Research Article

Coupled Fixed Point Results in Complete Partial Metric Spaces

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We establish some coupled fixed point theorems for a mapping satisfying some contraction conditions in complete partial metric spaces. Our consequences extend the results of H. Aydi (2011).

1. Introduction and Mathematical Preliminaries

The notion of a partial metric space (PMS) was introduced in 1992 by Matthews [1, 2]. Matthews proved a fixed point theorem on this spaces, analogous to the Banach’s fixed point theorem. Recently, many authors have focused on partial metric spaces and their topological properties (see e.g. [3–9]).

The definition of a partial metric space is given by Matthews (see [1, 2]) as follows:

**Definition 1.1.** Let \(X\) be a nonempty set and let \(p : X \times X \rightarrow \mathbb{R}^+\) satisfies

\[(P1)\] \(x = y \Leftrightarrow p(x, x) = p(y, y) = p(x, y), \) for all \(x, y \in X,\)

\[(P2)\] \(p(x, x) \leq p(x, y), \) for all \(x, y \in X,\)

\[(P3)\] \(p(x, y) = p(y, x), \) for all \(x, y \in X,\)

\[(P4)\] \(p(x, y) \leq p(x, z) + p(z, y) - p(z, z), \) for all \(x, y, z \in X.\)

Then the pair \((X, p)\) is called a partial metric space and \(p\) is called a partial metric on \(X.\)
The function \( d_p : X \times X \to \mathbb{R}^+ \) defined by
\[
d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\] (1.1)
satisfies the conditions of a metric on \( X \); therefore it is a (usual) metric on \( X \).

Remark 1.2. if \( x = y, p(x, y) \) may not be 0.

1. A famous example of partial metric spaces is the pair \((\mathbb{R}^+, p)\), where \( p(x, y) = \max\{x, y\} \) for all \( x, y \in \mathbb{R}^+ \). In this case, \( d_p \) is the Euclidian metric \( d_p(x, y) = |x - y| \).

2. Each partial metric \( p \) on \( X \) generates a \( T_0 \) topology \( \tau_p \) on \( X \) which has a base of open \( p \)-balls \( B_p(x, \varepsilon) \), where \( x \in X \) and \( \varepsilon > 0 \) \( B_p(x, \varepsilon) = \{ y \in X : p(x, y) < p(x, x) + \varepsilon \} \).

The following concepts has been defined as follows on a partial metric space.

Definition 1.3 (see e.g., \cite{1, 2}). (i) A sequence \( \{x_n\} \) in a PMS \((X, p)\) converges to \( x \in X \) if and only if \( p(x, x) = \lim_{n \to \infty} p(x, x_n) \).

(ii) A sequence \( \{x_n\} \) in a PMS \((X, p)\) is called Cauchy if and only if \( \lim_{n,m \to \infty} p(x_n, x_m) \) exists (and is finite).

(iii) A PMS \((X, p)\) is said to be complete if every Cauchy sequence \( \{x_n\} \) in \( X \) converges, with respect to \( \tau_p \), to a point \( x \in X \) such that \( p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m) \).

The concept of coupled fixed point have been introduced in \cite{10} by Bhaskar and Lakshmikantham as follows.

Definition 1.4 (see \cite{10}). An element \((x, y) \in X \times X\) is called a coupled fixed point of mapping \( F : X \times X \to X \) if \( x = F(x, y) \) and \( y = F(y, x) \).

Aydi in \cite{11} has obtained some coupled fixed point results for mappings satisfying different contractive conditions on complete partial metric spaces. Some of these results are the following cases.

Theorem 1.5 (see \cite{11, Theorem 2.1}). Let \((X, p)\) be a complete partial metric space. Suppose that the mapping \( F : X \times X \to X \) satisfies the following contractive condition:
\[
p(F(x, y), F(u, v)) \leq kp(x, u) + lp(y, v),
\] (1.2)
for all \( x, y, u, v \in X \), where \( k, l \) are nonnegative constants with \( k + l < 1 \). Then, \( F \) has a unique coupled fixed point.

