Research Article

Integral Formulae of Bernoulli and Genocchi Polynomials

Seog-Hoon Rim,1 Joung-Hee Jin,2 and Joohee Jeong1

1 Department of Mathematics Education, Kyungpook National University, Taegu 702-701, Republic of Korea
2 Department of Mathematics, Kyungpook National University, Taegu 702-701, Republic of Korea

Correspondence should be addressed to Seog-Hoon Rim, shrim@knu.ac.kr

Received 19 June 2012; Accepted 19 July 2012

Academic Editor: Taekyun Kim

Copyright © 2012 Seog-Hoon Rim et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Recently, some interesting and new identities are introduced in the work of Kim et al. (2012). From these identities, we derive some new and interesting integral formulae for Bernoulli and Genocchi polynomials.

1. Introduction

As it is well known, the Bernoulli polynomials are defined by generating functions as follows:

\[
\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\] (1.1)

(see [1–5]) with the usual convention about replacing \( B^n(x) \) by \( B_n(x) \). In the special case, \( x = 0 \), \( B_n(0) = B_n \) are called the \( n \)th Bernoulli numbers.

The Genocchi polynomials are also defined by

\[
\frac{2t}{e^t + 1} e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}
\] (1.2)

(see [1, 6–10]) with the usual convention about replacing \( G^n(x) \) by \( G_n(x) \). In the special case, \( x = 0 \), \( G_n(0) = G_n \) are called the \( n \)th Genocchi numbers.
From (1.1), we note that

\[ B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} \]  \hspace{1cm} (1.3)

(see [1–5]). Thus, by (1.3), we get

\[ \frac{d}{dx} B_n(x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_l x^{n-1-l} = nB_{n-1}(x) \]  \hspace{1cm} (1.4)

(see [2]). From (1.2), we note that

\[ G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \]  \hspace{1cm} (1.5)

From (1.5), we can derive the following equation:

\[ \frac{d}{dx} G_n(x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} G_l x^{n-1-l} = nG_{n-1}(x). \]  \hspace{1cm} (1.6)

By the definition of Bernoulli and Genocchi numbers, we get the following recurrence formulae:

\[ B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad G_0 = 0, \quad G_n(1) + G_n = 2\delta_{1,n}, \]  \hspace{1cm} (1.7)

where \( \delta_{n,k} \) is the Kronecker symbol (see [2]). From (1.4), (1.6), and (1.7), we note that

\[ \int_{0}^{1} B_n(x)dx = \frac{\delta_{0,n}}{n+1} \quad (n \geq 0), \quad \int_{0}^{1} G_n(x)dx = -\frac{2G_{n+1}}{n+1} \quad (n \geq 1). \]  \hspace{1cm} (1.8)

From the identities of Bernoulli and Genocchi polynomials, we derive some new and interesting integral formulae of an arithmetical nature on the Bernoulli and Genocchi polynomials.
2. Integral Formula of Bernoulli and Genocchi Polynomials

From (1.1) and (1.2), we note that

\[
\frac{t}{e^t - 1} e^{xt} = \frac{1}{2} \left( \frac{2te^{xt}}{e^t + 1} \right) + \frac{1}{t} \left( \frac{t}{e^t - 1} \right) \left( \frac{2te^{xt}}{e^t + 1} \right) 
\]

\[
= \frac{1}{2} \left( \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} \right) + \frac{1}{t} \left( \sum_{l=0}^{\infty} B_l \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} G_m(x) \frac{t^m}{m!} \right) 
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} + \frac{1}{t} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} G_l(x) B_{n-l} \frac{t^n}{n!} 
\]

\[
= \frac{1}{2} \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \left( -\frac{1}{2} G_n(x) + \sum_{l=0}^{n+1} \binom{n+1}{l} G_l(x) B_{n+1-l} \right) \frac{t^n}{n!} 
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n+1} \binom{n+1}{l} G_l(x) B_{n+1-l} \right) \frac{t^n}{n!}. 
\]

By comparing the coefficients on both sides of (2.1), we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+ \), one has

\[
B_n(x) = \sum_{l=0}^{n+1} \binom{n+1}{l} G_l(x) B_{n+1-l} \frac{t^n}{n+1}. 
\]

(2.2)

