Research Article

A Nice Separation of Some Seiffert-Type Means by Power Means

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Seiffert has defined two well-known trigonometric means denoted by \( \mathcal{P} \) and \( \mathcal{T} \). In a similar way it was defined by Carlson the logarithmic mean \( \mathcal{L} \) as a hyperbolic mean. Neuman and Sándor completed the list of such means by another hyperbolic mean \( \mathcal{M} \). There are more known inequalities between the means \( \mathcal{P}, \mathcal{T}, \) and \( \mathcal{L} \) and some power means \( \mathcal{A}_p \). We add to these inequalities two new results obtaining the following nice chain of inequalities \( \mathcal{A}_0 < \mathcal{L} < \mathcal{A}_{1/2} < \mathcal{P} < \mathcal{A}_1 < \mathcal{M} < \mathcal{A}_{3/2} < \mathcal{T} < \mathcal{A}_2 \), where the power means are evenly spaced with respect to their order.

1. Means

A mean is a function \( M : \mathbb{R}^2_+ \to \mathbb{R}_+ \), with the property

\[
\min(a, b) \leq M(a, b) \leq \max(a, b), \quad \forall a, b > 0.
\] (1.1)

Each mean is reflexive; that is,

\[
M(a, a) = a, \quad \forall a > 0.
\] (1.2)

This is also used as the definition of \( M(a, a) \).

We will refer here to the following means:

(i) the power means \( \mathcal{A}_p \), defined by

\[
\mathcal{A}_p(a, b) = \left[ \frac{a^p + b^p}{2} \right]^{1/p}, \quad p \neq 0;
\] (1.3)
(ii) the geometric mean \( \mathcal{G} \), defined as \( \mathcal{G}(a, b) = \sqrt{ab} \), but verifying also the property

\[
\lim_{p \to 0} \mathcal{A}_p(a, b) = \mathcal{A}_0(a, b) = \mathcal{G}(a, b); \tag{1.4}
\]

(iii) the first Seiffert mean \( \mathcal{P} \), defined in [1] by

\[
\mathcal{P}(a, b) = \frac{a - b}{2 \sin^{-1}((a - b)/(a + b))}, \quad a \neq b; \tag{1.5}
\]

(iv) the second Seiffert mean \( \mathcal{T} \), defined in [2] by

\[
\mathcal{T}(a, b) = \frac{a - b}{2 \tan^{-1}((a - b)/(a + b))}, \quad a \neq b; \tag{1.6}
\]

(v) the Neuman-Sándor mean \( \mathcal{M} \), defined in [3] by

\[
\mathcal{M}(a, b) = \frac{a - b}{2 \sinh^{-1}((a - b)/(a + b))}, \quad a \neq b; \tag{1.7}
\]

(vi) the Stolarsky means \( \mathcal{S}_{p,q} \) defined in [4] as follows:

\[
\mathcal{S}_{p,q}(a, b) = \begin{cases} 
\left[ \frac{q(a^p - b^p)}{p(a^q - b^q)} \right]^{1/(p-q)} , & p \neq q \\
\frac{1}{e^p} \left( \frac{a^p}{b^p} \right)^{1/(a^p - b^p)} , & p = q \neq 0 \\
\frac{1}{p} \left( \frac{a^p - b^p}{p(\ln a - \ln b)} \right)^{1/p} , & p \neq 0, q = 0 \\
\sqrt{ab}, & p = q = 0.
\end{cases} \tag{1.8}
\]

The mean \( \mathcal{A}_1 = \mathcal{A} \) is the arithmetic mean and the mean \( \mathcal{S}_{1,0} = \mathcal{L} \) is the logarithmic mean. As Carlson remarked in [5], the logarithmic mean can be represented also by

\[
\mathcal{L}(a, b) = \frac{a - b}{2 \tanh^{-1}((a - b)/(a + b))}; \tag{1.9}
\]
thus the means $P, T, M,$ and $L$ are very similar. In [3] it is also proven that these means can be defined using the nonsymmetric Schwab-Borchardt mean $SB$ given by

$$SB(a, b) = \begin{cases} 
\frac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)}, & \text{if } a < b \\
\frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & \text{if } a > b 
\end{cases}$$

(1.10)

(see [6, 7]). It has been established in [3] that

$$L = SB(\mathcal{A}, \mathcal{G}), \quad P = SB(\mathcal{G}, \mathcal{A}), \quad T = SB(\mathcal{A}, \mathcal{A}_2), \quad M = SB(\mathcal{A}_2, \mathcal{A}).$$

(1.11)

