Research Article

Asymmetric Information and Quantization in Financial Economics

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Received 29 June 2012; Accepted 27 September 2012

Academic Editor: Bernard Soifer

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We show how a quantum formulation of financial economics can be derived from asymmetries with respect to Fisher information. Our approach leverages statistical derivations of quantum mechanics which provide a natural basis for interpreting quantum formulations of social sciences generally and of economics in particular. We illustrate the utility of this approach by deriving arbitrage-free derivative-security dynamics.

1. Introduction

Asymmetric information lies at the heart of capital markets and how it induces information flow and economic dynamics is a key element to understanding the structure and function of economic systems generally and of price discovery in particular [1–3]. The information-theoretic underpinnings of economics also provide a common framework through which economics can leverage results in other fields, an example being the well-known use of statistical mechanics in financial economics (see, e.g., [44, 45] and references therein). In addition to the statistical-mechanics representation of financial economics, however, a quantum-mechanics representation has also emerged (see [5, 18–30, 42, 46–48]) and the purpose of this paper is to show that this quantum framework too can be derived from asymmetric information, thus providing a more comprehensive information-theoretic basis for financial economics.

Financial economics is unique among economic disciplines in the extent to which stochastic processes are employed as an explanatory framework, a ubiquity epitomized in the modeling of financial derivatives. Building on a history of shared metaphor between classical physics and neoclassical economics [4] and the initial use of simple diffusion processes, financial economics and statistical mechanics found a common language in stochastic dynamics with which statistical mechanics could be applied across a wide range of
economics including finance, macroeconomics, and risk management (see, e.g., [44, 45, 52–56]). Underlying that common language is a fundamental information-theoretic basis, a basis with which financial economics can be expressed as probability theory with constraints. It is from the perspective of financial economics as probability theory with constraints that we propose to show how and why financial economics can be expressed as a quantum theory. Financial economics as quantum theory has developed in a manner similar to that taken by statistical mechanics, exploiting formal similarities (see [5, 18–30, 42, 46–48]). Financial economics as a quantum theory, however, lacks the history of common metaphor that enabled statistical mechanics to achieve its reach as an explanatory framework for financial economics. The resulting lack of common language has proved a challenge to the quantum representation of financial economics on questions ranging from the interpretation of coefficients (e.g., Planck’s constant) to the ontological and epistemological content of the theory. We propose to develop a common language through the common structure of information theory in general and of Fisher information in particular. At a general level this paper follows naturally from research demonstrating the ubiquity of quantum theory generally [5] and the statistical origins of quantum mechanics in particular. At a specific level this paper shows how constrained Fisher information can be used to quantize financial economics.

To this end we continue in Section 2 with a derivation of quantized financial economics. In this section we extend our prior work on the principle of minimum Fisher information as the fundamental expression of the concept of asymmetric information in economics (see [10–15]) via a financial-economic interpretation and adaptation of Reginatto’s Fisher-information-based derivation of the time-dependent Schrödinger equation [6]. In Section 3 we demonstrate the utility of this approach by deriving the equilibrium and nonequilibrium probability densities associated with some canonical financial structures: forwards and options. Finally, we close in Section 4 with a discussion and summary.

2. Theory

It is our view that all things economic are information-theoretic in origin: economies are participatory, observer participancy gives rise to information, and information gives rise to economics. Dynamical laws follow from a perturbation of information flow which arises from the asymmetry between \( J \), the information that is intrinsic to the system, and \( I \), the measured Fisher information of the system: a natural consequence of the notion that any observation is a result of the \( J \rightarrow I \) information-flow process. In this manner financial-economic dynamics are a natural consequence of our information-theoretic approach.