Theorem 1.6 (see \cite{11, Theorem 2.4}). Let \((X, p)\) be a complete partial metric space. Suppose that the mapping \( F : X \times X \to X \) satisfies the following contractive condition:
\[
p(F(x, y), F(u, v)) \leq kp(F(x, y), x) + lp(F(u, v), u),
\] (1.3)
for all \( x, y, u, v \in X \), where \( k, l \) are nonnegative constants with \( k + l < 1 \). Then, \( F \) has a unique coupled fixed point.
Theorem 1.7 (see [11, Theorem 2.5]). Let \((X, p)\) be a complete partial metric space. Suppose that the mapping \(F : X \times X \to X\) satisfies the following contractive condition:

\[
p(F(x, y), F(u, v)) \leq kp(F(x, y), u) + lp(F(u, v), x),
\]

for all \(x, y, u, v \in X\), where \(k, l\) are nonnegative constants with \(k + l < 1\). Then, \(F\) has a unique coupled fixed point.

For a survey of fixed point theory, its applications, and related results in partial metric spaces we refer the reader to \([4, 5, 12–20]\) and the references mentioned therein. Also, many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the framework of partial metric spaces (see, e.g., \([21, 22]\)).

In this paper we establish some coupled fixed point results of contractive mappings in the framework of complete partial metric spaces. Our results extend and generalize the results of Aydi \([11]\).

2. Main Results

We recall three easy lemmas which have an essential role in the proof of the main result. These results can be derived easily (see, e.g., \([1, 2, 6]\)).

Lemma 2.1. (1) A sequence \(\{x_n\}\) is a Cauchy sequence in the PMS \((X, p)\) if and only if it is a Cauchy sequence in the metric space \((X, d_p)\).

(2) A PMS \((X, p)\) is complete if and only if the metric space \((X, d_p)\) is complete. Moreover,

\[
\lim_{n \to \infty} d_p(x, x_n) = 0 \iff p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m).
\]

Lemma 2.2 (see \([3]\)). Assume that \(x_n \to z\) as \(n \to \infty\) in a PMS \((X, p)\) such that \(p(z, z) = 0\). Then, \(\lim_{n \to \infty} p(x_n, y) = p(z, y)\), for every \(y \in X\).

Lemma 2.3 (see, e.g., \([3, 4]\)). Let \((X, p)\) be a complete PMS. Then,

(a) if \(p(x, y) = 0\) then, \(x = y\),

(b) if \(x \neq y\), then \(p(x, y) > 0\).

Throughout this paper, we assume that all of the constants are nonnegative. Our main result is the following. The method of the proof can be found in \([11]\).

Theorem 2.4. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \to X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \alpha_1 p(x, u) + \alpha_2 p(y, v) \\
+ \alpha_3 p(F(x, y), x) + \alpha_4 p(F(y, x), y) \\
+ \alpha_5 p(F(x, y), u) + \alpha_6 p(F(y, x), v) + \alpha_7 p(F(u, v), x) \\
+ \alpha_8 p(F(v, u), y) + \alpha_9 p(F(u, v), u) + \alpha_{10} p(F(v, u), v),
\]

(2.2)
for every pairs \((x, y), (u, v) \in X \times X\), where \(\sum_{i=1}^{10} \alpha_i < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

**Proof.** Let \(x_0, y_0 \in X\) be arbitrary. Define \(x_1, y_1 \in X\) such that \(x_1 = F(x_0, y_0)\) and \(y_1 = F(y_0, x_0)\) and in this way, we construct the sequences \(\{x_n\}\) and \(\{y_n\}\) as \(x_n = F(x_{n-1}, y_{n-1})\) and \(y_n = F(y_{n-1}, x_{n-1})\), for all \(n \geq 0\).

We will complete the proof in three steps.

Step I. Let \(\delta_n = p(x_{n-1}, x_n) + p(y_{n-1}, y_n)\). We will show that \(\lim_{n \to \infty} \delta_n = 0\).

Using (2.2) we obtain that

\[
p(x_n, x_{n+1}) = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n))
\leq \alpha_1 p(x_{n-1}, x_n) + \alpha_2 p(y_{n-1}, y_n) + \alpha_3 p(F(x_{n-1}, y_{n-1}), x_{n-1})
+ \alpha_4 p(F(y_{n-1}, x_{n-1}), y_{n-1})
+ \alpha_5 p(F(x_{n-1}, y_{n-1}), x_n) + \alpha_6 p(F(y_{n-1}, x_{n-1}), y_n) + \alpha_7 p(F(x_n, y_n), x_{n-1})
+ \alpha_8 p(F(y_n, x_n), y_{n-1}) + \alpha_9 p(F(x_n, y_n), x_n) + \alpha_{10} p(F(y_n, x_n), y_n)
= \alpha_1 p(x_{n-1}, x_n) + \alpha_2 p(y_{n-1}, y_n) + \alpha_3 p(x_{n-1}, x_n) + \alpha_4 p(y_{n-1}, y_n) + \alpha_5 p(x_{n-1}, x_n)
+ \alpha_6 p(y_n, y_n) + \alpha_7 p(x_{n+1}, x_n) + \alpha_8 p(y_{n+1}, y_{n-1}) + \alpha_9 p(x_{n+1}, x_n)
+ \alpha_{10} p(y_{n+1}, y_n).
\]

