Research Article

Pseudovaluations on WFI Algebras

Young Bae Jun,¹ Min Su Kang,² and Eun Hwan Roh³

¹ Department of Mathematics Education (and RINS), Gyeongsang National University, Chinju 660-701, Republic of Korea
² Department of Mathematics, Hanyang University, Seoul 133-791, Republic of Korea
³ Department of Mathematics Education, Chinju National University of Education, Chinju 660-756, Republic of Korea

Correspondence should be addressed to Min Su Kang, sinchangmyun@hanmail.net

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Using Busneag’s model, the notion of pseudovaluations (valuations) on a WFI algebra is introduced, and a pseudometric is induced by a pseudovaluation on WFI algebras. Given a valuation with additional condition, we show that the binary operation in WFI algebras is uniformly continuous.

1. Introduction

In 1990, Wu [1] introduced the notion of fuzzy implication algebras (FI algebra, for short) and investigated several properties. In [2], Li and Zheng introduced the notion of distributive (regular, and commutative, resp.) FI algebras and investigated the relations between such FI algebras and MV algebras. In [3], Jun discussed several aspects of WFI algebras. He introduced the notion of associative (normal and medial, resp.) WFI algebras and investigated several properties. He gave conditions for a WFI algebra to be associative/medial, provided characterizations of associative/medial WFI algebras, and showed that every associative WFI algebra is a group in which every element is an involution. He also verified that the class of all medial WFI algebras is a variety. Jun et al. [4] introduced the concept of ideals of WFI algebras, and gave relations between a filter and an ideal. Moreover, they provided characterizations of an ideal, and established an extension property for an ideal. Buşneag [5] defined pseudovaluation on a Hilbert algebra and proved that every pseudovaluation induces a pseudometric on a Hilbert algebra. Also, Buşneag [6] provided several theorems on extensions of pseudovaluations. Buşneag [7] introduced the notions of pseudovaluations...
(valuations) on residuated lattices, and proved some theorems of extension for these (using
the model of Hilbert algebras ([6])).

In this paper, using Buşneag’s model, we introduce the notion of pseudovaluations
valuations) on WFI algebras, and we induce a pseudometric by using a pseudovaluation
WFI algebras. Given a valuation with additional condition, we show that the binary operation
in WFI algebras is uniformly continuous.

2. Preliminaries

Let \( K(\tau) \) be the class of all algebras of type \( \tau = (2,0) \). By a WFI algebra, we mean an algebra
\((X; \odot, \theta) \in K(\tau)\) in which the following axioms hold:

\[
\begin{align*}
(a1) \quad & (\forall x \in X) \ (x \odot x = \theta), \\
(a2) \quad & (\forall x, y \in X) \ (x \odot y = y \odot x = \theta \Rightarrow x = y), \\
(a3) \quad & (\forall x, y, z \in X) \ (x \odot (y \odot z) = y \odot (x \odot z)), \\
(a4) \quad & (\forall x, y, z \in X) \ ((x \odot y) \odot ((z \odot y) \odot (x \odot z)) = \theta).
\end{align*}
\]

For the convenience of notation, we will write \([x, y_1, y_2, \ldots, y_n]\) for
\[
(\cdots ((x \odot y_1) \odot y_2) \odot \cdots ) \odot y_n.
\]

We define \([x, y]^0 = x\), and for \( n > 0 \), \([x, y]^n = [x, y, \ldots, y]\), where \( y \) occurs \( n \)-times.

