Research Article

The Multiple Gamma-Functions and the Log-Gamma Integrals

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Received 15 May 2012; Accepted 30 July 2012

Academic Editor: Shigeru Kanemitsu

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In this paper, which is a companion paper to [W], starting from the Euler integral which appears in a generalization of Jensen’s formula, we shall give a closed form for the integral of $\log \Gamma(\pm t)$. This enables us to locate the genesis of two new functions $A_{1/\alpha}$ and $C_{1/\alpha}$ considered by Srivastava and Choi. We consider the closely related function $A(\alpha)$ and the Hurwitz zeta function, which render the task easier than working with the $A_{1/\alpha}$ functions themselves. We shall also give a direct proof of Theorem 4.1, which is a consequence of [CKK, Corollary 1.1], though.

1. Introduction

If $f(z)$ is analytic in a domain $D$ containing the circle $C : |z| = r$ and has no zero on the circle, then the Gauss mean value theorem

\[
\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta
\]

(1.1)

is true. In [1, page 207] the case is considered where $f(z)$ has a zero $re^{i\theta_0}$ on the circle, and (1.1) turns out that the Euler integral

\[
\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2
\]

(1.2)

which is essential in proving a generalization of Jensen’s formula [1, pages 207-208].
Let $G$ denote the Catalan constant defined by the absolutely convergent series

$$
G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = L(2, \chi_4),
$$

(1.3)

where $\chi_4$ is the nonprincipal Dirichlet character mod 4.

As a next step from (1.2) the relation

$$
\int_{0}^{\pi/4} \log \sin t \, dt = -\frac{\pi}{4} \log 2 - \frac{1}{2} G
$$

(1.4)

holds true. In this connection, in [2] we obtained some results on $G$ viewing it as an intrinsic value to the Barnes $G$-function. The Barnes $G$-function (which is $\Gamma_2^{-1}$ in the class of multiple gamma functions) is defined as the solution to the difference equation (cf. (2.3))

$$
\log G(z + 1) - \log G(z) = \log \Gamma(z)
$$

(1.5)

with the initial condition

$$
\log G(1) = 0
$$

(1.6)

and the asymptotic formula to be satisfied

$$
\log G(z + N + 2) = \frac{N + 1 + z}{2} \log 2\pi
$$

$$
+ \frac{1}{2} \left( N^2 + 2N + 1 + B_2 + z^2 + 2(N + 1)z \right) \log N
$$

$$
- \frac{3}{4} N^2 - N - Nz - \log A + \frac{1}{12} + O(N^{-1}).
$$

(1.7)

$N \to \infty$, where $\Gamma(s)$ indicates the Euler gamma function (cf., e.g., [3]).

Invoking the reciprocity relation for the gamma function

$$
\Gamma(s) \sin \pi s = \frac{\pi}{\Gamma(1-s)},
$$

(1.8)

it is natural to consider the integrals of $\log \Gamma(\alpha + t)$ or of multiple gamma functions $\Gamma_r$ (cf., e.g., [4, 5]). Barnes’ theorem [6, page 283] reads

$$
\int_{0}^{a} \log \Gamma(\alpha + t) \, dt = -\log \left( \frac{G(\alpha + a)}{G(\alpha)} \right) - (1 - a) \log \left( \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)} \right)
$$

$$
+ a \log \Gamma(\alpha + a) - \frac{1}{2} a^2 - \frac{1}{2} \left( \log 2\pi + 1 - 2a \right) a
$$

(1.9)

valid for nonintegral values of $a$. 

In this paper, motivated by the above, we proceed in another direction to developing some generalizations of the above integrals considered by Srivastava and Choi [7]. For \( q \)-analogues of the results, compare the recent book of the same authors [8]. Our main result is Theorem 2.1 which gives a closed form for \( \int_0^a \log \Gamma(1 - t) \, dt \) and locates its genesis. A slight modification of Theorem 2.1 gives the counterpart of Barnes’ formula (1.9) which reads.