From (1.1) and (1.2), also note that

\[
\frac{2t}{e^t + 1} e^{xt} = \frac{1}{t} \left( \frac{2t(e^{xt} - 1)}{e^t + 1} \right) \left( \frac{te^{xt}}{e^t - 1} \right) = \frac{1}{t} \left( \frac{2t - 2}{e^t + 1} \right) \left( \frac{te^{xt}}{e^t - 1} \right) 
\]

\[
= \frac{1}{t} \left( 2t - 2 \sum_{l=0}^{\infty} G_l \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \right) 
\]

\[
= \frac{1}{t} \left( -2 \sum_{l=1}^{\infty} G_{l+1} \frac{t^{l+1}}{l+1} \right) \left( \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!} \right) 
\]

\[
= \sum_{n=1}^{\infty} \left( -2 \sum_{l=1}^{n} \binom{n}{l} G_{l+1} \frac{B_{n-l}(x)}{l+1} \right) \frac{t^n}{n!}. 
\]

By comparing the coefficients on both sides of (2.3), we obtain the following theorem.
Theorem 2.2. For \( n \in \mathbb{N} \), one has

\[
G_n(x) = -2 \sum_{l=1}^{n} \binom{n}{l} \frac{G_{l+1}}{l+1} B_{n-l}(x). \tag{2.4}
\]

Let one take the definite integral from 0 to 1 on both sides of Theorem 2.1. For \( n \geq 2 \),

\[
0 = -2 \sum_{l=1}^{n+1} \binom{n+1}{l} \frac{G_{l+1}}{n+1} B_{n+1-l} = -B_n G_2 - 2 \sum_{l=1}^{n} \binom{n}{l} \frac{B_{n-l} G_{l+2}}{(l+1)(l+2)}. \tag{2.5}
\]

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.3. For \( n \in \mathbb{N} \) with \( n \geq 2 \), one has

\[
B_n = 2 \sum_{l=1}^{n} \binom{n}{l} \frac{B_{n-l} G_{l+2}}{(l+1)(l+2)}. \tag{2.6}
\]

3. \( p \)-Adic Integral on \( \mathbb{Z}_p \) Associated with Bernoulli and Genocchi Numbers

Let \( p \) be a fixed odd prime number. Throughout this section, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) will denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively. Let \( v_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-v_p(p)} = 1/p \). Let \( \text{UD}(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in \text{UD}(\mathbb{Z}_p) \), the bosonic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by

\[
I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \tag{3.1}
\]

(see [2, 5, 11]). From (3.1), we can derive the following integral equation:

\[
I(f_n) = I(f) + \sum_{i=0}^{n-1} f'(i) \quad (n \in \mathbb{N}), \tag{3.2}
\]

where \( f_n(x) = f(x + n) \) and \( f'(i) = ((df(x))/dx)|_{x=i} \) (see [2]). Let us take \( f(y) = e^{(x+y)} \). Then we have

\[
\int_{\mathbb{Z}_p} e^{(x+y)} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{3.3}
\]
(see [2, 5]). From (3.3), we have
\[
\int_{\mathbb{Z}_p} (x + n)^n d\mu(y) = B_n(x), \quad \int_{\mathbb{Z}_p} y^n d\mu(y) = B_n
\] (3.4)
(see [2, 5]). Thus, by (3.2) and (3.4), we get
\[
\int_{\mathbb{Z}_p} (x + n)^m d\mu(x) = \int_{\mathbb{Z}_p} x^m d\mu(x) + m \sum_{l=0}^{n-1} l^{m-1},
\] (3.5)
(see [2]). From (3.5), we have
\[
B_m(n) - B_m = m \sum_{l=0}^{n-1} l^{m-1} \quad (n \in \mathbb{Z}_+)
\] (3.6)
(see [2]). The fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) is defined by Kim as follows [2, 8, 9]:
\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)(-1)^x.
\] (3.7)
From (3.7), we obtain the following integral equation:
\[
I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} f(l)
\] (3.8)
(see [2]), where \( f_n(x) = f(x + n) \). Thus, by (3.8), we have
\[
\int_{\mathbb{Z}_p} (x + n)^m d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-l-1} l^m
\] (3.9)
(see [2]). Let us take \( f(y) = e^{t(x+y)} \). Then we have
\[
t \int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_{-1}(y) = \frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}
\] (3.10)
From (3.10), we have
\[
\int_{\mathbb{Z}_p} (x + y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \quad \int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = \frac{G_{n+1}}{n+1}.
\] (3.11)
Thus, by (3.9) and (3.11), we get

\[ \frac{G_{m+1}(n)}{m + 1} = (-1)^n \left( \frac{G_{n+1}}{n + 1} + 2 \sum_{l=0}^{n-1} (-1)^{l-1} \right). \]  