**Proposition 2.1** (see [3]). In a WFI algebra \( X \), the following are true:

\[
\begin{align*}
(b1) \quad & x \odot [x, y]^2 = \theta, \\
(b2) \quad & \theta \odot x = x = \theta, \\
(b3) \quad & \theta \odot x = x, \\
(b4) \quad & x \odot y = \theta \Rightarrow (y \odot z) \odot (x \odot z) = \theta, \ (z \odot x) \odot (z \odot y) = \theta, \\
(b5) \quad & (x \odot y) \odot \theta = (x \odot \theta) \odot (y \odot \theta), \\
(b6) \quad & [x, y]^3 = x \odot y.
\end{align*}
\]

We define a relation “\( \leq \)” on \( X \) by \( x \leq y \) if and only if \( x \odot y = \theta \). It is easy to verify
that a WFI algebra is a partially ordered set with respect to \( \leq \). A nonempty subset \( S \) of a WFI
algebra \( X \) is called a subalgebra of \( X \) if \( x \odot y \in S \) whenever \( x, y \in S \). A nonempty subset \( F \) of
a WFI algebra \( X \) is called a filter of \( X \) if it satisfies:

\[
\begin{align*}
(c1) \quad & \theta \in F, \\
(c2) \quad & (\forall x \in F) \ (\forall y \in X) \ (x \odot y \in F \Rightarrow y \in F).
\end{align*}
\]

A filter \( F \) of a WFI algebra \( X \) is said to be closed (see [3]) if \( F \) is also a subalgebra of
\( X \). A nonempty subset \( I \) of a WFI algebra \( X \) is called an ideal of \( X \) (see [4]) if it satisfies the
condition \( (c1) \) and

\[
\begin{align*}
(c3) \quad & (\forall x, y \in X) \ (\forall z \in I) \ ((x \odot z) \odot y \in I \Rightarrow x \odot y \in I).
\end{align*}
\]

**Proposition 2.2** (see [3]). Let \( F \) be a filter of a WFI algebra \( X \). Then \( F \) is closed if and only if
\( x \odot \theta \in F \) for all \( x \in F \).
Proposition 2.3 (see [3]). In a finite WFI algebra, every filter is closed.

Note that every ideal of a WFI algebra is a closed filter (see [4, Theorem 4.3]). For a WFI algebra $X$, the set

$$S(X) := \{ x \in X \mid x \leq \theta \}$$

(2.2)

is called the simulative part of $X$.

3. WFI Algebras with Pseudovaluations

In what follows, let $X$ denote a WFI algebra unless otherwise specified.

Definition 3.1. A mapping $f : X \to \mathbb{R}$ is called a pseudovaluation on $X$ if it satisfies the following two conditions:

(i) $f(\theta) = 0$,

(ii) $(\forall x, y \in X) \ (f(x \oplus y) + f(x) \geq f(y))$.

A pseudovaluation $f$ on $X$ satisfying the following condition:

$$(\forall x \in X) \ (x \neq \theta \Rightarrow f(x) \neq 0)$$

(3.1)

is called a valuation on $X$.

Obviously, a mapping

$$f : X \to \mathbb{R}, \ x \mapsto 0$$

(3.2)

is a pseudovaluation on $X$, which is called the trivial pseudovaluation.

Example 3.2. Let $f : X \to \mathbb{R}$ be a mapping defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \theta, \\ k & \text{if } x \in X \setminus \{\theta\}, \end{cases}$$

(3.3)

where $k$ is a positive real number. Then, $f$ is a pseudovaluation on $X$. Moreover, it is a valuation on $X$.

Example 3.3. Let $\mathbb{Z}$ be the set of integers. Then, $(\mathbb{Z}; \ominus, \theta)$ is a WFI algebra, where $\theta = 0$ and $x \ominus y = y - x$ for all $x, y \in \mathbb{Z}$ (see [8]). Let $f : \mathbb{Z} \to \mathbb{R}$ be a mapping defined by

$$f(x) = \begin{cases} 0 & \text{if } x = \theta, \\ ax + b & \text{otherwise}, \end{cases}$$

(3.4)
Proof. Since \( x \) and Definition 3.1

Let \( Z \) be a pseudovaluation on \( X \) for all \( x \geq 0 \). Then, \( f \) is a pseudovaluation on \( Z \).