**Corollary 1.1.** Except for integral values of \( a \), one has

\[
\int_0^a \log \Gamma(a - t) \, dt = \log \frac{G(a - a)}{G(a)} + (1 - a) \log \frac{\Gamma(a - a)}{\Gamma(a)} + a \log \Gamma(a - a) + \frac{1}{2} a^2 + \frac{1}{2} (\log 2 \pi + 1 - 2a)a.
\] (1.10)

Srivastava and Choi introduced two functions \( \log A_{1/a} \) and \( \log C_{1/a} \) by (2.9) and (2.9) with formal replacement of \( 1/a \) by \(-1/a\), respectively. They state \( C_{1/a} = A_{-1/a} \), which is rather ambiguous as to how we interpret the meaning because (2.9) is defined for \( a > 0 \) [7, page 347, l.11]. They use this \( C_{1/a} \) function to express the integral \( \int_0^a \log \Gamma(1 - t) \, dt \), without giving proof. This being the case, it may be of interest to locate the integral of \( \log \Gamma(1 - t) \) [7, (13), page 349], thereby \( \log A_{1/a} \) [7, page 347].

For this purpose we use a more fundamental function \( A(a) \) than \( A_{1/a} \) defined by

\[
\log A(a) = -\zeta'(-1, a) + \frac{1}{12},
\] (1.11)

where \( \zeta(s, a) \) is the Hurwitz zeta-function

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad \text{Re } s = \sigma > 1
\] (1.12)

in the first instance. For its theory, compare, for instance, [3], [9, Chapter 3].

We shall prove the following corollary which gives the right interpretation of the function \( C_{1/a} \).

**Corollary 1.2.** For \( 0 < a < 1 \),

\[
\log C_{1/a} = \log A(1 - a) - \frac{1}{4} a^2,
\] (1.13)

or

\[
\log C_{1/a} = \log A_{1-1/a} + \frac{1}{4} (1 - a)^2 + (1 - a) \log(1 - a) - \frac{1}{4} a^2.
\] (1.14)
2. **Barnes Formula**

There is a generalization of (1.4) as well as (1.2) in the form [7, equation (28), page 31]:

\[
\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(1 + a)}{G(1 - a)}, \quad a \notin \mathbb{Z}.
\]  

(2.1)

Equation (2.1) is Barnes’ formula [6, page 279] which is equivalent to Kinkelin’s 1860 result [10] [7, equation (26), page 30]:

\[
\int_0^z \pi t \cot \pi t \, dt = \log \frac{G(1 - z)}{G(1 + z)} + z \log 2\pi.
\]  

(2.2)

Since (1.5) is equivalent to

\[G(z + 1) = G(z) \Gamma(z),\]

(2.3)

it follows that

\[
\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(a)}{G(1 - a)} + \log \Gamma(a).
\]  

(2.4)

Putting \(a = 1/2\), we obtain

\[
\pi^{-1} \int_0^{\pi/2} \log \sin x \, dx = \int_0^{1/2} \log \sin t \, dt = -\frac{1}{2} \log 2\pi + \log \Gamma\left(\frac{1}{2}\right) = -\frac{1}{2} \log 2,
\]  

(2.5)

which is (1.2).

The counterpart of (2.1) follows from the reciprocity relation (1.8), known as Alexeevsky’s Theorem [7, equation (42), page 32].

\[
\int_0^a \log \Gamma(1 + t) \, dt = \frac{1}{2} (\log 2\pi - 1) a - \frac{a^2}{2} + a \log \Gamma(a + 1) - \log G(a + 1),
\]  

(2.6)

which in turn is a special case of (1.9).

Indeed, in [7, page 207], only (1.9) and the integral of \(\log G(t + \alpha)\) are in closed form and the integral of \(\log \Gamma_3(t + \alpha)\) is not. A general formula is given by Barnes [4] with constants to be worked out. We shall state a concrete form for this integral in Section 3, using the relation [7, equation (455), page 210] between \(\log \Gamma_3(t + \alpha)\) and the integral of \(\varphi\) and appealing to a closed form for the latter in [11].

