Research Article

Integrability for Solutions of Anisotropic Obstacle Problems

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This paper deals with anisotropic obstacle problem for the $A$-harmonic equation

$$
\sum_{i=1}^{n} D_i(a_i(x,Du(x))) = 0.
$$

An integrability result is given under suitable assumptions, which show higher integrability of the boundary datum, and the obstacle force solutions $u$ have higher integrability as well.

1. Introduction and Statement of Result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. For $p_i > 1$, $i = 1, 2, \ldots, n$, we denote

$$
p_m = \max_{i=1,2,\ldots,n} p_i,
$$

and $\overline{p}$ is the harmonic mean of $p_i$, that is,

$$
\frac{1}{\overline{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}.
$$

The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined by

$$
W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \ldots, n \right\}.
$$

Let us consider solutions $u \in W^{1,(p_i)}(\Omega)$ of the following $A$-harmonic equation:

$$
\sum_{i=1}^{n} D_i(a_i(x,Du(x))) = 0.
$$
where $D = (D_1, D_2, \ldots, D_n)$ is the gradient operator, and the Carathéodory functions $a_i(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, satisfy
\begin{equation}
|a_i(x, z)| \leq c_2 (h(x) + |z|)^{p_i - 1}, \tag{1.4}
\end{equation}
for almost every $x \in \Omega$, for every $z \in \mathbb{R}^n$, and for any $i = 1, 2, \ldots, n$, and there exists $\tilde{v} \in (0, +\infty)$ such that
\begin{equation}
\tilde{v} \sum_{i=1}^n |z_i - \tilde{z}_i|^{p_i} \leq \sum_{i=1}^n (a_i(x, z) - a_i(x, \tilde{z})) (z_i - \tilde{z}_i), \tag{1.5}
\end{equation}
for almost every $x \in \Omega$, for any $z, \tilde{z} \in \mathbb{R}^n$. The integrability condition for $h(x) \geq 0$ in (1.4) will be given later.

Let $\psi$ be any function in $\Omega$ with values in $\mathbb{R} \cup \{\pm \infty\}$ and $\theta \in W^{1,p_i}(\Omega)$, and we introduce
\begin{equation}
\mathcal{K}_{\psi,\theta}(\Omega) = \left\{ v \in W^{1,p_i}(\Omega) : v \geq \psi, \text{ a.e. and } v - \theta \in W_{0}^{1,p_i}(\Omega) \right\}. \tag{1.6}
\end{equation}
Note that
\begin{equation}
W_{0}^{1,p_i}(\Omega) = \left\{ v \in W_{0}^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \ldots, n \right\}. \tag{1.7}
\end{equation}
The function $\psi$ is an obstacle and $\theta$ determines the boundary values.

**Definition 1.1.** A solution to the $\mathcal{K}_{\psi,\theta}$-obstacle problem is a function $u \in \mathcal{K}_{\psi,\theta}(\Omega)$ such that
\begin{equation}
\int_{\Omega} \sum_{i=1}^n a_i(x, Du(x)) (D_i v(x) - D_i u(x)) dx \geq 0, \tag{1.8}
\end{equation}
whenever $v \in \mathcal{K}_{\psi,\theta}(\Omega)$.

Higher integrability property is important among the regularity theories of nonlinear elliptic PDEs and systems, see the monograph [1] by Bensoussan and Frehse. Meyers and Elcrat [2] first considered the higher integrability for weak solutions of (1.3) in 1975. Iwaniec and Sbordone [3] obtained a regularity result for very weak solutions of the $\mathcal{A}$-harmonic equation (1.3) by using the celebrated Gehring’s Lemma. Global integrability for anisotropic equation is contained in [4]. As far as higher integrability of $\nabla u$ is concerned, in problems with nonstandard growth a delicate interplay between the regularity with respect to $x$ and the growth with respect to $\xi$ appears: see [5]. For a global boundedness result of anisotropic variational problems, see [6]. For other related works, see [7]. We refer the readers to the classical books by Ladyženskaya and Ural’ceva [8], Morrey [9], Gilbarg and Trudinger [10] and Giaquinta [11] for some details of isotropic cases.

