac{(2 - r_1)}{r_1^2}, \] (2.42)

\[ (ii) \int_0^1 \left| q(h_2, r_2, \lambda) \right| \, d\lambda = \frac{1}{2} - h_2 + h_2^2 + \frac{(2 - r_2)}{r_2^2}. \] (2.43)

Since \(|\partial^2 f / \partial t \partial \lambda|^q\) is a coordinated s-convex mapping in the second sense on \( \Delta = [a, b] \times [c, d] \), we have that, for \( t \in [0, 1] \),

\[ \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^{q} \, dtd\lambda \]

\[ \leq \int_0^1 \left| p(h_1, r_1, t)q(h_2, r_2, \lambda) \right| \times \left\{ t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^{q} + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^{q} \right. \]

\[ + (1-t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|^{q} + (1-t)^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|^{q} \left\} \, dtd\lambda \]
\[\begin{align*}
&= \left\{ \int_0^1 |p(h_1, r_1, t)|^{\frac{q}{2}} \, dt \right\} \left\{ \left( \int_0^1 |q(h_2, r_2, \lambda)|^{\frac{q}{2}} \, d\lambda \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, c) \right|^q \\
&\quad + \left( \int_0^1 |q(h_2, r_2, \lambda)|(1 - \lambda)^{\frac{q}{2}} \, d\lambda \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, d) \right|^q \right\} \\
&\quad + \left( \int_0^1 |p(h_1, r_1, t)|(1 - t)^{\frac{q}{2}} \, dt \right) \left\{ \left( \int_0^1 |q(h_2, r_2, s)|^{\frac{q}{2}} \, d\lambda \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, c) \right|^q \\
&\quad \quad + \left( \int_0^1 |q(h_2, r_2, \lambda)|(1 - \lambda)^{\frac{q}{2}} \, d\lambda \right) \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, d) \right|^q \right\} \\
&= \mu_1 \left\{ \mu_2 \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (a, d) \right|^q \right\} \\
&\quad + \nu_1 \left\{ \mu_2 \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial \lambda \partial \lambda} (b, d) \right|^q \right\}. 
\end{align*}\]

By (2.41)–(2.44), the assertion (2.40) holds. \(\square\)

References

