Research Article

On Prime-Gamma-Near-Rings with Generalized Derivations

Kalyan Kumar Dey, Akhil Chandra Paul, and Isamiddin S. Rakhimov

1 Department of Mathematics, Rajshahi University, Rajshahi 6205, Bangladesh
2 Department of Mathematics, Faculty of Science and Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 Serdang, Malaysia

Correspondence should be addressed to Kalyan Kumar Dey, kkdmath@yahoo.com

Received 26 January 2012; Accepted 19 March 2012

Copyright © 2012 Kalyan Kumar Dey et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let $N$ be a 2-torsion free prime $\Gamma$-near-ring with center $Z(N)$. Let $(f,d)$ and $(g,h)$ be two generalized derivations on $N$. We prove the following results:

(i) If $f([x,y]_\alpha) = 0$ or $f([x,y]_\alpha) = \pm [x,y]_\alpha$ for all $x,y \in N, \alpha \in \Gamma$, then $N$ is a commutative $\Gamma$-ring.

(ii) If $a \in N$ and $[f(x), a]_\alpha = 0$ for all $x \in N, \alpha \in \Gamma$, then $d(a) \in Z(N)$.

(iii) If $(fg,dh)$ acts as a generalized derivation on $N$, then $f = 0$ or $g = 0$.

1. Introduction

The derivations in $\Gamma$-near-rings have been introduced by Bell and Mason [1]. They studied basic properties of derivations in $\Gamma$-near-rings. Then Aşçi [2] obtained commutativity conditions for a $\Gamma$-near-ring with derivations. Some characterizations of $\Gamma$-near-rings and regularity conditions were obtained by Cho [3]. Kazaz and Alkan [4] introduced the notion of two-sided $\Gamma$-$\alpha$-derivation of a $\Gamma$-near-ring and investigated the commutativity of a prime and semiprime $\Gamma$-near-rings. Uçkun et al. [5] worked on prime $\Gamma$-near-rings with derivations and they found conditions for a $\Gamma$-near-ring to be commutative. In [6] Dey et al. studied commutativity of prime $\Gamma$-near-ring with generalized derivations.

In this paper, we obtain the conditions of a prime $\Gamma$-near-ring to be a commutative $\Gamma$-ring. If $a \in N$, and $[f(x), a]_\alpha = 0$ for all $x \in N, \alpha \in \Gamma$, then $d$ is central. Also we prove that if $(fg,dh)$ is the generalized derivation on $N$, then $f$ and $g$ are trivial.
2. Preliminaries

A \( \Gamma \)-near-ring is a triple \((N, +, \Gamma)\), where

(i) \((N, +)\) is a group (not necessarily abelian);

(ii) \(\Gamma\) is a nonempty set of binary operations on \(N\) such that for each \(\alpha \in \Gamma\), \((N, +, \alpha)\) is a left near-ring;

(iii) \(x\alpha(y\beta z) = (x\alpha y)\beta z\), for all \(x, y, z \in N\) and \(\alpha, \beta \in \Gamma\).

We will use the word \(\Gamma\)-near-ring to mean left \(\Gamma\)-near-ring. For a near-ring \(N\), the set \(N_0 = \{x \in N : 0\alpha x = 0, \alpha \in \Gamma\}\) is called the zero-symmetric part of \(N\). A \(\Gamma\)-near-ring \(N\) is said to be zero-symmetric if \(N = N_0\). Throughout this paper, \(N\) will denote a zero symmetric left \(\Gamma\)-near-ring with multiplicative centre \(Z(N)\). Recall that a \(\Gamma\)-near-ring \(N\) is prime if \(x\Gamma\Gamma y = 0\) implies \(x = 0\) or \(y = 0\). An additive mapping \(d : N \to N\) is said to be a derivation on \(N\) if \(d(x\alpha y) = x\alpha d(y) + \alpha d(x)\alpha y\) for all \(x, y \in \Gamma\), \(\alpha \in \Gamma\), or equivalently, as noted in [1], that \(d(x\alpha y) = d(x)\alpha y + x\alpha d(y)\) for all \(x, y \in N, \alpha \in \Gamma\). Further, an element \(x \in N\) for which \(d(x) = 0\) is called a constant. For \(x, y \in N, \alpha \in \Gamma\), the symbol \([x, y]_\alpha\) will denote the commutator \(x\alpha y - y\alpha x\), while the symbol \((x, y)\) will denote the additive-group commutator \(x + y - x - y\). An additive mapping \(f : N \to N\) is called a generalized derivation if there exists a derivation \(d\) of \(N\) such that \(f(x\alpha y) = f(x)\alpha y + x\alpha d(y)\) for all \(x, y \in N, \alpha \in \Gamma\). The concept of generalized derivation covers also the concept of a derivation.

3. Derivations on \(\Gamma\)-Near-Rings

In this section we prove that a few subsidiary results (Lemmas 3.1, 3.2, 3.4, 3.8, 3.9, 3.10 and 3.11) to use them for proving of our main results (Theorems 3.3, 3.5, 3.6, 3.12 and 3.13).

**Lemma 3.1.** Let \(d\) be an arbitrary derivation on a \(\Gamma\)-near-ring \(N\). Then \(N\) satisfies the following partial distributive law: \((a\alpha b) + d(a)\alpha b)c = a\alpha b\alpha c\) and \((a\alpha b + a\alpha d)\alpha c = d(a)\alpha b\alpha c\) for all \(a, b, c \in N, \alpha, \beta \in \Gamma\).

**Proof.** For all \(a, b, c \in N, \alpha, \beta \in \Gamma\), we get \(d((a\alpha b)c) = a\alpha b\beta d(c)\) and \((d(a\alpha b)c) = a\alpha (b\beta d(c) + d(b)\beta c) + d(a)\alpha b\beta c\). Equating these two relations for \(d(a\alpha b)c\) now yields the required partial distributive law.