(2.3)

Analogously, starting from \(p(x_{n+1}, x_n) = p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))\), we have

\[
p(x_{n+1}, x_n) = p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))
\leq \alpha_1 p(x_n, x_{n-1}) + \alpha_2 p(y_n, y_{n-1}) + \alpha_3 p(F(x_n, y_n), x_n) + \alpha_4 p(F(y_n, x_n), y_n)
+ \alpha_5 p(F(x_n, y_n), x_{n-1}) + \alpha_6 p(F(y_n, x_n), y_{n-1}) + \alpha_7 p(F(x_{n-1}, y_{n-1}), x_n)
+ \alpha_8 p(F(y_{n-1}, x_{n-1}), y_n) + \alpha_9 p(F(x_{n-1}, y_{n-1}), x_{n-1}) + \alpha_{10} p(F(y_{n-1}, x_{n-1}), y_{n-1})
= \alpha_1 p(x_{n-1}, x_n) + \alpha_2 p(y_{n-1}, y_n) + \alpha_3 p(x_{n+1}, x_n) + \alpha_4 p(y_{n+1}, y_n) + \alpha_5 p(x_{n+1}, x_n)
+ \alpha_6 p(y_{n+1}, y_n) + \alpha_7 p(x_{n+1}, x_n) + \alpha_8 p(y_{n+1}, y_{n-1}) + \alpha_9 p(x_{n+1}, x_{n-1}) + \alpha_{10} p(y_{n+1}, y_{n-1})
\leq \alpha_1 p(x_{n-1}, x_n) + \alpha_2 p(y_{n-1}, y_n) + \alpha_3 p(x_{n+1}, x_n) + \alpha_4 p(y_{n+1}, y_n)
+ \alpha_5 [p(x_{n+1}, x_n) + p(x_n, x_{n-1})] + \alpha_6 [p(y_{n+1}, y_n) + p(y_{n+1}, y_{n-1})]
\]
In a similar way, we have

\[ p(y_n, y_{n+1}) = p(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \]

\[ \leq \alpha_1 p(y_{n-1}, y_n) + \alpha_2 p(x_{n-1}, x_n) + \alpha_3 p(y_n, y_{n-1}) + \alpha_4 p(x_n, x_{n-1}) \]

\[ + (\alpha_5 - \alpha_7) p(y_n, y_n) + (\alpha_6 - \alpha_8) p(x_n, x_n) \]

\[ + \alpha_7 [p(y_{n+1}, y_n) + p(y_n, y_{n-1})] + \alpha_8 [p(x_{n+1}, x_n) + p(x_n, x_{n-1})] \]

\[ + \alpha_9 p(y_{n+1}, y_n) + \alpha_{10} p(x_{n+1}, x_n). \]

(2.5)

Analogously, starting from \( p(y_{n+1}, y_n) = p(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \), we have

\[ p(y_{n+1}, y_n) = p(F(y_n, x_n), F(y_{n-1}, x_{n-1})) \]

\[ \leq \alpha_1 p(y_{n-1}, y_n) + \alpha_2 p(x_{n-1}, x_n) + \alpha_3 p(y_n, y_{n-1}) + \alpha_4 p(x_n, x_{n-1}) \]

\[ + (\alpha_5 - \alpha_7) p(y_n, y_n) + (\alpha_6 - \alpha_8) p(x_n, x_n) \]

\[ + \alpha_7 [p(y_{n+1}, y_n) + p(y_n, y_{n-1})] + \alpha_8 [p(x_{n+1}, x_n) + p(x_n, x_{n-1})] \]

\[ + \alpha_9 p(y_{n+1}, y_n) + \alpha_{10} p(x_{n+1}, x_n). \]

(2.6)

Adding (2.3), (2.4), (2.5), and (2.6) we obtain that

\[ 2\delta_{n+1} \leq [2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}] \delta_n \]

\[ + [\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}] \delta_{n+1}, \]

(2.7)

or, equivalently,

\[ \delta_{n+1} \leq \lambda \delta_n, \]

(2.8)

where, \( \lambda = [2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}] / (2 - [\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}]). \)

Repeating the above mentioned process, we have

\[ \delta_{n+1} \leq \lambda \delta_n \leq \lambda^2 \delta_{n-1} \leq \cdots \leq \lambda^{n+1} \delta_0, \]

(2.9)

where, from our assumption about coefficients \( \alpha_i, \lambda \in [0, 1] \); hence,

\[ \lim_{n \to \infty} \delta_n = 0. \]

(2.10)
Step II. \{x_n\} and \{y_n\} are Cauchy.