Formula (2.6) is stated in the following form [7, equation (12), page 349]:

\[
\int_0^a \log \Gamma(1 + t) \, dt = \frac{1}{2} (\log 2\pi - 1) a - \frac{3}{4} a^2 + \log A - \log A_{1/4},
\]  

(2.7)
where $\log A$ is the Glaisher-Kinkelin constant defined by [7, equation (2), page 25]

$$\log A = \lim_{N \to \infty} \left( \sum_{n=1}^{N} n \log n - \frac{1}{2} \left( N^2 + N + B_2 \right) \log N + \frac{1}{4} N^2 \right), \quad (2.8)$$

and $\log A_{1/a}$ is defined by [7, equation (9), page 347]

$$\log A_{1/a} = \lim_{N \to \infty} \left( \sum_{n=1}^{N} (n + a) \log(n + a) \right)$$

$$- \frac{1}{2} \left( N^2 + (2a + 1) N + a^2 + a + B_2 \right) \log(N + a) + \frac{1}{4} N^2 + \frac{a}{2} N + \frac{a^2}{4} \log a, \quad (2.9)$$

for $a > 0$.

Comparing (2.6) and (2.7), we immediately obtain

$$\log A_{1/a} = \log G(a + 1) - a \log \Gamma(a + 1) + \log A - \frac{a^2}{4} + \log a$$

$$= \log G(a) + (1 - a) \log \Gamma(a) + \log a - \frac{a^2}{4} - a \log a, \quad (2.10)$$

on using the difference relation $\Gamma(a + 1) = a \Gamma(a)$.

Thus, in a sense we have located the genesis of the function $\log A_{1/a}$. although they prove (2.7) by an elementary method [7, page 348]. Indeed, $A_{1/a}$ and $A(a)$ are almost the same:

$$\log A_{1/a} = \log A(a) - \frac{1}{4} a^2 - a \log a, \quad (2.11)$$

a proof being given below. However, $\log A(a)$ is more directly connected with $\zeta(-1, a)$ for which we have rich resources of information as given in [9, Chapter 3].

We prove the following theorem which gives a closed form for $\int_{0}^{a} \log \Gamma(1-t) \; dt$, thereby giving the genesis of the constant $C_{1/a}$.

**Theorem 2.1.** For $a \notin \mathbb{Z}$, one has

$$\int_{0}^{a} \log \Gamma(1-t) \; dt = \log G(1-a) + a \log \Gamma(1-a) + \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1) a. \quad (2.12)$$

If $0 < a < 1$, then

$$\int_{0}^{a} \log \Gamma(1-t) \; dt = \log A(1-a) - \log A + \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1) a. \quad (2.13)$$
Proof. We evaluate the integral

\[ I = \int_0^a \log \Gamma(1 + t) \sin \pi t \, dt \quad (2.14) \]

in two ways. First,

\[ I = a \log \pi + a \log a - a \int_0^a \log \Gamma(1-t) \, dt. \quad (2.15) \]

On the other hand, noting that \( I \) is the sum of (2.1) and (2.7), we deduce that

\[ I = a \log \frac{\sin \pi a}{2\pi} + \log G(a+1) + \log A - \log G(1-a) \]
\[ + \frac{1}{2} (\log 2\pi - 1)a - \frac{3}{4} a^2 - \log A_{1/a}. \quad (2.16) \]

Substituting (1.5), we obtain

\[ I = a \log \frac{\sin \pi a}{2\pi} + a \log \Gamma(a) + \log A(a) - \log A_{1/a} \]
\[ - \log G(1-a) + \frac{1}{2} (\log 2\pi - 1)a - \frac{3}{4} a^2. \quad (2.17) \]

The first two terms on the right of (2.17) become

\[ a \log \frac{\Gamma(a) \sin \pi a}{2\pi} = a \log \frac{1}{2} \Gamma(1-a) = -a (\log 2 + \log \Gamma(1-a)), \quad (2.18) \]

while the 3rd and the 4th terms give, in view of (2.11), \((1/4)a^2 + a \log a\).

Hence, altogether

\[ I = -a \log 2 - a \log \Gamma(1-a) - \log G(1-a) + a \log a - \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1)a. \quad (2.19) \]

Comparing (2.15) and (2.19) proves (2.12), completing the proof. \(\square\)

Comparing (2.13) and [7, equation (13), page 349]

\[ \int_0^a \log \Gamma(1-t) \, dt = \log A(1-a) - \log A + \frac{3}{4} a^2 + \frac{1}{2} (\log 2\pi - 1)a, \quad (2.20) \]

we prove Corollary 1.2.