**Lemma 3.2.** Let \(d\) be a derivation on a \(\Gamma\)-near-ring \(N\) and suppose \(u \in N\) is not a left zero divisor. If \([u, d(u)]_\alpha = 0, \alpha \in \Gamma\), then \((x, u)\) is a constant for every \(x \in N\).

**Proof.** From \(ua(u + x) = uau + uax\), for all \(x \in N, \alpha \in \Gamma\), we obtain \(u\alpha d(u + x) + d(u)\alpha (u + x) = u\alpha d(u) + d(u)\alpha u + u\alpha d(x)\), which reduces \(u\alpha d(x) + d(u)\alpha u = d(u)\alpha u + u\alpha d(x)\), for all \(\alpha \in \Gamma\).

Since \(d(u)\alpha u = u\alpha d(u)\), \(\alpha \in \Gamma\), this equation is expressible as \(u\alpha (d(x) + d(u) - d(x) - d(u)) = 0 = u\alpha (d(x) - u)\). Thus \(d((x, u)) = 0\).

**Theorem 3.3.** Let \(N\) be a \(\Gamma\)-near-ring having no nonzero divisors of zero. If \(N\) admits a nontrivial commuting derivation \(d\), then \((N, +)\) is abelian.
Proof. Let \( c \) be any additive commutator. Then \( c \) is a constant by Lemma 3.2. Moreover, for any \( w \in N, \alpha \in \Gamma, wac \) is an additive commutator, hence also a constant. Thus, \( 0 = d(wac) = wad(c) + d(w)ac \) and \( d(w)ac = 0 \), for all \( \alpha \in \Gamma \). Since \( d(w) \neq 0 \) for all \( w \in N \), we conclude that \( c = 0 \).

\( \square \)

Lemma 3.4. Let \( N \) be a prime \( \Gamma \text{-near-ring} \).

(i) If \( z \in Z(N) \setminus \{0\} \), then \( z \) is not a zero divisor in \( N \).

(ii) If \( Z(N) \setminus \{0\} \) contains an element \( z \) for which \( z + z \in Z(N) \), then \( (N,+) \) is abelian.

(iii) Let \( d \) be a nonzero derivation on \( N \). Then \( xt_d(N)\{0\} \) implies \( x = 0 \), and \( d(N)\Gamma x = \{0\} \) implies \( x = 0 \).

(iv) If \( N \) is 2-torsion free and \( d \) is a derivation on \( N \) such that \( d^2 = 0 \), then \( d = 0 \).

Proof. (i) If \( z \in Z(N) \setminus \{0\} \) and \( zax = 0, x \in N, \alpha \in \Gamma \), then \( zax = 0, x, r \in N, \alpha \in \Gamma \). Thus we get \( z\Gamma_N x = 0 \), by primeness of \( N, x = 0 \).

(ii) Let \( z \in Z(N) \setminus \{0\} \) be an element such that \( z + z \in Z(N) \), and let \( x, y \in N, \alpha \in \Gamma \). Since \( z + z \) is distributive, we get \( (x + y)a(z + z) = xa(z + z) + ya(z + z) = xaz + xaz + yaz + yaz = za(x + x + y) \).

On the other hand, \( (x + y)a(z + z) = (x + y)az + (x + y)az = za(x + y + x + y) \). Thus, \( x + x + y = x + y + x + y \) and therefore \( x + y = y + x \). Hence \( (N,+) \) is abelian.

(iii) Let \( xt_d(N) = 0 \), and let \( r, s \) be arbitrary elements of \( N \) and \( \alpha, \beta \in \Gamma \). Then \( 0 = xad(r,b)s = xar\beta d(s) + xad(r)b\beta s = xar\beta d(s) \). Thus \( x\Gamma_N d(N) = \{0\} \), and since \( d(N) \neq \{0\} \), \( x = 0 \).

A similar argument works if \( d(N)\Gamma x = \{0\} \), since Lemma 3.1 provides enough distributivity to carry it through.

(iv) For arbitrary \( x, y \in N, \alpha \in \Gamma \), we have \( 0 = d^2(xy) = d(xad(y) + d(x)ay) = xad^2(y) + d(x)ad(y) + d(x)ad(y) + d^2(xy) = 2d(x)ad(y) \). Since \( N \) is 2-torsion free, \( d(x)ad(y) = 0, x, y \in N, \alpha \in \Gamma \). Thus \( d(x)\Gamma d(N) = \{0\} \) for each \( x \in N \), and (iii) yields \( d(a) = 0 \). Thus \( d = 0 \).

\( \square \)

Theorem 3.5. If a prime \( \Gamma \text{-near-ring} \) \( N \) admits a nontrivial derivation \( d \) for which \( d(N) \in Z(N) \), then \( (N,+) \) is abelian. Moreover, if \( N \) is 2-torsion free, then \( N \) is a commutative \( \Gamma \text{-ring} \).

Proof. Let \( c \) be an arbitrary constant, and let \( x \) be anon-constant. Then \( d(xac) = xad(c) + d(x)ac = d(x)ac \in Z(N), \alpha \in \Gamma \). Since \( d(x) \in Z(N) \setminus \{0\} \), it follows easily that \( c \in Z(N) \).

Since \( c + c \) is a constant for all constants \( c \), it follows from Lemma 3.4(ii) that \( (N,+) \) is abelian, provided that there exists a nonzero constant.

Assume, then, that \( 0 \) is the only constant. Since \( d \) is obviously commuting, it follows from Lemma 3.2 that all \( u \) which are not zero divisors belong to the center of \( (N,+) \), denoted by \( Z(N) \). In particular, if \( x \neq 0 \), \( d(x) \in Z(N) \). But then for all \( y \in N \), \( d(y)d(x) - d(y) - d(x) = d((y,x)) = 0 \), hence \( (y,x) = 0 \).