2.1. Fisher Information

We consider the price of an asset, liability, or more generally of a security which, for a given instant in time \( t_0 \), we write as \( x_0 \). At each point in time, however, the measured price is \( x_{\text{obs}} \) which is necessarily imperfect due to fluctuations \( x \), or

\[
x_{\text{obs}} = x_0 + x.
\]  

(2.1)

This fluctuation arises from both the inevitable uncertainty in the effective present time \( t_0 \) and the fact that the economy undergoes persistent random change. The greater is the fluctuation \( x \) the greater is one’s ignorance of the price. To represent this fluctuation we introduce the
probability amplitude $\psi(x)$ for the price fluctuation $x$, with the associated probability $P(x,t)$ given by

$$P(x,t) \equiv \psi(x,t)\psi^*(x,t) = |\psi(x,t)|^2,$$  \hspace{1cm} \text{(2.2)}

where the asterisk denotes the complex conjugate. Our use of probability amplitudes is at this stage independent of that seen in quantum theory: Fisher and Mather [7], for example, introduced them as a vehicle for simplifying calculations. Assuming that the statistics of $x_{\text{obs}}$ are independent of the price level $x_0$, the likelihood law for the process obeys

$$P(x_{\text{obs}}| x_0,t) \equiv P(x_{\text{obs}} - x_0,t) = P(x,t)$$ \hspace{1cm} \text{(2.3)}

by (2.1).

The classical Fisher information $I$ for this one-dimensional problem is

$$I \equiv \left\langle \left( \frac{\partial \ln(p(x_{\text{obs}}| x_0,t))}{\partial x_0} \right)^2 \right\rangle,$$ \hspace{1cm} \text{(2.4)}

where the angle bracket denotes an expectation over all possible data $x_{\text{obs}}$ [8, 9]. This is a universal form that applies to all data acquisition problems and measures the information in the data irrespective of the intrinsic nature of that data. For our problem this can be reduced to

$$I = \int \frac{1}{P} \left( \frac{\partial P}{\partial t} \right)^2 dx dt = \int |\psi'(x)|^2 dx dt,$$ \hspace{1cm} \text{(2.5)}

where $|\psi'|^2 = \psi'\psi''$ and $\psi' = d\psi/dx$. Fisher information corresponds well to our intuition regarding price discovery given in (2.1): if the price fluctuation $x$ is small then the level of information in the observation should be high, whereas if $x$ is large the information should be low. This notion is expressed in (2.5) by the shape of the probability density $P$ and since for small price fluctuations $P$ must be narrowly peaked about $x = 0$ implying high gradient $dP/dx$, a consequent high gradient $d\psi/dx$, and a large value for $I$. Similarly, if $x$ has many large values then $P$ will be broad; it and $\psi$ will have low gradients and there will be a small value for $I$.

### 2.2. Intrinsic Information $J$

The intrinsic information $J$ is the present, most complete and perfectly knowable collection of information concerning the system that is relevant to the measurement exercise [8, 9]. One example of this information is exact knowledge characterized by a unitary transform between the observation space and some conjugate space, a situation we exploited in our derivation of Tobin’s $Q$-theory [10, 11]. Another example of $J$ is empirical price data which we have employed in the application of Fisher-information-based statistical mechanics to economics [10–15]. To empirical data one can also add assumptions such as the conservation law for probability [6] and it is this approach that we will employ in this paper.

To develop $J$ for financial economics we employ three well-known assumptions: the first being that the price of a cash flow is the probability-weighted, discounted (or present)
value of that cash flow and that the price of any security is the sum of the price of the
costuent cash flows. This assumption is the basis for valuation in financial economics
[16]. Future cash flows are contingent on the future state of the economy, either implicitly
(e.g., the ability of a corporation to make future bond coupon and principal payments) or
explicitly (e.g., insurance coverage). Discounted state-contingent payments are also known
as derivative securities. Thus, derivatives, the fundamental securities of an economy, will be
our focus below.

Our second assumption is that a cash-flow price can be represented by a probability
distribution \( P(x,t) \). This assumption is central to the literature of derivative securities in
which all underlying-asset prices are represented by time-dependent probability distribu-
tions. Our final assumption is closely related to our second assumption and is that a set of
cash-flow price trajectories forms a coherent system [6, 17].

These three assumptions coalesce in the information \( J \) associated with empirical
observations of prices in an economy as observed derivative-security prices \( d_1, \ldots, d_N = \{d_n\} \)
are averages of the discounted payoff functions \( \{f_n(x,t)\} \),

\[
d_n = \int f_n(x,t)P(x,t)dx \, dt \quad n = 1, \ldots, N.
\tag{2.6}
\]

Our focus on derivative securities complements Haven’s [18–23] analysis of derivatives from
a de Broglie-Bohm perspective by providing (as we shall see presently) an information-
theoretic basis for the de Broglie-Bohm approach in financial economics and generalizes
the work of Khrennikov and Choustova [5, 24–30] on equity prices in the de Broglie-Bohm
framework as equity can be viewed as a derivative security, namely, a call option on the assets
of the issuing firm [31–33].