Hence the relation between \( C_{1/a} \) and \( A_{1/a} \) is (1.14), that is, one between \( C_{1/a} \) and \( A_{1-1/a} \) rather than \( C_{1/a} = A_{-1/a} \) as Srivastava and Choi state.
At this point we shall dwell on the underlying integral representation for (the derivative of) the Hurwitz zeta-function, which makes the argument rather simple and lucid as in [12] and gives some consequences.

**Proof of (2.11).** Consider that

\[
\zeta'(s, a) - \frac{1}{12} = -\frac{1}{2} a^2 \log a - \frac{1}{4} a^2 - \frac{1}{2} a \log a - \frac{B_2}{2} \log a - \frac{1}{3!} \int_0^\infty B_5(t)(t + a)^{-2} dt
\]  
(2.21)

[9, (3.15), page 59], where the last integral may be also expressed as

\[
-\frac{1}{2!} \int_0^\infty B_2(t)(t + a)^{-1} dt,
\]  
(2.22)

and where \( B_k(t) \) is the \( k \)th periodic Bernoulli polynomial. Then

\[
-\zeta'(-1, a) = \sum_{0 \leq n \leq x} (n + a) \log(n + a) - \frac{1}{2} (x + a)^2 \log(x + a)
\]  
(2.23)

\[
+ \frac{1}{4} (x + a)^2 + \bar{B}_1(x)(x + a) - \frac{1}{2} \bar{B}_2(x)(x + a) + O(x^{-1} \log x);
\]

whence in particular, we have the generic formula for \( \zeta'(-1, a) \) and consequently for \( \log A(a) \) through (1.11):

\[
\log A(a) = \lim_{N \to \infty} \left( \sum_{n=0}^\infty (n + a) \log(n + a) - \frac{1}{2} \log(N + a) \right.
\]  
(2.24)

\[
\times \left( (N + a)^2 + N + a + B_2 \right) + \frac{1}{4} (N + a)^2 \right).
\]

This may be slightly modified in the form

\[
\log A(a) = \lim_{N \to \infty} \left( \sum_{n=0}^N (n + a) \log(n + a)
\]  
(2.25)

\[
- \frac{1}{2} \left( N^2 + (2a + 1)N + a^2 + a + B_2 \right) \log(N + a) + \frac{1}{4} N^2 + \frac{1}{2} aN
\]

\[
+ \frac{1}{4} a^2 + a \log a.
\]

Comparing (2.9) and (2.25), we verify (2.11). \( \square \)
The merit of using $A(a)$ is that by way of \(\zeta'(-1,a)\), we have a closed form for it:

\[
\log A(a) = \frac{1}{2}a^2 \log a - \frac{1}{4}a^2 + \frac{1}{2}a \log a + \frac{B_2}{2} \log a + \sum_{n=1}^{\infty} \frac{\Gamma(1+n)}{n! 2^n} \int_0^\infty B_2(t)(t+a)^{-1} \, dt.
\]

(2.26)

In the same way, via another important relation [7, equation (23), page 94],

\[
\log G(a) = -\left(\zeta'(-1,a) - \frac{1}{12}\right) - \log A - (1-a) \log \Gamma(a).
\]

(2.27)

Equation (2.21) gives a closed form for \(\log G(a)\), too. We also have from (1.11) and (2.27)

\[
\log A(a) = \log G(a) + (1-a) \log \Gamma(a) + \log A
\]

\[= \log G(a+1) - a \log \Gamma(a) + \log A.\]

(2.28)

There are some known expressions not so handy as given by (2.27). For example, [7, page 25] and [7, equation (440), page 206], one of which reads

\[
\frac{G'}{G} (1+z) = \sum_{n=1}^{\infty} \left( \frac{n}{z+n} - 1 + \frac{z}{n} \right) + \frac{1}{2} (\log 2\pi - 1) - (1+\gamma)z,
\]

(2.29)

with \(\gamma\) designating the Euler constant. Equation (2.29) is a basis of (2.2) (cf. proof of [2, Lemma 1]).

**Remark 2.2.** The Glaisher-Kinkelin constant \(A\) is connected with \(A(1)\) and \(A_1\) as follows:

\[
\log A = \log A(1) = \log A_1 + \frac{1}{4}
\]

(2.30)

This can also be seen from Vardi’s formula [7, (31), page 97]:

\[
\log A = -\zeta'(-1) + \frac{1}{12}
\]

(2.31)

which is (1.11) with \(a = 1\).