Example 3.4. Let \( X = \{\theta, a, b\} \) be a set with the following Cayley table:

\[
\begin{array}{ccc}
\ominus & \theta & a & b \\
\theta & \theta & a & b \\
a & \theta & \theta & b \\
b & b & b & \theta \\
\end{array}
\] (3.5)

Then, \( (X; \ominus, \theta) \) is a WFI algebra (see [3]). Define a mapping \( f : X \rightarrow \mathbb{R} \) by \( f(\theta) = 0 \), \( f(a) = 2 \) and \( f(b) = 9 \). Then, \( f \) is a pseudovaluation on \( X \). Also, it is a valuation on \( X \).

Proposition 3.5. Every pseudovaluation \( f \) on \( X \) satisfies the following conditions:

1. \( \forall x, y \in X \) \( (x \leq y \Rightarrow f(x) \geq f(y)) \).
2. \( \forall x, y, z \in X \) \( (f(x \ominus z) \leq f(x \ominus y) + f(y \ominus z)) \).
3. \( \forall x, y \in X \) \( (f(x \ominus y) + f(y \ominus x) \geq 0) \).

Proof. (1) Let \( x, y \in X \) be such that \( x \leq y \). Then, \( x \ominus y = \theta \), and so

\[ f(y) \leq f(x \ominus y) + f(x) = f(\theta) + f(x) = 0 + f(x) = f(x) \]. (3.6)

(2) Using (a4), we have \( x \ominus y \leq (y \ominus z) \ominus (x \ominus z) \) for all \( x, y, z \in X \). It follows from (1) and Definition 3.1(ii) that

\[ f(x \ominus y) \geq f((y \ominus z) \ominus (x \ominus z)) \geq f(x \ominus z) - f(y \ominus z) \], (3.7)

so that \( f(x \ominus z) \leq f(x \ominus y) + f(y \ominus z) \) for all \( x, y, z \in X \).

(3) Let \( x, y \in X \). Using Definition 3.1(ii), we have \( f(x \ominus y) + f(x) \geq f(y) \) and \( f(y \ominus x) + f(y) \geq f(x) \); that is, \( f(x \ominus y) \geq f(y - f(x)) \) and \( f(y \ominus x) \geq f(x - f(y)) \). It follows that \( f(x \ominus y) + f(y \ominus x) \geq 0 \). \( \square \)

Corollary 3.6. Let \( f : X \rightarrow \mathbb{R} \) be a pseudovaluation on \( X \). Then, \( f(x) \geq 0 \) for all \( x \in \mathcal{S}(X) \).

Proof. Since \( x \leq \theta \) for all \( x \in \mathcal{S}(X) \), we have \( f(x) \geq f(\theta) = 0 \) by Proposition 3.5(1) and Definition 3.1(i). \( \square \)

The following example shows that the converse of Corollary 3.6 may not be true.

Example 3.7. Let \( X \) be a WFI algebra which is considered in Example 3.4. Let \( g : X \rightarrow \mathbb{R} \) be a mapping defined by

\[ g = \begin{pmatrix} \theta & a & b \\ 0 & -3 & 2 \end{pmatrix} \]. (3.8)
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Then, \( S(X) = \{\theta, b\} \), \( g(\theta) = 0 \) and \( g(b) = 2 \geq 0 \). But \( g \) is not a pseudovaluation on \( X \), since

\[
    g(a \odot \theta) + g(a) = g(\theta) + g(a) = -3 \neq 0 = g(\theta).
\]