In the present paper, we consider integrability for solutions of anisotropic obstacle problems of the $\mathcal{A}$-harmonic equation (1.3), which show higher integrability of the boundary
datum, and the obstacle force solutions $u$, have higher integrability as well. The idea of this paper comes from [4], and the result can be considered as a generalization of [4, Theorem 2.1].

**Theorem 1.2.** Let $u \in \mathcal{K}^{(p)}_{\varphi,\theta}(\Omega)$ be a solution to the $\mathcal{K}^{(p)}_{\varphi,\theta}$ obstacle problem and $\theta \in W^{1,(q_i)}(\Omega)$, $q_i \in (p_i,+\infty)$, $i = 1, 2, \ldots, n$, $0 \leq h \in L^{q_i}(\Omega)$ with $q_m = \max_{i=1,\ldots,n} q_i$, $\varphi \in [-\infty, +\infty]$ is such that $\theta_* = \max\{\varphi, \theta\} \in \theta + W^{1,(q_i)}_0(\Omega)$. Moreover, $\bar{p} < n$. Then

$$u \in \theta_* + L^1_{\text{weak}}(\Omega),$$

where

$$t = \frac{p^* \bar{p}}{1 - (b \frac{p^*}{\bar{p}}) (p_m / p - 1)} > \bar{p},$$

and $b$ is any number verifying

$$0 < b \leq \min_{j=1,\ldots,n} \left(1 - \frac{p_j}{q_j} \right) \left(1 - \frac{1}{p} \right),$$

$$b < \frac{p_m - 1}{p_m - \bar{p}}.$$

**Remark 1.3.** Take the obstacle function $\varphi$ to be minus infinity in Theorem 1.2, and the condition (1.4) replaced by

$$|a_i(x,z)| \leq c_2(1 + |z|)^{p_i - 1}$$

for almost every $x \in \Omega$, for every $z \in \mathbb{R}^n$, and for any $i = 1, 2, \ldots, n$, then we arrive at Theorem 2.1 in [4].

### 2. Proof of the Main Theorem

**Proof of Theorem 1.2.** Let $u \in \mathcal{K}^{(p)}_{\varphi,\theta}(\Omega)$ be a solution to the $\mathcal{K}^{(p)}_{\varphi,\theta}$-obstacle problem. Take $\theta_* = \max\{\varphi, \theta\} \in \theta + W^{1,(q_i)}_0(\Omega)$. Let us consider $L \in (0, +\infty)$ and

$$v = \begin{cases} 
\theta_* - L, & \text{for } u - \theta_* < -L, \\
u, & \text{for } -L \leq u - \theta_* \leq L, \\
\theta_* + L, & \text{for } u - \theta_* > L.
\end{cases}$$

Then $v \in \mathcal{K}^{(p)}_{\varphi,\theta}(\Omega)$. Indeed, for the second and the third cases of the above definition for $v$, we obviously have $v \geq \varphi$, and for the first case, $u - \theta_* < -L$, we have $\theta_* > u + L \geq \varphi + L$; this
implies \( v = \theta_\ast - L \geq \varphi \). Since \( u = \theta_\ast = \theta \) on \( \partial \Omega \), then \( v = u \) on \( \partial \Omega \), this implies \( v = \theta \) on \( \partial \Omega \). By Definition 1.1, one has

\[
0 \leq \int_{\{|u-\theta|>L\}} \sum_{i=1}^{n} a_i(x,Du(x))(D_i v(x) - D_i u(x))dx
\]

\[
= \int_{\{|u-\theta|>L\}} \sum_{i=1}^{n} a_i(x,Du(x))(D_i \theta_\ast(x) - D_i u(x))dx. \tag{2.2}
\]