We may also give another direct proof of Corollary 1.2.

**Proof of Corollary 1.2 (another proof).** \(\log C_{1/a}\) is the limit of the expression

\[
S_N = \sum_{k=1}^{N} (k-1+a) \log(k-1+a) - \left( \frac{1}{2} N^2 + \left( a - \frac{1}{2} \right) N + \frac{1}{2} B_2(a) \right) \log(N-1+a) + \frac{1}{4} N^2 + \frac{N}{2} (a-1)
\]

(2.32)
where $\alpha = 1 - a$. Let $N = M + 1$. Then

$$S_N = \sum_{k=0}^{M} (k + \alpha) \log(k + \alpha) - \left( \frac{1}{2} (M + 1)^2 + \left( \alpha - \frac{1}{2} \right)(M + 1) + \frac{1}{2} B_2(\alpha) \right) \times \log(M + \alpha) + \frac{1}{4} (M + 1)^2 + \frac{M + 1}{2} \alpha - \frac{M + 1}{2}.$$  

(2.33)

Hence, simplifying, we find that

$$S_N = \sum_{k=1}^{M} (k + \alpha) \log(k + \alpha) - \left( \frac{1}{2} M^2 + \left( \alpha + \frac{1}{2} \right) M + \frac{1}{2} (\alpha^2 + \alpha + B_2) \right) \times \log(M + \alpha) + \frac{1}{4} M^2 + \frac{1}{2} \alpha M + \alpha \log \alpha - \frac{(\alpha - 1)^2}{4} + \frac{1}{4} \alpha^2.$$  

(2.34)

Hence

$$\log C_{1/a} = \log A_a + \alpha \log \alpha - \frac{(\alpha - 1)^2}{4} + \frac{1}{4} \alpha^2,$$  

(2.35)

which is (1.14). This completes the proof.

As an immediate consequence of Corollary 1.2, we prove (2.36) as can be found in [7, pages 350–351].

$$A_{1/a} = \left( \frac{\pi a}{\sin \pi a} \right)^a \frac{G(1 + a)}{G(1 - a)} C_{1/a}, \quad 0 < a < 1.$$  

(2.36)

Proof of (2.36). From (2.28), (1.5), and (1.8), we obtain

$$\log A(a) - \log A(1 - a) = \log \frac{G(1 + a)}{G(1 - a)} - a \log \frac{\pi}{\sin \pi a}.$$  

(2.37)

On the other hand, by (2.11) and (1.13), we see that the left-hand side of (2.37) is

$$\log \frac{A_{1/a}}{C_{1/a}} + a \log a,$$  

(2.38)

whence we conclude that

$$\log \frac{A_{1/a}}{C_{1/a}} = \log \frac{G(1 + a)}{G(1 - a)} - a \log \frac{\pi a}{\sin \pi a}.$$  

(2.39)

On exponentiating, (2.37) leads to (2.36).
3. Polygamma Function of Negative Order

In this section we introduce the function \( \tilde{A}_k(q) \) [13]:

\[
\tilde{A}_k(q) = k\zeta'(1-k,q),
\]  \hspace{1cm} (3.1)

which is closely related to the polygamma function of negative order and states some simple applications. We recall some properties of \( \tilde{A}_k(q) \):

\[
\tilde{A}_2(q+1) = \tilde{A}_2(q) + 2q \log q,
\]

\[
\tilde{A}_2\left(\frac{1}{2}\right) = -\zeta'(-1) - \frac{1}{12} \log 2,
\]  \hspace{1cm} (3.2)

\[
\tilde{A}_2\left(\frac{1}{4}\right) = -\frac{1}{4} \zeta'(-1) + \frac{G}{2\pi},
\]

\[
\tilde{A}_2\left(\frac{3}{4}\right) = -\frac{1}{2} \zeta'(-1) - \tilde{A}_2\left(\frac{1}{4}\right).
\]  \hspace{1cm} (3.3)

Equation (3.3) is [2, equation (2.31)], which is used in proving [2, Theorem 2] and can be read off from the distribution property [9, equation (3.72), page 76] as follows:

\[
\sum_{a=1}^{4} \zeta\left(s, \frac{a}{4}\right) = 4^s \zeta(s).
\]  \hspace{1cm} (3.4)