Let \( f : X \to \mathbb{R} \) be a pseudovaluation on \( X \). If \( x_1 \odot x = \theta \), that is, \( x_1 \leq x \), for all \( x, x_1 \in X \), then \( f(x) \leq f(x_1) \) by Proposition 3.5(1). If \( x_2 \ominus (x_1 \odot x) = \theta \) for all \( x, x_1, x_2 \in X \), then \( x_2 \leq x_1 \ominus x \), and so \( f(x_2) \geq f(x_1 \ominus x) \geq f(x) - f(x_1) \) by Proposition 3.5(1) and Definition 3.1(ii). Hence, \( f(x) \leq f(x_1) + f(x_2) \). Now, if \( x_3 \ominus (x_2 \ominus (x_1 \ominus x)) = \theta \) for all \( x, x_1, x_2, x_3 \in X \), then \( x_3 \leq x_2 \ominus (x_1 \ominus x) \). It follows from Proposition 3.5(1) and Definition 3.1(ii) that

\[
    f(x_3) \geq f(x_2 \ominus (x_1 \ominus x)) \geq f(x_1 \ominus x) - f(x_2) \geq f(x) - f(x_1) - f(x_2),
\]

so that \( f(x) \leq f(x_1) + f(x_2) + f(x_3) \). Continuing this process, we have the following proposition.

**Proposition 3.8.** Let \( f : X \to \mathbb{R} \) be a pseudovaluation on \( X \). For any elements \( x, x_1, x_2, \ldots, x_n \) of \( X \), if \( x_n \ominus (\cdots \ominus (x_2 \ominus (x_1 \ominus x)) \cdots) = \theta \), then \( f(x) \leq \sum_{k=1}^{n} f(x_k) \).

**Theorem 3.9.** Let \( F \) be a filter of \( X \), and let \( f_F : X \to \mathbb{R} \) be a mapping defined by

\[
    f_F(x) = \begin{cases} 
        0 & \text{if } x \in F, \\ 
        k & \text{if } x \notin F,
    \end{cases}
\]

where \( k \) is a positive real number. Then, \( f_F \) is a pseudovaluation on \( X \). In particular, \( f_F \) is a valuation on \( X \) if and only if \( F = \{\theta\} \).

**Proof.** Straightforward. \( \square \)

We say \( f_F \) is a pseudovaluation induced by a filter \( F \).

**Theorem 3.10.** If a mapping \( f : X \to \mathbb{R} \) is a pseudovaluation on \( X \), then the set

\[
    F_f := \{x \in X \mid f(x) \leq 0\}
\]

is a filter of \( X \).

**Proof.** Obviously, \( \theta \in F_f \). Let \( x, y \in X \) be such that \( x \in F_f \) and \( x \ominus y \in F_f \). Then, \( f(x) \leq 0 \) and \( f(x \ominus y) \leq 0 \). It follows from Definition 3.1(ii) that \( f(y) \leq f(x \ominus y) + f(x) \leq 0 \) so that \( y \in F_f \). Hence, \( F_f \) is a filter of \( X \). \( \square \)

We say \( F_f \) is a filter induced by a pseudovaluation \( f \) on \( X \).

**Corollary 3.11.** If a mapping \( f : X \to \mathbb{R} \) is a pseudovaluation on a finite WFI algebra \( X \), then the set

\[
    F_f := \{x \in X \mid f(x) \leq 0\}
\]

is a closed filter of \( X \).
Proof. It follows from Proposition 2.3 and Theorem 3.10.

Remark 3.12. A filter induced by a pseudovaluation on X may not be closed. Indeed, in Example 3.3, if we take \( a = 1 \) and \( b = 0 \), then \( f : \mathbb{Z} \to \mathbb{R}, x \mapsto x \), is a pseudovaluation on \( \mathbb{Z} \). Then, \( F_f = \{ \theta \} \cup \{ k \in \mathbb{Z} \mid k < \theta \} \) which is a filter but not a subalgebra of \( \mathbb{Z} \), since \( (-3) \oplus (-1) = -1 - (-3) = 2 \notin F_f \). Hence, \( F_f \) is not a closed filter of \( \mathbb{Z} \).

Theorem 3.13. For any pseudovaluation \( f : X \to \mathbb{R} \), if \( F \) is a filter of \( X \), then \( F_f = F \).