Now we assume that \( N \) is 2-torsion free. By Lemma 3.1, \( aad(b) + d(a)ab\beta c = aad(b)\beta c + d(a)ab\beta c \) for all \( a, b, c \in N, \alpha, \beta \in \Gamma \), and using the fact that \( d(aab) \in Z(N) \), \( a, \alpha \in \Gamma \), we get \( caad(b) + cad(a)\beta b = aad(b)\beta c + aad(b)\beta c, a, \beta \in \Gamma \). Since \( (N,+) \) is abelian and \( d(N) \subseteq Z(N) \), this equation can be rearranged to yield \( d(b)a[c, a]_\beta = d(a)a[b, c]_\beta \) for all \( a, b, c \in N, \alpha, \beta \in \Gamma \).

Suppose now that \( N \) is not commutative. Choosing \( b, c \in N \), with \( [b, c]_\beta \neq 0, \beta \in \Gamma \), and letting \( a = d(x) \), we get \( d^2(x)a[b, c]_\beta = 0 \), for all \( x \in N, \alpha, \beta \in \Gamma \), and since the central
elements $d^2(x)$ cannot be a nonzero divisor of zero, we conclude that $d^2(x) = 0$ for all $x \in N$. But by Lemma 3.4(iv), this cannot happen for nontrivial $d$.

**Theorem 3.6.** Let $N$ be a prime $\Gamma$-near-ring admitting a nontrivial derivation $d$ such that $[d(x), d(y)]_{\alpha} = 0$ for all $x, y \in N, \alpha \in \Gamma$. Then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free, then $N$ is a commutative $\Gamma$-ring.

**Proof.** By Lemma 3.4(ii), if both $z$ and $z + z$ commute element-wise with $d(N)$, then $zad(c) = 0, \alpha \in \Gamma$, for all additive commutators $c$. Thus, taking $z = d(x)$, we get $d(x)ad(c) = 0$ for all $x \in N, \alpha \in \Gamma$, so $d(c) = 0$ by Lemma 3.4(iii). Since $wac$ is also an additive commutator for any $w \in N, \alpha \in \Gamma$, we have $d(wac) = 0 d(w)ac$, and another application of Lemma 3.4(iii) gives $c = 0$.

Now we assume that $N$ is 2-torsion free. By the partial distributive law, $d(d(x)ay\beta d(z)) = d(x)ad(y)\beta d(z) + d^2(x)ay\beta d(z)$ for all $x, y, z \in N, \alpha, \beta \in \Gamma$, hence, $d^2(x)ay\beta d(z) = d(d(x)ay\beta d(z) - d(x)ad(y)\beta d(z) = d(x)ad(y)\beta d(z) = d(z)ad(x\beta y) - d(x)ad(y)\beta d(z)) = d(z)d^2(x)\beta y = d^2(x)ad(z)\beta y, \alpha, \beta \in \Gamma$. Thus $d^2(x)\alpha(y\beta d(z) - d(z)\beta y) = 0$ for all $x, y, z \in N, \alpha, \beta \in \Gamma$.

Replacing $y\delta t, \delta \in \Gamma$, we obtain $d^2(x)ay\delta t\beta d(z) = d^2(x)ad(z)\beta y\delta t = d^2(x)ay\beta d(z)\delta t$, for all $x, y, z, t \in N, \alpha, \beta, \delta \in \Gamma$, so that $d^2(x)ay\beta [t, d(z)]_{\delta} = 0$ for all $x, y, z, t \in N, \alpha, \beta, \delta \in \Gamma$. The primeness of $N$ shows that either $d^2 = 0$ or $d(N) \subseteq Z(N)$, and since the first of these conditions is impossible by Lemma 3.4(iv), the second must hold and $N$ is a commutative $\Gamma$-ring by Theorem 3.5.

**Definition 3.7.** Let $N$ be a $\Gamma$-near-ring and $d$ a derivation of $N$. An additive mapping $f : N \to N$ is said to be a right generalized derivation of $N$ associated with $d$ if

$$f(xy) = f(x)ay + xad(y) \quad \forall x, y \in R, \quad \alpha \in \Gamma,$$

and $f$ is said to be a left generalized derivation of $N$ associated with $d$ if

$$f(xy) = d(x)ay + xaf(y) \quad \forall x, y \in R, \quad \alpha \in \Gamma.$$

Finally, $f$ is said to be a generalized derivation of $N$ associated with $d$ if it is both a left and right generalized derivation of $N$ associated with $d$.

**Lemma 3.8.** Let $f$ be a right generalized derivation of a $\Gamma$-near ring $N$ associated with $d$. Then

(i) $f(xy) = xad(y) + f(x)ay$ for all $x, y \in N, \alpha \in \Gamma$;

(ii) $f(xy) = xaf(y) + d(x)ay$ for all $x, y \in N, \alpha \in \Gamma$.

**Proof.** (i) For any $x, y \in N, \alpha \in \Gamma$, we get

$$f(xa(y + y)) = f(x)a(y + y) + xad(y + y) = f(x)ay + f(x)ay + xad(y) + xad(y),$$

$$f(xy + xay) = f(x)ay + xad(y) + f(x)ay + xad(y).$$

(3.3)
Comparing these two expressions, we obtain

\[ f(x)ay + xad(y) = xad(y) + f(x)ay \quad \forall x, y \in N, \ a \in \Gamma, \]  

and so,

\[ f(xay) = xad(y) + f(x)ay \quad \forall x, y \in N, \ a \in \Gamma. \]  

(ii) In a similar way.

Lemma 3.9. Let \( f \) be a right generalized derivation of a \( \Gamma \)-near ring \( N \) associated with \( d \). Then

(i) \( (f(x)ay + xad(y))\beta z = f(x)ay\beta z + xad(y)\beta z \), for all \( x, y \in N, \ a, \beta \in \Gamma \).

(ii) \( (d(x)ay + xaf(y))\beta z = d(x)ay\beta z + xaf(y)\beta z \), for all \( x, y \in N, \ a, \beta \in \Gamma \).