Differentiation gives

\[
\sum_{n=1}^{4} \zeta'\left(s, \frac{a}{4}\right) = 4^s ((\log 4) \zeta(s) + \zeta'(s)).
\]  \hspace{1cm} (3.5)

Putting \( s = -1 \), we obtain

\[
\zeta'(-1) + \zeta'\left(-1, \frac{1}{2}\right) + \zeta'\left(-1, \frac{1}{4}\right) + \zeta'\left(-1, \frac{3}{4}\right) = 4^{-1} ((\log 4) \zeta(-1) + \zeta'(-1)) \hspace{1cm} (3.6)
\]

which we solve in \( \zeta'(-1,3/4) \):

\[
\zeta'\left(-1, \frac{3}{4}\right) = \frac{1}{4} ((2 \log 2) \zeta(-1) + \zeta'(-1))
\]

\[
- \zeta'(-1) - \frac{1}{2} \tilde{A}_2\left(\frac{1}{2}\right) - \zeta'\left(-1, \frac{1}{4}\right).
\]  \hspace{1cm} (3.7)
Substituting (3.2) and \(\zeta(-1) = -B_2/2 = -1/12\) and simplifying, we conclude that
\[
\zeta^{'}(-1, \frac{3}{4}) = -\frac{1}{4}\zeta^{'}(-1) - \zeta^{'}(-1, \frac{1}{4})
\]
(3.8)
and that
\[
\bar{A}_2\left(\frac{3}{4}\right) = 2\zeta^{'}\left(-1, \frac{3}{4}\right) = -\frac{1}{2}\zeta^{'}(-1) - 2\zeta^{'}(-1, \frac{1}{4}),
\]
(3.9)
whence (3.3).

Using these, we deduce from (2.37) the following.

**Example 3.1.**

\[
\log A_{1/4} = \frac{5}{64} + \frac{1}{2}\log 2 - \frac{1}{8}\log A - \frac{G}{2\pi}.
\]
(3.10)

**Proof.** By (1.11) and (3.1), for \(q > 0\),
\[
\log A(q) = -\frac{1}{2} \bar{A}_2(q) + \frac{1}{12}.
\]
(3.11)

Since \(\log A(1/4) - \log A(3/4) = -1/2(\bar{A}_2(1/4) - \bar{A}_2(3/4))\), it follows from (3.3) that the left-hand side of (2.37) is
\[
-\bar{A}_2\left(\frac{1}{4}\right) - \frac{1}{4}\zeta^{'}(-1),
\]
(3.12)
which is
\[
2\log A\left(\frac{1}{4}\right) - \frac{1}{6} + \frac{1}{4}\left(\log A - \frac{1}{12}\right)
\]
(3.13)
where we used (2.31).

The right-hand side of (2.37), \(\log(G(5/4)/G(3/4)) - 1/4 \log(\pi/\sin(\pi/4))\), becomes \(-G/2\pi\), in view of known values of \(G\) [7, page 30].

Hence, altogether, (2.37) with \(a = 1/4\) reads
\[
\frac{G}{2\pi} = 2\log A\left(\frac{1}{4}\right) + \frac{1}{4}\log A - \frac{3}{16}.
\]
(3.14)

Invoking (2.11), this becomes (3.10).

We note that (3.14) gives a proof of the third equality in (3.2). Both (2.36) and (3.10) are contained in [14, 1999a] and are given as exercises in [7].
4. The Triple Gamma Function

For general material, we refer to [7, page 42]. As can been seen on [7, page 207], the important integral \( \int_0^z \log \Gamma_3(t + a) \, dt \) is not in closed form. Recently, Chakraborty-Kanemitsu-Kuzumaki [5, Corollary 1.1] have given a general expressions for all the integrals in \( \log \Gamma_3 \), by appealing to Barnes’ original results.