Proof. We have \( F_f = \{ x \in X \mid f(x) \leq 0 \} = \{ x \in X \mid x \in F \} = F \).

The following example shows that the converse of Theorem 3.10 may not be true; that is, there exist a WFI algebra \( X \) and a mapping \( f : X \to \mathbb{R} \) such that

(1) \( f \) is not a pseudovaluation on \( X \),

(2) \( F_f := \{ x \in X \mid f(x) \leq 0 \} \) is a filter of \( X \).

Example 3.14. Let \( X = \{ \theta, 1, 2, a, b \} \) be a set with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>( \theta )</th>
<th>1</th>
<th>2</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta )</td>
<td>( \theta )</td>
<td>1</td>
<td>2</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>1</td>
<td>( \theta )</td>
<td>( \theta )</td>
<td>2</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>2</td>
<td>( \theta )</td>
<td>( \theta )</td>
<td>( \theta )</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>( \theta )</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>a</td>
<td>a</td>
<td>a</td>
<td>( \theta )</td>
<td>( \theta )</td>
</tr>
</tbody>
</table>

Then \( (X; \oplus, \theta) \) is a WFI algebra. Let \( f : X \to \mathbb{R} \) be a mapping defined by

\[
f = \begin{pmatrix} \theta & 2 & a & b \\ 0 & -4 & 3 & -2 & 5 \end{pmatrix}.
\]

Then, \( F_f = \{ \theta, 1, a \} \) is a filter of \( X \). But \( f \) is not a pseudovaluation on \( X \), since

\[
f(a \oplus b) + f(a) = 1 \not\geq 5 = f(b).
\]

Definition 3.15. A pseudovaluation (or, valuation) \( f \) on \( X \) is said to be positive if \( f(x) \geq 0 \) for all \( x \in X \).

The pseudovaluation \( f \) on \( X \) which is given in Example 3.4 is positive.

Theorem 3.16. If a pseudovaluation \( f \) on \( X \) is positive, then the set

\[
F_f^+ := \{ x \in X \mid f(x) = 0 \}
\]

is a filter of \( X \).
Proof. Clearly, \( \theta \in F_f^x \). Let \( x, y \in X \) be such that \( x \in F_f^x \) and \( x \oplus y \in F_f^x \). Then, \( f(x) = 0 \) and \( f(x \oplus y) = 0 \). Since \( f \) is positive, it follows from Definition 3.1(ii) that
\[
0 \leq f(y) \leq f(x \oplus y) + f(x) = 0,
\]
so that \( f(y) = 0 \), that is, \( y \in F_f^x \). Hence, \( F_f^x \) is a filter of \( X \).

The following example shows that two distinct pseudovaluations induce the same filter.

Example 3.17. Consider a WFI algebra \( X = \{\theta, 1, 2, a, b\} \) which is given in Example 3.14. Let \( g \) and \( h \) be mappings from \( X \) to \( \mathbb{R} \) defined by
\[
g = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0 & 0 & 4 & 3 & 5 \end{pmatrix},
\]
\[
h = \begin{pmatrix} \theta & 1 & 2 & a & b \\ 0 & 0 & 4 & 2 & 3 \end{pmatrix}.
\]
Then, \( g \) and \( h \) are pseudovaluations on \( X \), and \( F_g = \{\theta, 1\} = F_h \).

For a mapping \( f : X \to \mathbb{R} \), define a mapping \( d_f : X \times X \to \mathbb{R} \) by \( d_f(x, y) = f(x \oplus y) + f(y \ominus x) \) for all \( (x, y) \in X \times X \). Note that \( d_f(x, y) \geq 0 \) for all \( (x, y) \in X \times X \).

Theorem 3.18. If \( f : X \to \mathbb{R} \) is a pseudovaluation on \( X \), then \( d_f \) is a pseudometric on \( X \), and so \( (X, d_f) \) is a pseudometric space.

We say \( d_f \) is called the pseudometric induced by pseudovaluation \( f \).