The last two of our three assumptions imply that the velocity \( \nu \) of a cash-flow at price
point \( x \) can be related to a real function \( S(x,t) \) by an expression of the form [6]:

\[
\nu = \frac{1}{m_e} \frac{\partial S}{\partial x},
\tag{2.7}
\]

where \( m_e \) is the effective mass of the price represented by market turnover.\(^5\) It follows that
the probability distribution must satisfy a conservation law of the form

\[
\frac{\partial P}{\partial t} + \frac{1}{m_e} \frac{\partial}{\partial x} \left( P \frac{\partial S}{\partial x} \right) = 0,
\tag{2.8}
\]

and, as discussed by Reginatto [6], (2.8) can be derived from a variational principle, by
minimization of the expression

\[
\int \left( \frac{\partial S}{\partial t} + \frac{1}{2m_e} \left( \frac{\partial S}{\partial x} \right)^2 \right)dx \, dt,
\tag{2.9}
\]

with respect to \( S \).
2.3. Information Asymmetry and Dynamics

To minimize the Fisher information $I$ in a manner consistent with our information $J$ one can form the information asymmetry Lagrangian $\mathcal{L} = I - J$ [6]:

$$\mathcal{L} = \lambda_0 \int \left( \frac{1}{P} \left( \frac{\partial P}{\partial t} \right)^2 \right) dx \, dt - \int P \left( \frac{\partial S}{\partial t} + \frac{1}{2m_e} \left( \frac{\partial S}{\partial x} \right)^2 \right) dx \, dt - \sum_{n=1}^{N} \lambda_n \left[ \int f_n P \, dx \, dt \right].$$

Variation of the information asymmetry $\mathcal{L}$ with respect to $S$ and $P$ yields

$$\frac{\partial P}{\partial t} + \frac{1}{m_e} \frac{\partial}{\partial x} \left( P \frac{\partial S}{\partial x} \right) = 0,$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m_e} \left( \frac{\partial S}{\partial x} \right)^2 + V(x,t) - \frac{\lambda_0}{m_e} \left( 2 \frac{\partial^2 P}{P \partial x^2} - \frac{1}{P^2} \left( \frac{\partial P}{\partial x} \right)^2 \right) = 0,$$

where

$$V(x,t) = \sum_{n=1}^{N} \lambda_n f_n(x,t).$$

Equations (2.11) and (2.12) are the Madelung hydrodynamic equations [34] that, via the Madelung transform $\psi = \sqrt{P} \exp(iS/\sqrt{8\lambda_0})$ where $i \equiv \sqrt{-1}$, are the real and imaginary parts of

$$i \frac{\partial \psi}{\partial t} = - \frac{\sqrt{2\lambda_0}}{m_e} \left[ \frac{\partial^2 \psi}{\partial x^2} - V(x,t) \right] \psi;$$

a Schrödinger-like wave equation with a potential function $V$ that is a linear combination of the payoff functions of the derivative securities in the economy.

To complete the model we need Lagrange multipliers that are consistent with our information $J$. The Lagrange multiplier $\lambda_0$ can be resolved in a manner that relates this quantum approach to the traditional stochastic representation of financial economics through the use of Nelson’s stochastic mechanics [35, 36]:

$$dx(t) = b(x(t),t) dt + dw(t),$$

where

$$b = \frac{\sqrt{2\lambda_0}}{m_e} \left( \frac{1}{2} \ln P + S \right).$$

$w(t)$ is a Wiener process, and $dw(t)$ is Gaussian with zero mean and product expectation $Edw_i(t)dw_j(t) = 2\nu \delta_{ij} dt$ where $\nu = \sqrt{2\lambda_0/m_e}$ is the diffusion coefficient and $\delta_{ij}$ is the
Kronecker delta function. From this it follows that the Lagrange multiplier \( \lambda_0 \) is related to turnover and to the diffusivity of the economy by \( \lambda_0 = m_2^2 \nu^2 / 2 \).