Proof. (i) For any \( x, y, z \in N, \ a, \beta \in \Gamma \), we get \( f((xay)\beta z) = f(xay)\beta z + xay\beta d(z) \). On the other hand,

\[ f(xa(y\beta z)) = f(x)ay\beta z + xad(y\beta z) = f(x)ay\beta z + xad(y)\beta z + xay\beta d(z). \]  

From these two expressions of \( f(xay\beta z) \), we obtain that, for all \( x, y, z \in N, \ a, \beta \in \Gamma \),

\[ (f(x)ay + xad(y))\beta z = f(x)ay\beta z + xad(y)\beta z. \]  

(ii) The proof is similar.

Lemma 3.10. Let \( N \) be a prime \( \Gamma \)-near-ring, \( f \) a nonzero generalized derivation of \( N \) associated with the nonzero derivation \( d \) and \( a \in N \). (i) If \( a\Gamma f(N) = 0 \), then \( a = 0 \). (ii) If \( f(N)\Gamma a = 0 \), then \( a = 0 \).

Proof. (i) For any \( x, y \in N, \ a, \beta \in \Gamma \), we get \( 0 = a\beta f(xay) = a\beta f(x)ay + a\beta xad(y) = a\beta xad(y) \). Hence \( a\Gamma N\Gamma d(N) = 0 \). Since \( N \) is a prime \( \Gamma \)-near-ring and \( d \neq 0 \), we obtain \( a = 0 \).

(ii) A similar argument works if \( f(N)\Gamma a = 0 \).

Lemma 3.11. Let \( N \) be a prime \( \Gamma \)-near-ring. Let \( f \) be a generalized derivation of \( N \) associated with the nonzero derivation \( d \). If \( N \) is a 2-torsion free \( \Gamma \)-near-ring and \( f^{2} = 0 \), then \( f = 0 \).

Proof. (i) For any \( x, y \in N, \ a \in \Gamma \), we get

\[ 0 = f^{2}(xay) = f(f(xay)) = f(f(x)ay + xad(y)) = f^{2}(x)ay + 2f(x)ad(y) + xad^{2}(y). \]  

By the hypothesis,

\[ 2f(x)ad(y) + xad^{2}(y) = 0 \quad \forall x, y \in N, \ a \in \Gamma. \]  

Writing \( f(x) \) by \( x \) in (3.9), we get \( f(x)ad^{2}(y) = 0 \), for all \( x, y \in N, \ a \in \Gamma \).
By Lemma 3.9(ii), we obtain that $d^2(N) = 0$ or $f = 0$. If $d^2(N) = 0$ then $d = 0$ from Lemma 3.4(iv), a contradiction. So we find $f = 0$. \hfill \Box

**Lemma 3.9.** Let $N$ be a prime $\Gamma$-near-ring with a nonzero generalized derivation $f$ associated with $d$. If $f(N) \subseteq Z(N)$, then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free, then $N$ is commutative $\Gamma$-ring.

**Proof.** Suppose that $a \in N$, such that $f(a) \neq 0$. So, $f(a) \in Z(N) - \{0\}$ and $f(a) + f(a) \in Z(N) - \{0\}$. For all $x, y \in N$, $\alpha \in \Gamma$, we have $(x + y)\alpha f(a + f(a)) = (f(a + f(a))\alpha(x + y)$. That is, $xaf(a) + xaf(a) + yaf(a) + yaf(a) = f(a)ax + f(\alpha)af(a) + f(\alpha)ay + f(\alpha)ay$, for all $x, y \in N$, $\alpha \in \Gamma$.

Since $f(a) \in Z(N)$, we get $f(a)ax + f(\alpha)af(a) + f(\alpha)ay = f(\alpha)ay + f(\alpha)ax$, and so, $f(a)a(x, y) = 0$ for all $x, y \in N$, $\alpha \in \Gamma$.

Since $f(a) \in Z(N) - \{0\}$ and $N$ is a prime $\Gamma$-near-ring, it follows that $x, y = 0$, for all $x, y \in N$. Thus $(N, +)$ is abelian.

Using the hypothesis, for any $x, y, z \in N, \alpha, \beta \in \Gamma$, $zaf(x\beta y) = f(x\beta y)\alpha z$. By Lemma 3.4(ii), we have $zad(x)\beta y + zaf(y) = d(x)\alpha y\beta z + xaf(y)\beta z$. Using $f(N) \subseteq Z(N)$ and $(N, +)$ being abelian, we obtain that

$$zad(x)\beta y - d(x)\alpha y\beta z = [x, z]_\alpha \beta f(y), \quad \forall x, y \in N, \alpha, \beta \in \Gamma. \quad (3.10)$$

Substituting $f(z)$ for $z$ in (3.10), we get $f(z)\beta[d(x), y]_\alpha = 0$ for all $x, y \in N, \alpha, \beta \in \Gamma$.

Since $f(z) \in Z(N)$ and $f$ a nonzero generalized derivation with associated with $d$, we get $d(N) \subseteq Z(N)$. So, $N$ is a commutative $\Gamma$-ring by Theorem 3.3. \hfill \Box

**Theorem 3.13.** Let $N$ be a prime $\Gamma$-near-ring with a nonzero generalized derivation $f$ associated with $d$. If $[f(N), f(N)]_\alpha = 0, \alpha \in \Gamma$, then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free, then $N$ is commutative $\Gamma$-ring.

**Proof.** By the same argument as in Theorem 3.12, it is shown that if both $z$ and $z + z$ commute elementwise with $f(N)$, then we have

$$zaf(x, y) = 0 \quad \forall x, y \in N, \alpha \in \Gamma. \quad (3.11)$$

Substituting $f(t), t \in N$ for $z$ in (3.11), we get $f(t)\alpha f(x, y) = 0, \alpha \in \Gamma$. By Lemma 3.9(i), we obtain that $f(x, y) = 0$ for all $x, y \in N, \alpha \in \Gamma$. For any $w \in N, \beta \in \Gamma$, we have $0 = f(w\beta x, w\beta y) = f(w\beta(x, y)) = d(w)\beta(x, y) + w\beta f(x, y)$ and so, we obtain $d(w)\beta(x, y) = 0$, for any $w \in N, \beta \in \Gamma$. From Lemma 3.4(iii), we get $(x, y) = 0$ for any $x, y \in N$.