Proof. Let \( x, y, z \in X \). Then, \( d_f(x, y) = f(x \oplus y) + f(y \ominus x) \geq 0 \) by Proposition 3.5(3), and obviously, \( d_f(x, y) = d_f(y, x) \) and \( d_f(x, x) = 0 \). Now,
\[
d_f(x, y) + d_f(y, z) = [f(x \oplus y) + f(y \ominus x)] + [f(y \ominus z) + f(z \oplus y)]
= [f(x \oplus y) + f(y \ominus z)] + [f(z \ominus y) + f(y \oplus x)]
\geq f(x \ominus z) + f(z \ominus x) = d_f(x, z).
\]
Therefore, \( (X, d_f) \) is a pseudometric space.

Proposition 3.19. Every pseudometric \( d_f \) induced by pseudovaluation \( f \) satisfies the following inequalities:
\begin{align*}
(1) \quad d_f(x, y) & \geq d_f(x \ominus a, y \ominus a), \\
(2) \quad d_f(x, y) & \geq d_f(a \ominus x, a \ominus y), \\
(3) \quad d_f(x \ominus y, a \ominus b) & \leq d_f(x \ominus y, a \ominus y) + d_f(a \ominus y, a \ominus b),
\end{align*}
for all \( x, y, a, b \in X \).
Proof. (1) Let $x, y, a \in X$. Since $(x \odot y) \oplus ((y \odot a) \odot (x \odot a)) = \theta$ and $(y \odot x) \oplus ((x \odot a) \odot (y \odot a)) = \theta$, it follows from Proposition 3.5(1) that $f(x \odot y) \geq f((y \odot a) \odot (x \odot a))$ and $f(y \odot x) \geq f((x \odot a) \odot (y \odot a))$ so that

$$d_f(x, y) = f(x \odot y) + f(y \odot x)$$

$$\geq f((y \odot a) \odot (x \odot a)) + f((x \odot a) \odot (y \odot a))$$

$$= d_f(x \odot a, y \odot a). \quad (3.21)$$

(2) It is similar to the proof of (1).

(3) Using Proposition 3.5(2), we have

$$f((x \odot y) \odot (a \odot b)) \leq f((x \odot y) \odot (a \odot y)) + f((a \odot y) \odot (a \odot b)), \quad (3.22)$$

$$f((a \odot b) \odot (x \odot y)) \leq f((a \odot b) \odot (a \odot y)) + f((a \odot y) \odot (x \odot y)),$$

for all $x, y, a, b \in X$. Hence,

$$d_f(x \odot y, a \odot b) = f((x \odot y) \odot (a \odot b)) + f((a \odot b) \odot (x \odot y))$$

$$\leq [f((x \odot y) \odot (a \odot y)) + f((a \odot y) \odot (a \odot b))]$$

$$+ [f((a \odot b) \odot (a \odot y)) + f((a \odot y) \odot (x \odot y))]$$

$$= [f((x \odot y) \odot (a \odot y)) + f((a \odot y) \odot (x \odot y))]$$

$$+ [f((a \odot b) \odot (a \odot y)) + f((a \odot y) \odot (a \odot b))]$$

$$= d_f(x \odot y, a \odot y) + d_f(a \odot y, a \odot b)$$

for all $x, y, a, b \in X$. \hfill $\square$

**Theorem 3.20.** Let $f : X \to \mathbb{R}$ be a pseudovaluation on $X$ such that $F_f = \{ x \in X \mid f(x) \leq 0 \}$ is a closed filter of $X$. If $d_f$ is a metric on $X$, then $f$ is a valuation on $X$.