2. Interlacing Property of Power Means

Given two means $M$ and $N$, we will write $M < N$ if

$$M(a, b) < N(a, b), \quad \text{for } a \neq b. \quad (2.1)$$

It is known that the family of power means is an increasing family of means, thus

$$\mathcal{A}_p < \mathcal{A}_q, \quad \text{if } p < q. \quad (2.2)$$

Of course, it is more difficult to compare two Stolarsky means, each depending on two parameters. To present the comparison theorem given in [8, 9], we have to give the definitions of the following two auxiliary functions:

$$k(x, y) = \begin{cases} 
\frac{|x| - |y|}{x - y}, & x \neq y \\
\text{sign}(x), & x = y 
\end{cases}$$

$$l(x, y) = \begin{cases} 
\mathcal{L}(x, y), & x > 0, \ y > 0 \\
0, & x \geq 0, \ y \geq 0, \ xy = 0 
\end{cases}$$

(2.3)

**Theorem 2.1.** Let $p, q, r, s \in \mathbb{R}$. Then the comparison inequality

$$S_{p,q} \leq S_{r,s}$$

holds true if and only if $p + q \leq r + s$, and (1) $l(p, q) \leq l(r, s)$ if $0 \leq \min(p, q, r, s)$, (2) $k(p, q) \leq k(r, s)$ if $\min(p, q, r, s) < 0 < \max(p, q, r, s)$, or (3) $-l(p, -q) \leq -l(-r, -s)$ if $\max(p, q, r, s) \leq 0$.

We need also in what follows an important double-sided inequality proved in [3] for the Schwab-Borchardt mean:

$$\sqrt[3]{ab^2} < SB(a, b) < \frac{a + 2b}{3}, \quad a \neq b.$$
Being rather complicated, the Seiffert-type means were evaluated by simpler means, first of all by power means. The evaluation of a given mean $M$ by power means assumes the determination of some real indices $p$ and $q$ such that $A_p < M < A_q$. The evaluation is optimal if $p$ is the greatest and $q$ is the smallest index with this property. This means that $M$ cannot be compared with $A_r$ if $p < r < q$.

For the logarithmic mean in [10], it was determined the optimal evaluation

$$A_0 < L < A_{1/3}. \tag{2.6}$$

For the Seiffert means, there are known the evaluations

$$A_{1/3} < P < A_{2/3}, \tag{2.7}$$

proved in [11] and

$$A_1 < T < A_2, \tag{2.8}$$

given in [2]. It is also known that

$$A_1 < M < T, \tag{2.9}$$

as it was shown in [3]. Moreover in [12] it was determined the optimal evaluation

$$A_{\ln 2/\ln \pi} < P < A_{2/3}. \tag{2.10}$$

Using these results we deduce the following chain of inequalities:

$$A_0 < L < A_{1/2} < P < A_1 < M < T < A_2. \tag{2.11}$$

To prove the full interlacing property of power means, our aim is to show that $A_{3/2}$ can be put between $M$ and $T$. We thus obtain a nice separation of these Seiffert-type means by power means which are evenly spaced with respect to their order.

### 3. Main Results

We add to the inequalities (2.11) the next results.

**Theorem 3.1.** The following inequalities

$$M < A_{3/2} < T \tag{3.1}$$

are satisfied.
Proof. First of all, let us remark that $A_{3/2} = S_{3/2}$. So, for the first inequality in (3.1), it is sufficient to prove that the following chain of inequalities

$$\mathcal{M} < \frac{A_2 + 2A}{3} < S_{3,1} < S_{3,3/2}$$

is valid. The first inequality in (3.2) is a simple consequence of the property of the mean $\mathcal{M}$ given in (1.11) and the second inequality from (2.5). The second inequality can be proved by direct computation or by taking $a = 1 + t$, $b = 1 - t$, $(0 < t < 1)$ which gives

$$\frac{\sqrt{1 + t^2} + 2}{3} < \sqrt{\frac{3 + t^2}{3}},$$

which is easy to prove. The last inequality in (3.2) is given by the comparison theorem of the Stolarsky means. In a similar way, the second inequality in (3.1) is given by the relations

$$S_{3,3/2} < S_{4,1} = \sqrt{\mathcal{A}_2^2} < T.$$

The first inequality is again given by the comparison theorem of the Stolarsky means. The equality in (3.4) is shown by elementary computations, and the last inequality is a simple consequence of the property of the mean $\mathcal{T}$ given in (1.11) and the first inequality from (2.5).

Corollary 3.2. The following two-sided inequality

$$\frac{x}{\sinh^{-1}x} < A_{3/2}(1 - x, 1 + x) < \frac{x}{\tan^{-1}x},$$

is valid for all $0 < x < 1$.

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References


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