Now we assume that $N$ is 2-torsion free. By the assumption $[f(N), f(N)]_\alpha = 0, \alpha \in \Gamma$, we have

$$f(z)af(f(x)\beta y) = f(f(x)\beta y)af(z) \quad \forall x, y, z \in N, \alpha, \beta \in \Gamma. \quad (3.12)$$

Hence we get

$$f(z)ad(f(x))\beta y + f(z)\alpha f(x)\beta f(y) = d(f(x))\alpha y\beta f(z) + f(x)(\alpha f(y))\beta f(z),$$

$$f(z)ad(f(x))\beta y + f(x)(\alpha f(z)\beta f(y)) = d(f(x))\alpha y\beta f(z) + f(x)(\alpha f(z)\beta f(y)). \quad (3.13)$$
and so,

\[ f(z)ad(f(x)βy) = d(f(x))αyβf(z) \quad ∀x,y,z ∈ N, \ α, β ∈ \Gamma. \] (3.14)

If we take \( yδω \) instead of \( y \) in (3.14), then

\[ d(f(x))αyδωβf(z) = f(z)ad(f(x)βyδω) = d(f(x))αyδf(z)βω \]

∀x, y, z ∈ N, α, β, δ ∈ Γ,

and so,

\[ d(f(x))αyδωβf(z) - d(f(x))αyδf(z)βω = d(f(x))αyδ[f(z), ω]_δ = 0 \]

∀x, y, z ∈ N, α, β, δ ∈ Γ.

Thus we get \( d(f(x))Γ\Gamma[f(z), ω]_δ = 0 \), for all \( x, y, z ∈ N, \ α, β, δ ∈ Γ \). Since \( N \) is a prime \( \Gamma \)-near-ring, we have \( d(f(N)) = 0 \) or \( f(N) ⊂ Z(N) \). Let us assume that \( d(f(N)) = 0 \). Then

\[ 0 = d(f(xay)) = d(d(x)ay + xaf(y)) \] (3.17)

and so,

\[ d^2(x)ay + d(x)ad(y) + d(x)af(y) = 0, \ ∀x, y ∈ N, \ α ∈ \Gamma. \] (3.18)

Replacing \( y \) by \( yβz, β ∈ \Gamma \), in (3.18), we get

\[ 0 = d^2(x)ayβz + d(x)ad(yβz) + d(x)af(yβz) \]

\[ = d^2(x)ayβz + d(x)ad(y)βz + d(x)ayβd(z) + d(x)af(y)βz + d(x)ayβd(z) \]

\[ = \left\{ d^2(x)ay + d(x)ad(y) + d(x)af(y) \right\} βz + 2d(x)ayβd(z) \quad ∀x, y, z ∈ N, \ α, β ∈ \Gamma. \] (3.19)

Using (3.18) and \( N \) being 2-torsion free \( \Gamma \)-near-ring, we get \( d(N)\Gamma\Gamma d(N) = 0 \).

Thus we obtain that \( d = 0 \). It contradicts by \( d ≠ 0 \). The theorem is proved.

\[ \square \]

**4. Generalized Derivations of \( \Gamma \)-Near-Rings**

We denote a generalized derivation \( f : N → N \) determined by a derivation \( d \) of \( N \) by \( (f,d) \).

We assume that \( d \) is a nonzero derivation of \( N \) unless stated otherwise.

**Theorem 4.1.** Let \( (f,d) \) be a generalized derivation of \( N \). If \( f([x,y]_α) = 0 \) for all \( x, y ∈ N, α ∈ \Gamma \), then \( N \) is a commutative \( \Gamma \)-ring.
Proof. Assume that \( f([x, y]_a) = 0 \) for all \( x, y \in N, \alpha \in \Gamma \). Substitute \( x\beta y \) instead of \( y \), obtaining
\[
f([x, x\beta y]_a) = f(x\beta[x, y]_a) = d(x)\beta[x, y]_a + x\beta f([x, y]_a) = 0.
\]
(4.1)
Since the second term is zero, it is clear that
\[
d(x)ax\beta y = d(x)ay\beta x \quad \forall x, y \in N, \alpha, \beta \in \Gamma.
\]
(4.2)
Replacing \( y \) by \( y\delta z \) in (4.2) and using this equation, we get
\[
d(x)ay\beta[z, \delta] = 0 \quad \forall x, y, z \in N, \alpha, \beta, \delta \in \Gamma.
\]
(4.3)
Hence either \( x \in Z(N) \) or \( d(x) = 0 \). Let \( L = \{x \in N \mid d(x) = 0\} \). Then \( Z(N) \) and \( L \) are two additive subgroups of \((N, +) = Z(N) \cup L\). However, a group cannot be the union of proper subgroups, hence either \( N = Z(N) \) or \( N = L \). Since \( d \neq 0 \), we are forced to conclude that \( N \) is a commutative \( \Gamma \)-ring.

**Theorem 4.2.** Let \((f, d)\) be a generalized derivation of \( N \). If \( f([x, y]_a) = \pm[x, y]_a \) for all \( x, y \in N, \alpha \in \Gamma \), then \( N \) is a commutative \( \Gamma \)-ring.