Proof. Suppose that $f$ is not a valuation on $X$. Then, there exists $x \in X$ such that $x \neq \theta$ and $f(x) = 0$. Thus, $\theta, x \in F_f$ and so $x \odot \theta \in F_f$, since $F_f$ is a closed filter of $X$. It follows that $f(x \odot \theta) \leq 0$ so that

$$0 = f(\theta) \leq f(x \odot \theta) + f(x) = f(x \odot \theta) \leq 0. \quad (3.24)$$

Hence, $f(x \odot \theta) = 0$, and thus $d_f(x, \theta) = f(x \odot \theta) + f(\theta \odot x) = f(x \odot \theta) + f(x) = 0$. Thus, $x = \theta$ since $d_f$ is a metric on $X$. This is a contradiction. Therefore, $f$ is a valuation on $X$. \hfill $\square$
Consider the pseudovaluation \( f \) on \( \mathbb{Z} \) which is described in Example 3.3. If \( a = -1 \), then

\[
f(x) = \begin{cases} 
0 & \text{if } x = \theta, \\
-x + b & \text{otherwise,}
\end{cases}
\]

(3.25)

for all \( x \in \mathbb{Z} \), and \( F_f = \{ x \in \mathbb{Z} | b \leq x \} \cup \{ \theta \} \) which is not a closed filter of \( \mathbb{Z} \). Since \( f \) is a pseudovaluation on \( \mathbb{Z} \), we know that \((\mathbb{Z}, d_f)\) is a pseudometric space by Theorem 3.18. If \( x \neq y \in \mathbb{Z} \), then

\[
d_f(x, y) = f(x \circ y) + f(y \circ x) = f(y - x) + f(x - y)
= -y + x + b - x + y + b = 2b \neq 0.
\]

Hence, \((\mathbb{Z}, d_f)\) is a metric space. But \( f(b) = 0 \), and so, \( f \) is not a valuation on \( \mathbb{Z} \). This shows that Theorem 3.20 may not be true when \( F_f \) is not a closed filter of \( X \).

**Theorem 3.21.** For a mapping \( f : X \to \mathbb{R} \), if \( d_f \) is a pseudometric on \( X \), then \((X \times X, d_f^*)\) is a pseudometric space, where

\[
d_f^*((x, y), (a, b)) = \max\{d_f(x, a), d_f(y, b)\}
\]

(3.27)

for all \((x, y), (a, b) \in X \times X\).

**Proof.** Suppose \( d_f \) is a pseudometric on \( X \). For any \((x, y), (a, b) \in X \times X\), we have

\[
d_f^*((x, y), (x, y)) = \max\{d_f(x, x), d_f(y, y)\} = 0,
\]

\[
d_f^*((x, y), (a, b)) = \max\{d_f(x, a), d_f(y, b)\}
= \max\{d_f(a, x), d_f(b, y)\}
= d_f^*((a, b), (x, y)).
\]

(3.28)

Now, let \((x, y), (a, b), (u, v) \in X \times X\). Then,

\[
d_f^*((x, y), (u, v)) + d_f^*((u, v), (a, b)) = \max\{d_f(x, u), d_f(y, v)\} + \max\{d_f(u, a), d_f(v, b)\}
\geq \max\{d_f(x, u) + d_f(u, a), d_f(y, v) + d_f(v, b)\}
\geq \max\{d_f(x, a), d_f(y, b)\}
= d_f^*((x, y), (a, b)).
\]

(3.29)

Therefore, \((X \times X, d_f^*)\) is a pseudometric space. \(\square\)
Corollary 3.22. If \( f : X \to \mathbb{R} \) is a pseudovaluation on \( X \), then \((X \times X, d_f^*)\) is a pseudometric space.

It is natural to ask that if \( f : X \to \mathbb{R} \) is a valuation on \( X \), then \((X, d_f)\) a metric space. But, we see that it is incorrect in the following example.

Example 3.23. For a WFI algebra \((\mathbb{Z}, \oplus, \theta)\), a mapping \( f : \mathbb{Z} \to \mathbb{R} \) defined by \( f(x) = (1/2)x \) for all \( x \in \mathbb{Z} \) is a valuation on \( \mathbb{Z} \). Then, \( d_f \) is a pseudometric on \( \mathbb{Z} \). Note that \( d_f(-2, 3) = f(-2 \oplus 3) + f(3 \oplus (-2)) = 0 \), but \(-2 \neq 3\). Hence, \((X, d_f)\) is not a metric space.