(3.12)

Let us consider the following \( p \)-adic integral on \( \mathbb{Z}_p \):

\[ K_1 = \int_{\mathbb{Z}_p} B_n(x) d\mu(x) = \sum_{l=0}^{n} \left( \frac{n}{l} \right) B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu(x) = \sum_{l=0}^{n} \left( \frac{n}{l} \right) B_{n-l} B_l. \]  

(3.13)

From Theorem 2.1 and (3.13), one has

\[ K_1 = \sum_{k=0}^{n+1} \left( \frac{n+1}{k} \right) \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \left( \frac{k}{l} \right) G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \]

\[ = \sum_{k=0}^{n+1} k \int_{\mathbb{Z}_p} B_{n+1-k} B_l G_{k-l}. \]  

(3.14)

Therefore, by (3.13) and (3.14), we obtain the following theorem.

**Theorem 3.1.** For \( n \in \mathbb{Z}_p \), one has

\[ \sum_{l=0}^{n} \left( \frac{n}{l} \right) B_{n-l} B_l = \sum_{k=0}^{n+1} \left( \frac{n+1}{k} \right) \frac{B_{n+1-k} G_{k-l}}{n+1}. \]  

(3.15)

Now, one sets

\[ K_2 = \int_{\mathbb{Z}_p} B_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \left( \frac{n}{l} \right) B_{n-l} \frac{G_{l+1}}{l+1}. \]  

(3.16)

By Theorem 2.1, one gets

\[ K_2 = \sum_{k=0}^{n+1} \left( \frac{n+1}{k} \right) \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \left( \frac{k}{l} \right) G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \]

\[ = \sum_{k=0}^{n+1} k \int_{\mathbb{Z}_p} B_{n+1-k} G_{k-l} \frac{G_{l+1}}{(n+1)(l+1)}. \]  

(3.17)

Therefore, by (3.16) and (3.17), we obtain the following theorem.
Theorem 3.2. For \( n \in \mathbb{Z}_+ \), one has
\[
\sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} = \sum_{k=0}^{n} \sum_{l \neq n} \binom{k}{l} \binom{n+1}{k+1} \frac{B_{n+1-k} G_{k-1} G_{l+1}}{(n+1)(l+1)}.
\] (3.18)

Let us consider the following \( p \)-adic integral on \( \mathbb{Z}_p \):
\[
K_3 = \int_{\mathbb{Z}_p} G_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}.
\] (3.19)

From Theorem 2.2, one has
\[
K_3 = -2 \sum_{l=1}^{n} \binom{n}{l} \frac{G_{l+1}}{l+1} \sum_{k=0}^{n-l} \binom{n-l}{k} B_{n-l-k} \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x)
\]
\[
= -2 \sum_{l=1}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} B_{n-l-k} \frac{G_{l+1} G_{k+1}}{(l+1)(k+1)}.
\] (3.20)

Therefore, by (3.19) and (3.20), we obtain the following theorem.

Theorem 3.3. For \( n \in \mathbb{Z}_+ \), one has
\[
\sum_{l=0}^{n} \binom{n}{l} \frac{G_{n-l} G_{l+1}}{l+1} = -2 \sum_{l=1}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \frac{B_{n-l-k} G_{l+1} G_{k+1}}{(l+1)(k+1)}.
\] (3.21)

Now, one sets
\[
K_4 = \int_{\mathbb{Z}_p} G_n(x) d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} B_l.
\] (3.22)

By Theorem 2.2, one gets
\[
K_4 = -2 \sum_{l=1}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \frac{G_{l+1} B_{n-l-k} B_k}{l+1}.
\] (3.23)

Therefore, by (3.22) and (3.23), we obtain the following corollary.

Corollary 3.4. For \( n \in \mathbb{Z}_+ \), one has
\[
\sum_{l=0}^{n} \binom{n}{l} G_{n-l} B_l = -2 \sum_{l=1}^{n} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} \frac{G_{l+1} B_{n-l-k} B_k}{l+1}.
\] (3.24)
Acknowledgement

This research was supported by Kyungpook National University research Fund, 2012.

References