**Proof.** Assume that \( f([x, y]_a) = \pm[x, y]_a \) for all \( x, y \in N, \alpha \in \Gamma \). Replacing \( y \) by \( x\beta y, \beta \in \Gamma \), in the hypothesis, we have
\[
f([x, x\beta y]_a) = \pm(xax\beta y - xay\beta x) = \pm x\beta[x, y]_a.
\]
(4.4)
On the other hand,
\[
f([x, x\beta y]_a) = f(x\beta[x, y]_a) = d(x)\beta[x, y]_a + x\beta f([x, y]_a) = d(x)\beta[x, y]_a + x\beta(\pm[x, y]_a).
\]
(4.5)
It follows from the two expressions for \( f([x, x\beta y]_a) \) that
\[
d(x)ax\beta y = d(x)ay\beta x \quad \forall x, y \in N, \alpha, \beta \in \Gamma.
\]
(4.6)
Using the same argument as in the proof of Theorem 4.1, we get that \( N \) is a commutative \( \Gamma \)-ring.

**Theorem 4.3.** Let \((f, d)\) be a nonzero generalized derivation of \( N \). If \( f \) acts as a homomorphism on \( N \), then \( f \) is the identity map.

**Proof.** Assume that \( f \) acts as a homomorphism on \( N \). Then one obtains
\[
f(xay) = f(x)af(y) = d(x)ay + xaf(y) \quad \forall x, y \in N, \alpha \in \Gamma.
\]
(4.7)
Replacing $y$ by $y\beta z$ in (4.7), we arrive at

$$f(x)\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha f(y\beta z).$$

(4.8)

Since $(f,d)$ is a generalized derivation and $f$ acts as a homomorphism on $N$, we deduce that

$$f(x\alpha y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z).$$

(4.9)

By Lemma 3.9(ii), we get

$$d(x)\alpha y\beta f(z) + x\alpha f(y)\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z),$$

(4.10)

and so

$$d(x)\alpha y\beta f(z) + x\alpha f(y\beta z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z).$$

(4.11)

That is,

$$d(x)\alpha y\beta f(z) + x\alpha d(y)\beta z + x\alpha y\beta f(z) = d(x)\alpha y\beta z + x\alpha d(y)\beta z + x\alpha y\beta f(z).$$

(4.12)

Hence, we deduce that

$$d(x)\alpha y\beta (f(z) - z) = 0 \quad \forall x, y, z \in N, \quad \alpha, \beta \in \Gamma.$$  

(4.13)

Because $N$ is prime and $d \neq 0$, we have $f(z) = z$ for all $z \in N$. Thus, $f$ is the identity map.  

**Theorem 4.4.** Let $(f,d)$ be a nonzero generalized derivation of $N$. If $f$ acts as an antihomomorphism on $N$, then $f$ is the identity map.

**Proof.** By the hypothesis, we have

$$f(x\alpha y) = f(y)f(x\alpha) = d(x)\alpha y + x\alpha f(y) \quad \forall x, y \in N, \quad \alpha \in \Gamma.$$  

(4.14)

Replacing $y$ by $x\beta y$ in the last equation, we obtain

$$f(x\beta y)\alpha f(x) = d(x)\beta x\alpha y + x\beta f(x\alpha y).$$  

(4.15)

Since $(f,d)$ is a generalized derivation and $f$ acts as an antihomomorphism on $N$, we get

$$(d(x)\beta y + x\beta f(y))\alpha f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x).$$  

(4.16)

By Lemma 3.9(ii), we conclude that

$$d(x)\alpha y\beta f(x) + x\alpha f(y)\beta f(x) = d(x)\alpha x\beta y + x\alpha f(y)\beta f(x),$$  

(4.17)
and so
\[ d(x)yβf(x) = d(x)xβy \quad \forall x, y \in N, \ a, β \in Γ. \] (4.18)

Replacing \( y \) by \( yδz \) and using this equation, we have
\[ d(x)yβ[ f(x), z ]_a = 0 \quad \forall x, z \in N, \ a, β \in Γ. \] (4.19)

Hence we obtain the following alternatives: \( d(x) = 0 \) or \( f(x) \in Z(N) \), for all \( x \in N \). By a standard argument, one of these must hold for all \( x \in N \). Since \( d \neq 0 \), the second possibility gives that \( N \) is commutative \( Γ \)-ring by Theorem 3.12, and so we deduce that \( f \) is the identity map by Theorem 4.3.

Theorem 4.5. Let \( (f, d) \) be a generalized derivation of \( N \) such that \( d(Z(N)) \neq 0 \), and \( a \in N \). If \( [f(x), a]_a = 0 \) for all \( x \in N, \ a, β \in Γ \), then \( a \in Z(N) \).

Proof. Since \( d(Z(N)) \neq 0 \), there exists \( c \in Z(N) \) such that \( d(c) \neq 0 \). Furthermore, as \( d \) is a derivation, it is clear that \( d(c) \in Z(N) \). Replacing \( x \) by \( cβx, \ β \in Γ \), in the hypothesis and using Lemma 3.9(ii), we have
\[ f(cβx)aa = aaf(cβx), \] (4.20)
\[ d(c)xβa + caf(x)βa = aad(c)βx + aacβf(x). \]

Since \( c \in Z(N) \) and \( d(c) \in Z(N) \), we get
\[ d(c)xβy = 0 \quad ∀ y \in N, \ a, β, δ \in Γ. \] (4.21)

By the primeness of \( N \) and \( 0 \neq d(c) \in Z(N) \), we obtain that \( a \in Z(N) \).

Theorem 4.6. Let \( (f, d) \) be a generalized derivation of \( N \), and \( a \in N \). If \( [f(x), a]_a = 0 \) for all \( x \in N, \ a \in N \) then \( d(a) \in Z(N) \).

Proof. If \( a = 0 \), then there is nothing to prove. Hence, we assume that \( a \neq 0 \).