Determining the Lagrange multipliers \( \lambda_1, \ldots, \lambda_N \) is a straightforward exercise as the value of each is set by the requirement that the corresponding observed derivative price as expressed in (2.6) is recovered. A convenient consequence of this approach is that the calculated \( P(x, t) \) is consistent with all observed security prices (i.e., is arbitrage free) by construction. A measure of the price uncertainty in the economy \( \sigma^2 \) can be had through the use of the Cramer-Rao inequality \( \sigma^2 \geq 1 / I \) [37, 38] with which a lower bound of the price variance, or the notion of implied volatility employed widely in derivatives trading, is seen to be the inverse of the Fisher information [9, 12].

3. Example: Derivative Securities

A particular advantage of our Fisher information approach to dynamics in financial economics is the natural way by which both time dependence and departures from equilibrium arise. Time dependence of the probability density and, by implication, security prices is expressed in (2.14). The simplest example of equilibrium and departures therefrom can be seen by considering the price of a forward \( d_{\text{fwd}} \) in a zero-rate environment which is the first moment:

\[
d_{\text{fwd}} = \int_0^\infty x \rho(x, t) \delta(T - t) dx dt, \tag{3.1}
\]

where \( T \) is the expiration date of the forward contract and \( \delta(x) \) is the Dirac delta function. The solution of (2.14) that vanishes at the zero-price boundary for this linear payoff function is known to be a linear combination of the the Airy function \( \text{Ai}(x) \) [39]:

\[
q_n(x) \propto \text{Ai}(\xi_n(x)), \tag{3.2}
\]

where

\[
\xi_n(x) = \left[ \frac{m_2 \lambda_1}{4 \lambda_0} \right]^{1/3} x - a_n, \tag{3.3}
\]

where \( a_n \) is the \( n \)th zero of the Airy function. The equilibrium solution for the forward price corresponds to the first zero of the Airy function \( a_1 = -2.33811 \), with the value of \( \lambda_1 \) chosen to reproduce the observed forward price. Disequilibrium forward prices can be represented as linear combinations of the solutions corresponding to the zeros of the Airy function for \( n \geq 1 \). This complete description of the temporal evolution of the forward price in and about the equilibrium state in terms of a collection of eigenfunctions is the information-theoretic first quantization of financial economics.
Extending this to call and put options which are the partial moments,

\[
d_{\text{call}} = \int_0^\infty \max(x - k, 0) P(x, t) \delta(T - t) dx \, dt,
\]

\[
d_{\text{put}} = \int_0^\infty \max(k - x, 0) P(x, t) \delta(T - t) dx \, dt,
\]

(3.4)

where \( k \) is the strike price is straightforward. The simple linear potential function of the forward shown in (3.1) is replaced by a piecewise linear potential resulting from the linear combination of the payoff functions of each derivative weighted by their corresponding Lagrange multiplier.

### 4. Discussion and Summary

From the perspective of information theory, the quantum representation of financial economics is a natural outcome of the process of inference using Fisher information. This approach also complements the work of Haven, Choustova and Khrennikov [5, 18–30] concerning the quantum potential introduced by Bohm [40, 41]. Bohm interpreted (2.12) as a Hamilton-Jacobi equation and the final two terms on the left-hand side of this equation as a potential that acts, in our case, on the security. The first of these two terms \( V \) is the usual physical potential that in financial economics is the potential associated with the contractual state-contingent payoff structure of a derivative security (cf. (2.13)). The other potential term in (2.12), identified by Bohm [40, 41] as the quantum potential, was later reassessed by Reginatto [6] in light of its dependence on the probability to reflect the inferential nature of the quantum formalism: specifically, he noted that the average of the quantum potential is proportional to Fisher information. Thus, the potential in financial economics naturally splits into one component that originates in the specifics of the economy and another component that arises from the manner by which the market infers prices. This derivation also provides a clear distinction between the ontological and epistemological content of the quantum theory of financial economics. The epistemological content of this theory is the use of the minimization of Fisher information to choose the probability distribution that describes the price of securities. The economic content of the theory is the assumption that the price-space motion of securities is a coherent structure and the existence of observed security prices.\(^7\)