If \(\delta_0 = 0\) then, \(p(x_0,x_1) + p(y_0,y_1) = 0\). Hence, we get \(x_0 = x_1 = F(x_0,y_0)\) and \(y_0 = y_1 = F(y_0,x_0)\); that is, \((x_0,y_0)\) is a coupled fixed point of \(F\). Now, let \(\delta_0 > 0\). For each \(m \geq n\), we have

\[
p(x_m, x_n) + p(y_m, y_n) \leq p(x_m, x_{m-1}) + p(y_m, y_{m-1})
+ p(x_{m-1}, x_{m-2}) + p(y_{m-1}, y_{m-2})
+ \ldots
+ p(x_{n+1}, x_n) + p(y_{n+1}, y_n)
= \delta_m + \delta_{m-1} + \cdots + \delta_{n+1}
\leq \left[\lambda^m + \lambda^{m-1} + \cdots + \lambda^{n+1}\right] \delta_0
\leq \frac{\lambda^{n+1}}{1 - \lambda} \delta_0.
\] (2.11)

So, we have \(\lim_{n,m \to \infty} p(x_n, x_m) + p(y_n, y_n) = 0\). This proves that \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequences in \((X, p)\) and hence \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequences in the metric space \((X, d_p)\). From Lemma 2.1, \((X, d_p)\) is complete, so \(\{x_n\}\) and \(\{y_n\}\) converge to some \(x, y \in X\), respectively; that is, \(\lim_{n \to \infty} d_p(x_n,x) = 0\) and \(\lim_{n \to \infty} d_p(y_n,y) = 0\). Therefore, from Lemma 2.1 and (2.10), we have

\[
p(x,x) = \lim_{n \to \infty} p(x_n,x) = \lim_{n,m \to \infty} p(x_n, x_m) = 0, \tag{2.12}
\]

\[
p(y,y) = \lim_{n \to \infty} p(y_n,y) = \lim_{n,m \to \infty} p(y_n, y_m) = 0. \tag{2.13}
\]

Step III. We will show that \(F\) has a unique coupled fixed point.

From the above step,

\[
\lim_{n \to \infty} p(F(x_n, y_n), x) = \lim_{n \to \infty} p(F(y_n, x_n), y) = 0. \tag{2.14}
\]

Next, we will prove that \(x = F(x,y)\) and \(y = F(y,x)\).

We have

\[
p(x, F(x,y)) \leq p(x, x_{n+1}) + p(x_{n+1}, F(x,y)) - p(x_{n+1}, x_{n+1}). \tag{2.15}
\]

Taking the limit as \(n \to \infty\) in the above inequality, as \(x_{n+1} = F(x_n, y_n)\) and using triangle inequality and (2.12), we have

\[
p(x, F(x,y)) \leq \lim_{n \to \infty} p(x, x_{n+1}) + \lim_{n \to \infty} p(F(x_n, y_n), F(x,y))
= \lim_{n \to \infty} p(F(x_n, y_n), F(x,y)). \tag{2.16}
\]
Remark 2.5.

But, for all \( n \geq 0 \), from (2.2),

\[
p(F(x_n, y_n), F(x, y)) \leq \alpha_1 p(x_n, x) + \alpha_2 p(y_n, y) \\
+ \alpha_3 p(F(x_n, y_n), x_n) + \alpha_4 p(F(y_n, x_n), y_n) \\
+ \alpha_5 p(F(x_n, y_n), x) + \alpha_6 p(F(y_n, x_n), y) + \alpha_7 p(F(x, y), x_n) \\
+ \alpha_8 p(F(y, x), y_n) + \alpha_9 p(F(x, y), x) + \alpha_{10} p(F(y, x), y) \\
= \alpha_1 p(x_n, x) + \alpha_2 p(y_n, y) + \alpha_3 p(x_{n+1}, x) \\
+ \alpha_4 p(y_n+1, y) + \alpha_5 p(x_{n+1}, x) + \alpha_6 p(y_{n+1}, y) + \alpha_7 p(F(x, y), x_n) \\
+ \alpha_8 p(F(y, x), y_n) + \alpha_9 p(F(x, y), x) + \alpha_{10} p(F(y, x), y).
\]  