Monotonicity (1.5) allows us to write

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_\ast|>L\}} \ |D_i u(x) - D_i \theta_\ast(x)|^p dx
\]

\[
\leq \int_{\{|u-\theta_\ast|>L\}} \sum_{i=1}^{n} (a_i(x,Du(x)) - a_i(x,D\theta_\ast(x)))(D_i u(x) - D_i \theta_\ast(x))dx,
\]

which together with (2.2) implies

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_\ast|>L\}} \ |D_i u(x) - D_i \theta_\ast(x)|^p dx
\]

\[
\leq - \int_{\{|u-\theta_\ast|>L\}} \sum_{i=1}^{n} a_i(x,D\theta_\ast)(D_i u(x) - D_i \theta_\ast(x))dx. \tag{2.4}
\]

We now use anisotropic growth (1.4) and the Hölder inequality in (2.4), obtaining that

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_\ast|>L\}} |D_i u - D_i \theta_\ast|^p dx
\]

\[
\leq - \sum_{i=1}^{n} \int_{\{|u-\theta_\ast|>L\}} a_i(x,D\theta_\ast)(D_i u - D_i \theta_\ast)dx
\]

\[
\leq c_2 \sum_{i=1}^{n} \int_{\{|u-\theta_\ast|>L\}} (h + |D_i \theta_\ast|)^{p-1}|D_i u - D_i \theta_\ast|dx
\]

\[
\leq c_2 \sum_{i=1}^{n} \left( \int_{\{|u-\theta_\ast|>L\}} (h + |D_i \theta_\ast|)^p dx \right)^{(p-1)/p_i} \left( \int_{\{|u-\theta_\ast|>L\}} |D_i u - D_i \theta_\ast|^p dx \right)^{1/p_i}. \tag{2.5}
\]

Let \( t_i \) be such that

\[
p_i < t_i \leq q_i, \tag{2.6}
\]
for every $i = 1, \ldots, n$; $t_i$ will be chosen later. We use the Hölder inequality as follows:

\[
\left( \int_{\{|u-\theta_\ast| > L\}} (h + |D_i\theta_\ast|)^{p_i} \, dx \right)^{(p_i-1)/p_i} \leq \left( \int_{\{|u-\theta_\ast| > L\}} (h + |D_i\theta_\ast|)^{t_i} \, dx \right)^{(p_i-1)/t_i} (||u - \theta_\ast|| > L)^{(t_i-p_i)/(p_i-1)}. \tag{2.7}
\]

The following proof is similar to that of [4, Theorem 2.1]; we only list the necessary changes: instead of [4, (3.14)] by

\[
\left( \int_{\{|u-\theta_\ast| > L\}} (h + |D_i\theta_\ast|)^{p_i} \, dx \right)^{(p_i-1)/p_i} \leq \left( \int_{\{|u-\theta_\ast| > L\}} (h + |D_i\theta_\ast|)^{t_i} \, dx \right)^{(p_i-1)/t_i} \leq M||u - \theta_\ast||^b, \tag{2.8}
\]

where

\[
M = \max_{j=1, \ldots, n} \left( \int_{\Omega} (h + |D_j\theta_\ast|)^{t_j} \, dx \right)^{(p_j-1)/t_j} < \infty, \tag{2.9}
\]

and instead of [4, (3.19)] we use anisotropic Sobolev Embedding Theorem for $v - u$,

\[
\left( \int_{\Omega} |v - u|^{p_\ast} \, dx \right)^{1/p_\ast} \leq c_\ast \left[ \prod_{i=1}^{n} \left( \int_{\Omega} |D_i(v - u)|^{p_i} \, dx \right)^{1/p_i} \right]^{1/n} \leq c_\ast \left[ \prod_{i=1}^{n} \left( \int_{\{|u-\theta_\ast| > L\}} |D_i u - D_i \theta_\ast|^{p_i} \, dx \right)^{1/p_i} \right]^{1/n}. \tag{2.10}
\]

By $|v - u| = (|u-\theta_\ast| - L)1_{\{|u-\theta_\ast| > L\}}$, we obtain

\[
\left( \int_{\{|u-\theta_\ast| > L\}} (|u-\theta_\ast| - L)^{p_\ast} \, dx \right)^{1/p_\ast} = \left( \int_{\Omega} |v - u|^{p_\ast} \, dx \right)^{1/p_\ast}. \tag{2.11}
\]

Following the idea of the proof of Theorem 2.1 in [4], we complete the proof of Theorem 1.2. □
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