Theorem 3.24. If \( f : X \to \mathbb{R} \) is a positive valuation on \( X \), then \((X, d_f)\) is a metric space.

Proof. Suppose that \( f \) is a positive valuation on \( X \). Then, \((X, d_f)\) is a pseudometric space by Theorem 3.18. Let \( x, y \in X \) be such that \( d_f(x, y) = 0 \). Then, \( 0 = d_f(x, y) = f(x \oplus y) + f(y \ominus x) \), and so \( f(x \oplus y) = 0 \) and \( f(y \ominus x) = 0 \), since \( f \) is positive. Also, since \( f \) is a valuation on \( X \), it follows that \( x \ominus y = \theta \) and \( y \ominus x = \theta \) so from (a2) that \( x = y \). Therefore, \((X, d_f)\) is a metric space. \( \square \)

Corollary 3.25. If \( f : X \to \mathbb{R} \) is a valuation on \( X \) such that \( F_f = \{\theta\} \), then \((X, d_f)\) is a metric space.

Theorem 3.26. If \( f : X \to \mathbb{R} \) is a positive valuation on \( X \), then \((X \times X, d_f^*)\) is a metric space.

Proof. Note from Corollary 3.22 that \((X \times X, d_f^*)\) is a pseudometric space. Let \((x, y), (a, b) \in X \times X \) be such that \( d_f^*((x, y), (a, b)) = 0 \). Then,

\[
0 = d_f^*((x, y), (a, b)) = \max\{d_f(x, a), d_f(y, b)\},
\]

and so \( d_f(x, a) = 0 = d_f(y, b) \), since \( d_f(x, y) \geq 0 \) for all \((x, y) \in X \times X \). Hence,

\[
0 = d_f(x, a) = f(x \oplus a) + f(a \ominus x),
\]

\[
0 = d_f(y, b) = f(y \ominus b) + f(b \oplus y).
\]

Since \( f \) is positive, it follows that \( f(x \ominus a) = 0 = f(a \ominus x) \) and \( f(y \ominus b) = 0 = f(b \oplus y) \) so that \( x \ominus a = \theta = a \ominus x \) and \( y \ominus b = \theta = b \oplus y \). Using (a2), we have \( a = x \) and \( b = y \), and so \((x, y) = (a, b) \). Therefore, \((X \times X, d_f^*)\) is a metric space. \( \square \)

Theorem 3.27. If \( f \) is a positive valuation on \( X \), then the operation \( \ominus : X \times X \to X \) is uniformly continuous. (Suppose that \((X, d)\) and \((Y, \rho)\) are metric spaces and \( f : X \to Y \). We say that \( f \) is uniformly continuous provided that for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any points \( x_1 \) and \( x_2 \) in \( X \), if \( d(x_1, x_2) < \delta \), then \( \rho(f(x_1), f(x_2)) < \varepsilon \).)

Proof. For any \( \varepsilon > 0 \), if \( d_f^*((x, y), (a, b)) < \varepsilon /2 \), then \( d_f(x, a) < \varepsilon /2 \), and \( d_f(y, b) < \varepsilon /2 \). Using Proposition 3.19, we have

\[
d_f(x \ominus y, a \ominus b) \leq d_f(x \ominus y, a \ominus y) + d_f(a \ominus y, a \ominus b) \leq d_f(x, a) + d_f(y, b) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Therefore, the operation \( \ominus : X \times X \to X \) is uniformly continuous. \( \square \)
Corollary 3.28. If $f$ is a valuation on $X$ such that $F_f = \{\theta\}$, then the operation $\ominus : X \times X \to X$ is uniformly continuous.

References

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