Quantum theory is an attractive formalism to use in the treatment of economics generally and of financial economics in particular due to the manner in which uncertainty arises [5, 18–30, 42, 43]. In this paper we have adapted an information-theoretic approach—the use of the principle of minimum Fisher information—to the derivation of quantum mechanics to illuminate aspects of this formalism that have proved challenging to the use of the quantum formalism in financial economics, challenges due in part to the lack of a historic common language. This approach is consistent with existing foundational assumptions in financial economics, incorporates time-dependence as a natural consequence of conservation of probability, and yields both equilibrium and disequilibrium solutions for security prices. Finally, the clarity of expression and formal link with the statistical origins of quantum mechanics recommends this quantum formalism as a useful approach to the treatment of problems in financial economics.
Acknowledgments

The authors thank Professor Emmanuel Haven for bringing the use of the de Broglie-Bohm approach in financial economics to our attention through his engaging presentation and our subsequent conversations at the 2011 Winter Workshop on Economic Heterogeneous Interacting Agents held at Tianjin University. The authors thank Professor Ewan Wright for many helpful conversations regarding information theory and the statistical basis of quantum mechanics, and his insight and suggestions that materially improved this paper. The authors also thank Dr. Dana Hobson and Dr. Minder Cheng for insight concerning price momentum.

Endnotes

1. The physical basis for employing stochastic dynamics, uncertainty, has been a part of economics generally for some time since the pioneering work of Keynes and Knight [49–51].

2. The information-theoretic basis of statistical mechanics is discussed in [57–61]. The relationship between information theory and economics is discussed in detail in [11] and references therein. In recent communications we have shown that the dynamics of economic systems can be derived from information asymmetry with respect to Fisher information and that this form of asymmetric information yields a powerful explanatory statistical mechanical framework for financial economics [10–15]. With a common information-theoretic basis for both statistical mechanics and financial economics the notion of statistical mechanics as probability theory with constraints (see, e.g., [62]) suggests that financial economics can be expressed as probability theory with constraints.

3. For a review of the statistical origins of quantum mechanics, see [63].

4. A discrete-time illustration of this assumption expresses observed price data as [64]

\[ d = \sum_{t=1}^{T} \frac{(P_t + P_{t-1}D_t\mu)C_t}{(1+r)^t}, \]  

where \( T \) is the number of cash flows expected of the security, \( P_t \) is the probability that the \( t \)th cash flow \( C_t \) is received, \( D_t \) is the probability of default at time \( t \), \( \mu \) is the fraction of the cash flow \( C_t \) that is received in the event of default, \( r \) is the interest rate, and \( (1+r)^{-t} \) is the discount factor associated with cash flow \( C_t \). If the cash flows are dividends then this is the dividend-discount model for stock prices. If \( D_t = 0 \), the cash flows \( C_t \) for \( t < T \) are coupon payments, and the final cash flow \( C_T \) is a coupon and principal payment; this is the model for a government bond: with \( D_t > 0 \) this becomes the model for a corporate bond. The price of a security is also shown in (\(*)\) to be the sum of the probability weighted \( P_t \) discounted by \( (1+r)^{-t} \) state-contingent future payments \( C_t \) or \( D_t\mu C_t \).

5. Turnover is the ratio of the amount traded (or volume) to the average amount traded during a given time period [65]. As a financial-economic representation of the notion of effective mass, it represents the ease with which a security traverses price space within a market and is a function of the trading environment in the market. The use of turnover in this context of price momentum \( vme \) is suggested in the economics literature [66–68] and found formally in the econophysics literature with the amount traded introduced by
Khrennikov and Choustova [5, 24–30] and turnover introduced by Ausloos and Ivanova [65].

6. The identity of the Lagrange multiplier $\lambda_0$ depends on the physical or economic nature of the associated derivation. In a quantum mechanics, Reginatto identified $\lambda_0$ in terms of Plank’s constant as $\hbar = \sqrt{8\lambda_0}$ [6]. Derivations in optics or acoustics often identify $\lambda_0$ in terms of a wavelength (see, e.g., [69] and references therein). In the current paper we have identified it as a function of economic variables.

7. These last two sentences paraphrase and adapt the original observations of Reginatto regarding the ontological and epistemological content of quantum theory [6].

References