Replacing \( x \) by \( aβx \) in the hypothesis, we have
\[ f(aβx)aa = aaf(aβx), \] (4.22)
\[ d(a)xβa + aaf(x)βa = aad(a)βx + aacβf(x). \]

Using \( f(x)aa = aaf(x) \), we have
\[ d(a)xβa = aad(a)βx \quad ∀ x \in N, \ a, β \in Γ. \] (4.23)

Taking \( xδy \) instead of \( x \) in the last equation and using this, we conclude that
\[ d(a)xNβ[a, y]_a = 0 \quad ∀ y \in N, \ a, β \in Γ. \] (4.24)
Since $N$ is a prime $\Gamma$-near-ring, we have either $d(a) = 0$ or $a \in Z(N)$. If $0 \neq a \in Z(N)$, then $(N, +)$ is abelian by Lemma 3.2(ii). Thus

\[ f(xaa) = f(aa x) \]

\[ f(x)aa + xad(a) = d(a)ax + af(x) \]

and so

\[ [d(a), x] = 0 \quad \forall x \in N, \quad a \in \Gamma. \]

That is, $d(a) \in Z(N)$. Hence in either case we have $d(a) \in Z(N)$. This completes the proof. \(\square\)

**Theorem 4.7.** Let $(f, d)$ be a generalized derivation of $N$. If $N$ is a 2-torsion free $\Gamma$-near-ring and $f^2(N) \subseteq Z(N)$, then $N$ is a commutative $\Gamma$-ring.

**Proof.** Suppose that $f^2(N) \subseteq Z(N)$. Then we get

\[ f^2(xy) = f^2(x)y + 2f(x)ad(y) + xad^2(y) \in Z(N) \quad \forall x, y \in N, \quad a \in \Gamma. \]  

(4.27)

In particular, $f^2(x)ac + 2f(x)ad(c) + xad^2(c) \in Z(N)$ for all $x \in N, c \in Z(N), a \in \Gamma$. Since the first summand is an element of $Z(N)$, we have

\[ 2f(x)ad(c) + xad^2(c) \in Z(N) \quad \forall x \in N, \quad c \in Z(N), \quad a \in \Gamma. \]  

(4.28)

Taking $f(x)$ instead of $x$ in (4.28), we obtain that

\[ 2f^2(x)ad(c) + f(x)ad^2(c) \in Z(N) \quad \forall x \in N, \quad c \in Z(N), \quad a \in \Gamma. \]  

(4.29)

Since $d(c) \in Z(N), f^2(x) \in Z(N), and so $f^2(x)ad(c) \in Z(N)$ for all $x \in N, c \in Z(N), a \in \Gamma$, we conclude $f(x)ad^2(c) \in Z(N)$ for all $x \in N, c \in Z(N), a \in \Gamma$.

Since $N$ is prime, we get $d^2(Z(N)) = 0$ or $f(N) \subseteq Z(N)$. If $f(N) \nsubseteq Z(N)$, then $N$ is a commutative $\Gamma$-ring by Lemma 3.8. Hence, we assume $d^2(Z) = 0$. By (4.28), we get $2f(x)ad(c) \in Z(N)$ for all $x \in N, c \in Z(N), a \in \Gamma$.

Since $N$ is a 2-torsion free near-ring and $d(c) \in Z(N)$, we obtain that either $f(N) \subseteq Z(N)$ or $d(Z(N)) = 0$. If $f(N) \subseteq Z(N)$, then we are already done. So, we may assume that $d(Z(N)) = 0$. Then

\[ f(cax) = f(xac), \]

\[ f(c)ax + cad(x) = f(x)ac + xad(c), \]  

(4.30)

and so

\[ f(c)ax + cad(x) = f(x)ac \quad \forall x \in N, \quad c \in Z(N). \]  

(4.31)
Now replacing $x$ by $f(x)$ in (4.31), and using the fact that $f^2(N) \subseteq Z(N)$, we get

$$f(c)af(x) + cad(f(x)) = f^2(x)ac \quad \forall x \in N, \ c \in Z(N).$$  \hspace{1cm} (4.32)

That is,

$$f(c)af(x) + cad(f(x)) \in Z \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma.$$  \hspace{1cm} (4.33)

Again taking $f(x)$ instead of $x$ in this equation, one can obtain

$$f(c)af^2(x) + cad(f^2(x)) \in Z \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma.$$  \hspace{1cm} (4.34)

The second term is equal to zero because of $d(Z) = 0$. Hence we have $f(c)af^2(x) \in Z(N)$ for all $x \in N, c \in Z(N), a \in \Gamma$.

Since $f^2(N) \subseteq Z(N)$ by the hypothesis, we get either $f^2(N) = 0$ or $f(Z(N)) \subseteq Z(N)$. If $f^2(N) = 0$, then the theorem holds by Definition 3.7. If $f(Z) \subseteq Z(N)$, then $f(xaf(c)) = f(f(c)ax)$ for all $x \in N, c \in Z(N)$, and so

$$d(x)af(x) = f(c)af(x) \quad \forall x \in N, \ c \in Z(N).$$ \hspace{1cm} (4.35)

Using $f(c) \in Z(N)$, we now have $f(c)a(d(x) - f(x)) = 0$ for all $x \in N, c \in Z(N), a \in \Gamma$. Since $f(Z(N)) \subseteq Z(N)$, we have either $f(Z(N)) = 0$ or $d = f$. If $d = f$, then $f$ is a derivation of $N$ and so $N$ is commutative $\Gamma$-ring by Lemma 3.11.

Now assume that $f(Z(N)) = 0$. Returning to the equation (4.31), we have

$$ca(d(x) - f(x)) = 0 \quad \forall x \in N, \ c \in Z(N), \ a \in \Gamma.$$ \hspace{1cm} (4.36)

Since $c \in Z(N)$, we have either $d = f$ or $Z(N) = 0$. Clearly, $d = f$ implies the theorem holds. If $Z(N) = 0$, then $f^2(N) = 0$ by the hypothesis, and so $N$ is a commutative $\Gamma$-ring by Lemma 3.4(iv). Hence, the proof is completed. \hfill \Box

**Corollary 4.8.** Let $N$ be a 2-torsion free near-ring, and $(f, d)$ a nonzero generalized derivation of $N$. If $[f(N), f(N)]_a = 0, a \in \Gamma$, then $N$ is a commutative $\Gamma$-ring.