(2.17)

In the above inequality, if \( n \to \infty \), using (2.12) and Lemma 2.2 we have

\[
\lim_{n \to \infty} p(F(x_n, y_n), F(x, y)) \leq (\alpha_7 + \alpha_9) p(F(x, y), x) + (\alpha_8 + \alpha_{10}) p(F(y, x), y).
\]  

(2.18)

Analogously,

\[
p(y, F(y, x)) \leq p(y, y_{n+1}) + p(y_{n+1}, F(y, x)) - p(y_{n+1}, y_{n+1}).
\]  

(2.19)

Taking the limit as \( n \to \infty \) in the above inequality, since \( y_{n+1} = F(y_n, x_n) \) and using triangle inequality and (2.13), we have

\[
p(y, F(y, x)) \leq \lim_{n \to \infty} p(y, y_{n+1}) + \lim_{n \to \infty} p(F(y_n, x_n), F(y, x))
\]
\[
= \lim_{n \to \infty} p(F(y_n, x_n), F(y, x)).
\]  

(2.20)

Similar to (2.17), we have

\[
\lim_{n \to \infty} p(F(y_n, x_n), F(y, x)) \leq (\alpha_7 + \alpha_9) p(F(y, x), y) + (\alpha_8 + \alpha_{10}) p(F(x, y), x).
\]  

(2.21)

Adding (2.18) and (2.21) and using (2.15) and (2.19), we obtain that

\[
p(x, F(x, y)) + p(y, F(y, x)) \\
\leq (\alpha_7 + \alpha_9 + \alpha_8 + \alpha_{10}) [p(x, F(x, y)) + p(y, F(y, x))].
\]  

(2.22)

Therefore, \( p(x, F(x, y)) + p(y, F(y, x)) = 0 \); that is, \( F(x, y) = x \) and \( F(y, x) = y \). \( \square \)

Remark 2.5. (1) If in the above theorem, we assume that \( \alpha_i = 0 \), for all \( 3 \leq i \leq 10 \), then we obtain the result of Aydi in [11] which is noted here in Theorem 1.5.
(2) If in the above theorem, \( \alpha_i = 0 \), for all \( 1 \leq i \leq 10 \), unless \( i = 3, 9 \), then we obtain the result of Aydi in [11] which is mentioned here in Theorem 1.6.

(3) If in the above theorem, we assume that \( \alpha_i = 0 \), for all \( 3 \leq i \leq 10 \), except that \( i \neq 5, 7 \), then we obtain the result of Aydi in [11] (Theorem 1.7).

Many results can be deduced from the above theorem as follows.

**Corollary 2.6.** Let \((X, p)\) be a complete partial metric space and \( F : X \times X \to X \) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \alpha_1 p(F(x, y), x) + \alpha_2 p(F(y, x), y) + \alpha_3 p(F(u, v), u) + \alpha_4 p(F(v, u), v),
\]

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(\sum_{i=1}^{4} \alpha_i < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

**Corollary 2.7.** Let \((X, p)\) be a complete partial metric space and \( F : X \times X \to X \) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \alpha_1 p(F(x, y), u) + \alpha_2 p(F(y, x), v) + \alpha_3 p(F(u, v), x) + \alpha_4 p(F(v, u), y),
\]

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(\sum_{i=1}^{4} \alpha_i < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

**Corollary 2.8.** Let \((X, p)\) be a complete partial metric space and \( F : X \times X \to X \) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \alpha_1 p(F(x, y), x) + \alpha_2 p(F(y, x), y) + \alpha_3 p(F(u, v), x) + \alpha_4 p(F(v, u), y),
\]

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

**Corollary 2.9.** Let \((X, p)\) be a complete partial metric space and \( F : X \times X \to X \) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \alpha_1 p(F(x, y), u) + \alpha_2 p(F(y, x), v) + \alpha_3 p(F(u, v), u) + \alpha_4 p(F(v, u), v),
\]

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(\sum_{i=1}^{4} \alpha_i < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