In this section, we shall give a direct derivation of a closed form by combining [7, (455), page 210] and [11, Corollary 3] (with \( \lambda = 3 \)). The first reads

\[
2 \int_0^z \log \Gamma_3(t + a) \, dt = - \int_0^z t^3 \varphi(t + a) \, dt + 2z \log \Gamma_3(z + a) \\
- 2(2a - 3) \frac{\log \Gamma_3(z + a)}{\log \Gamma_3(a)} + \left( 3a^2 - 9a + 7 \right) \frac{\log G(z + a)}{\log G(a)} \\
- (a - 1)^3 \frac{\log \Gamma(z + a)}{\log \Gamma(a)} + \frac{3}{8} z^4 + \frac{1}{3} (1 - \log 2\pi) z^3 \\
+ \left( - \frac{3}{4} a^2 + \frac{7}{4} a - \frac{9}{8} + \frac{1}{4} (2a - 3) \log 2\pi + \log A \right) z^2, \\
+ \left( a^2 - \frac{3}{2} a + \frac{1}{4} (a - 2 - 3a + 2) \log 2\pi + 2(3 - 2a) \log A \right) z,
\]

while the second reads (cf. also [15])

\[
\int_0^z t^3 \log \varphi(t + a) \, dt = - \sum_{r=0}^3 C_3(r, a) \log \frac{\Gamma_{r+1}(a + z)}{\Gamma_{r+1}(a)} \\
- \sum_{r=0}^3 (-1)^r \left( \begin{array}{l} 3 \\ r \end{array} \right) (l - 3) \zeta'(l - 3) + \frac{B_4(t)}{l(l - 1)} \right) z^j + \frac{11}{24} z^4,
\]

where \( C_3(r, a) \) are defined by

\[
C_3(r, a) = (-1)^r r! \sum_{m=r}^3 \left( \begin{array}{l} 3 \\ m \end{array} \right) (-1)^m S(m, n) (a - 1)^{3-m}
\]

and where \( S(m, n) \) are the Stirling numbers of the second kind [7, page 58]. To express the values of \( \zeta'(l - 3) \), we appeal to [7]

(i) \( \zeta'(0) = -(1/2) \log 2\pi [7, (20), page 92] \)

(ii) \( \zeta'(-2) = \log B = \zeta(3) / 4\pi^2 [7, pages 99-100] \)
and (2.31). After some elementary but long calculations, we arrive at

\[
\begin{align*}
\int_0^z t^3 \log q(t + a) \, dt &= -3! \log \frac{\Gamma_4(a + z)}{\Gamma_4(a)} - 6(a - 2) \log \frac{\Gamma_3(a + z)}{\Gamma_3(a)} \\
&\quad - (3a^2 - 9a + 7) \log \frac{\Gamma_2(a + z)}{\Gamma_2(a)} - (a - 1)^3 \log \frac{\Gamma(a + z)}{\Gamma(a)} \frac{11}{24} z^4 \\
&\quad + \left( -\frac{1}{2} \log 2\pi + \frac{1}{3} B_1(a) \right) z^3 - \left( \frac{1}{4} - 3 \log A + \frac{1}{4} B_2(a) \right) z^2 \\
&\quad + 3 \left( \log B + \frac{1}{3} B_3(a) \right) z.
\end{align*}
\] (4.4)

Combining we have the following.

**Theorem 4.1** (see [5, Example 2.3]). *Except for the singularities of the multiple gamma function, one has*

\[
\begin{align*}
\int_0^z \log \Gamma_3(t + a) \, dt &= 3! \log \frac{\Gamma_4(a + z)}{\Gamma_4(a)} + z \log \Gamma_3(z + a) \\
&\quad + (a - 3) \log \frac{\Gamma_3(a + z)}{\Gamma_3(a)} - \frac{1}{24} z^4 - \frac{1}{6} \left( a - \frac{3}{2} - \frac{1}{2} \log 2\pi \right) z^3 \\
&\quad + \frac{1}{8} \left( -2a^2 + 6a - \frac{10}{3} + (2a - 3) \log 2\pi - 8 \log A \right) z^2 \\
&\quad - \frac{1}{2} \left( a^3 - \frac{5}{2} a^2 + 2a - \frac{1}{4} + \frac{1}{2} (a^2 - 3a + 2) \log 2\pi \right) \\
&\quad + 2(2a - 3) \log A + 3 \log B \right) z.
\end{align*}
\] (4.5)

This theorem enables us to put many formulas in [7] in closed form including, for instance, [7, (698), page 245]. Compare [5].

**Acknowledgment**

The authors would like to express their hearty thanks to Professor S. Kanemitsu for his enlightening supervision and encouragement.

**References**