**Lemma 4.9.** Let $(f, d)$ and $(g, h)$ be two generalized derivations of $N$. If $h$ is a nonzero derivation on $N$ and $f(x)ah(y) = -g(x)ad(y)$ for all $x, y \in N$, then $(N, +)$ is abelian.

**Proof.** Suppose that $f(x)ah(y) + g(x)ad(y) = 0$ for all $x, y \in N, a \in \Gamma$.

We substitute $y + z$ for $y$, thereby obtaining

$$f(x)ah(y) + f(x)ah(z) + g(x)ad(y) + g(x)ad(z) = 0.$$ \hspace{1cm} (4.37)

Using the hypothesis, we get

$$f(x)ah(y, z) = 0 \quad \forall x, y, z \in N, \ a \in \Gamma.$$ \hspace{1cm} (4.38)
It follows by Lemma 3.10(ii) that \( h(y, z) = 0 \) for all \( y, z \in N \). For any \( w \in N \), we have \( h(wa(y, z)) = h(w) a(y, z) + wah(y, z) = 0 \) and so \( h(w) a(y, z) = 0 \) for all \( w, y, z \in N, a \in \Gamma \).

An appeal to Lemma 3.4(iii) yields that \((N, +)\) is abelian.

**Theorem 4.10.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of \( N \). If \( N \) is 2-torsion free and \( f(x)ah(y) = g(x)ad(y) \) for all \( x, y \in N, a \in \Gamma \), then \( f = 0 \) or \( g = 0 \).

**Proof.** If \( h = 0 \) or \( d = 0 \), then the proof of the theorem is obvious. So, we may assume that \( h \neq 0 \) and \( d \neq 0 \). Therefore, we know that \((N, +)\) is abelian by Lemma 4.9.

Now suppose that

\[
f(x)ah(y) + g(x)ad(y) = 0 \quad \forall x, y \in N, \ a \in \Gamma. \tag{4.39}
\]

Replacing \( x \) by \( u^\beta v \) in this equation and using the hypothesis, we get

\[
f(u^\beta v)ah(y) + g(u^\beta v)ad(y)
\]

\[
= uaf(v)^\beta h(y) + d(u)a\alpha v^\beta h(y) + uag(v)^\beta d(y) + h(u)a\alpha v^\beta d(y) \tag{4.40}
\]

\[
= 0,
\]

and so

\[
d(u)a\alpha v^\beta h(y) = -h(u)a\alpha v^\beta d(y) \quad \forall u, v, y \in N, \ a \in \Gamma. \tag{4.41}
\]

Taking \( y^\delta t \) instead of \( y \) in the above relation, we obtain

\[
d(u)a\alpha v^\beta h(y)^\delta t + d(u)a\alpha v^\beta \delta h(t) = -h(u)a\alpha v^\beta d(y)^\delta t - h(u)a\alpha v^\beta \delta d(t). \tag{4.42}
\]

That is,

\[
d(u)a\alpha v^\beta \gamma \delta h(t) = -h(u)a\alpha v^\beta \gamma \delta d(t) \quad \forall u, v, y, t \in N, \ a, \beta, \delta \in \Gamma. \tag{4.43}
\]

Replacing \( y \) by \( h(y) \) in (4.43) and using this relation, we have

\[
h(u)aN^\beta (d(y)^\gamma \delta h(t) - h(y)a^\delta d(t)) = 0 \quad \forall u, y, t \in N. \tag{4.44}
\]

Since \( N \) is a prime \( \Gamma \)-near-ring and \( h \neq 0 \), we obtain that

\[
d(y)^\alpha h(t) = h(y)a^\delta d(t), \quad \forall y, t \in N. \tag{4.45}
\]

Now again taking \( u^\lambda v \) instead of \( x \) in the initial hypothesis, we get

\[
f(u)a\alpha v^\beta h(y) + uad(v)^\beta h(y) + g(u)a\alpha v^\beta d(y) + uah(v)^\beta d(y) = 0. \tag{4.46}
\]
Using (4.45) yields that
\[
f(u)av\beta h(y) + 2uah(v)\beta d(y) + g(u)av\beta d(y) = 0 \quad \forall u, v, y \in N, \tag{4.47}
\]
Taking \(h(v)\) instead of \(v\) in this equation, we arrive at
\[
f(u)ah(v)\beta h(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y) = 0. \tag{4.48}
\]
By the hypothesis and (4.45), we have
\[
0 = -g(u)ad(v)\beta h(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y)
\]
\[
= -g(u)ah(v)\beta d(y) + 2uah^2(v)\beta d(y) + g(u)ah(v)\beta d(y), \tag{4.49}
\]
and so
\[
2uah^2(v)\beta d(y) = 0 \quad \forall u, v, y \in N, \quad \alpha, \beta \in \Gamma. \tag{4.50}
\]
Since \(N\) is a 2-torsion free prime \(\Gamma\)-near-ring, we obtain that \(h^2(N)\Gamma d(N) = 0\). An appeal to Lemma 3.4(iii) and (iv) gives that \(h = 0\). This contradicts by our assumption. Thus the proof is completed.

**Theorem 4.11.** Let \((f, d)\) and \((g, h)\) be two generalized derivations of \(N\). If \((fg, dh)\) acts as a generalized derivation on \(N\), then \(f = 0\) or \(g = 0\).

**Proof.** By calculating \(fg(xy)\) in two different ways, we see that \(g(x)ad(y) + f(x)ah(y) = 0\) for all \(x, y \in N, \alpha \in \Gamma\). The proof is completed by using Theorem 4.10.

**Acknowledgments**

The paper was supported by Grant 01-12-10-978FR MOHE Malaysia. The authors are thankful to the referee for valuable comments.

**References**

Submit your manuscripts at
http://www.hindawi.com