Also, we have the following results, when the constants in the above corollaries are equal.
Corollary 2.10. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \rightarrow X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(x, u) + p(y, v)] + \frac{l}{2} [p(F(x, y), x) + p(F(y, x), y)] + \frac{r}{2} [p(F(x, y), u) + p(F(y, x), v)] + \frac{s}{2} [p(F(u, v), x) + p(F(v, u), y)] + \frac{t}{2} [p(F(u, v), u) + p(F(v, u), v)],
\]

(2.27)

for every pairs \((x, y), (u, v) \in X \times X\), where \(k + l + r + s + t < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

Corollary 2.11. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \rightarrow X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(F(x, y), x) + p(F(y, x), y)] + \frac{l}{2} [p(F(u, v), u) + p(F(v, u), v)],
\]

(2.28)

for every pairs \((x, y), (u, v) \in X \times X\), where \(k + l < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

Corollary 2.12. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \rightarrow X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(F(x, y), u) + p(F(y, x), v)] + \frac{l}{2} [p(F(u, v), x) + p(F(v, u), y)],
\]

(2.29)

for every pairs \((x, y), (u, v) \in X \times X\), where \(k + l < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

Corollary 2.13. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \rightarrow X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(F(x, y), x) + p(F(y, x), y)] + \frac{l}{2} [p(F(u, v), x) + p(F(v, u), y)],
\]

(2.30)

for every pairs \((x, y), (u, v) \in X \times X\), where \(k + l < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).
Corollary 2.14. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \to X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(x, u) + p(y, v)]
+ \frac{l}{4} [p(F(x, y), x) + p(F(y, x), y) + p(F(u, v), u) + p(F(v, u), v)]
+ \frac{r}{4} [p(F(x, y), u) + p(F(y, x), v) + p(F(u, v), x) + p(F(v, u), y)],
\]

(2.31)

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(k + l + r < 1\). Then, \(F\) has a unique coupled fixed point in \(X\).

Corollary 2.15. Let \((X, p)\) be a complete partial metric space and \(F : X \times X \to X\) be a mapping such that

\[
p(F(x, y), F(u, v)) \leq \frac{k}{2} [p(F(x, y), u) + p(F(y, x), v)]
+ \frac{l}{4} [p(F(u, v), u) + p(F(v, u), v)],
\]

(2.32)

for every pairs \((x, y), (u, v)\) \(\in X \times X\), where \(k + l < 1\). Then \(F\) has a unique coupled fixed point in \(X\).

Example 2.16. Let \(X = [0, \infty)\) and \(p\) on \(X\) be given as \(p(a, b) = \max\{a, b\}\). Obviously, the partial metric space \((X, p)\) is complete (see, e.g., Example 2.3 of [11]).

Define \(F : X \times X \to X\) as \(F(x, y) = (x + y)/30\), for all \(x, y \in X\).

Now, we have

\[
p(F(x, y), F(u, v)) = \frac{1}{30} \max\{x + y, u + v\}
\leq \frac{1}{27} [\max\{x, u\} + \max\{y, v\}]
\leq \frac{1}{27} [\max\{x, u\} + \max\{y, v\}]
+ \frac{1}{27} \left[ \max\left\{ \frac{x + y}{30}, x \right\} + \max\left\{ \frac{y + x}{30}, y \right\} \right]
+ \frac{1}{27} \left[ \max\left\{ \frac{x + y}{30}, u \right\} + \max\left\{ \frac{y + x}{30}, v \right\} \right]
+ \frac{1}{27} \left[ \max\left\{ \frac{u + v}{30}, x \right\} + \max\left\{ \frac{v + u}{30}, y \right\} \right]
\]
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\[ + \frac{1}{27} \left[ \max \left\{ \frac{u + v}{3}, u \right\} + \max \left\{ \frac{v + u}{3}, v \right\} \right] \]

\[ = \alpha_1 p(x, u) + \alpha_2 p(y, v) + \alpha_3 p(F(x, y), x) + \alpha_4 p(F(y, x), y) \]
\[ + \alpha_5 p(F(x, y), u) + \alpha_6 p(F(y, x), v) + \alpha_7 p(F(u, v), x) \]
\[ + \alpha_8 p(F(v, u), y) + \alpha_9 p(F(u, v), u) + \alpha_{10} p(F(v, u), v). \]

(2.33)

Thus, (2.2) is satisfied with \( \alpha_i = 1/27 \). Obviously, all the conditions of Theorem 2.4 are satisfied. Moreover, \((0,0)\) is the unique coupled fixed point of \( F \).

References


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