Research Article

Identities on the Bernoulli and Genocchi Numbers and Polynomials

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Received 9 June 2012; Accepted 9 August 2012

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$. The $p$-adic norm on $\mathbb{C}_p$ is normalized so that $|p|_p = p^{-1}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x$$

(1.1)

(see [1–16]). From (1.1), we have

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0)$$

(1.2)

(see [1–16]), where $f_1(x) = f(x + 1)$.
Let us take \( f(x) = e^{xt} \). Then, by (1.2), we get

\[
\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} \frac{G_n t^n}{n!},
\]

(1.3)

where \( G_n \) are the \( n \)th ordinary Genocchi numbers (see [8, 15]).

From the same method of (1.3), we can also derive the following equation:

\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},
\]

(1.4)

where \( G_n(x) \) are called the \( n \)th Genocchi polynomials (see [14, 15]).

By (1.3), we easily see that

\[
G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}
\]

(1.5)

(see [15]). By (1.3) and (1.4), we get Witt’s formula for the \( n \)th Genocchi numbers and polynomials as follows:

\[
\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \quad \text{for } n \in \mathbb{Z}_+.
\]

(1.6)

From (1.2), we have

\[
\int_{\mathbb{Z}_p} (x+1)^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = 2\delta_{0,n},
\]

(1.7)

where the symbol \( \delta_{0,n} \) is the Kronecker symbol (see [4, 5]).

Thus, by (1.5) and (1.7), we get

\[
(G+1)^n + G_n = 2\delta_{1,n}
\]

(1.8)

(see [15]). From (1.4), we can derive the following equation:

\[
\int_{\mathbb{Z}_p} (1-x+y)^n d\mu_{-1}(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y).
\]

(1.9)

By (1.6) and (1.9), we see that

\[
\frac{G_{n+1}(1-x)}{n+1} = (-1)^n \frac{G_{n+1}(x)}{n+1}.
\]

(1.10)

Thus, by (1.10), we get \( G_{n+1}(2)/(n+1) = (-1)^n(G_{n+1}(-1)/(n+1)). \)
From (1.5) and (1.8), we have

\[
\frac{G_{n+1}(2)}{n+1} = 2 - \frac{G_{n+1}(1)}{n+1} = 2 + \frac{G_{n+1}}{n+1} - 2\delta_{1,n+1}. \tag{1.11}
\]

The Bernoulli polynomials \(B_n(x)\) are defined by

\[
\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.12}
\]

(see [6, 9, 12]) with the usual convention about replacing \(B''(x)\) by \(B_n(x)\).

In the special case, \(x = 0\), \(B_n(0) = B_n\) is called the \(n\)-th Bernoulli number. By (1.12), we easily see that

\[
B_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l = (B + x)^n \tag{1.13}
\]

(see [6]). Thus, by (1.12) and (1.13), we get reflection symmetric formula for the Bernoulli polynomials as follows:

\[
B_n(1 - x) = (-1)^n B_n(x), \tag{1.14}
\]

\[
B_0 = 1, \quad (B + 1)^n - B_n = \delta_{1,n} \tag{1.15}
\]

(see [6, 9, 12]). From (1.14) and (1.15), we can also derive the following identity:

\[
(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n}. \tag{1.16}
\]

In this paper, we investigate some properties of the fermionic \(p\)-adic integrals on \(\mathbb{Z}_p\). By using these properties, we give some new identities on the Bernoulli and the Euler numbers which are useful in studying combinatorics.

## 2. Identities on the Bernoulli and Genocchi Numbers and Polynomials

Let us consider the following fermionic \(p\)-adic integral on \(\mathbb{Z}_p\) as follows:

\[
I_1 = \int_{\mathbb{Z}_p} B_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{n+1}}{l+1}, \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2.1}
\]
On the other hand, by (1.14) and (1.15), we get
\[ I_1 = (-1)^n \int_{\mathbb{Z}_p} B_n(1-x) d\mu_{-1}(x) \]
\[ = (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \]
\[ = (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1} \]
\[ = (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \left( 2 + \frac{G_{l+1}}{l+1} - 2\delta_{l,1} \right) \]
\[ = 2(-1)^n (B_n + \delta_{1,n}) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} + 2(-1)^{n+1} B_n. \]

Equating (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** For \( n \in \mathbb{Z}_+ \), one has
\[ \left( 1 + (-1)^{n+1} \right) \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} = 2(-1)^n \delta_{1,n}. \]  \hspace{1cm} (2.3)

By using the reflection symmetric property for the Euler polynomials, we can also obtain some interesting identities on the Euler numbers.

Now, we consider the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) for the polynomials as follows:
\[ I_2 = \int_{\mathbb{Z}_p} G_n(x) d\mu_{-1}(x) \]
\[ = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \]
\[ = \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}, \quad \text{for} \ n \in \mathbb{Z}_+. \] \hspace{1cm} (2.4)

On the other hand, by (1.8), (1.10), and (1.11), we get
\[ I_2 = (-1)^{n-1} \int_{\mathbb{Z}_p} B_n(1-x) d\mu_{-1}(x) \]
\[ = (-1)^{n-1} \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \]
\[ = (-1)^{n-1} \sum_{l=0}^{n} \binom{n}{l} G_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1} \]
Equating (2.4) and (2.5), we obtain the following theorem.

**Theorem 2.2.** For \( n \in \mathbb{Z}_+ \), one has

\[
(1 + (-1)^n) \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1} = 4(-1)^n G_n + 4(-1)^{n+1} \delta_{1,n}.
\]  

Let us consider the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) for the product of \( B_n(x) \) and \( G_n(x) \) as follows:

\[
I_3 = \int_{\mathbb{Z}_p} B_n(x) G_n(x) d\mu_{-1}(x) \\
= \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-1}(x) \\
= \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1}.
\]  

On the other hand, by (1.10) and (1.14), we get

\[
I_3 = \int_{\mathbb{Z}_p} B_n(x) G_n(x) d\mu_{-1}(x) \\
= (-1)^{n+m-1} \int_{\mathbb{Z}_p} B_m(1-x) G_n(1-x) d\mu_{-1}(x) \\
= (-1)^{n+m-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \int_{\mathbb{Z}_p} (1-x)^{k+l} d\mu_{-1}(x) \\
= 2(-1)^{n+m-1} B_m(1) G_n(1) + 2(-1)^{n+m} B_m G_n \\
+ (-1)^{n+m-1} \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1}.
\]
By (2.7) and (2.8), we easily see that

\[
\left(1 + (-1)^{n+m+1}\right) \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{k+l+1} \\
= 2(-1)^{m+n-1} (\delta_{1,m} + B_m)(2\delta_{1,n} - G_n) + 2(-1)^{m+n} B_m G_n \\
= 4(-1)^{m+n-1} B_m \delta_{1,n} + 2(-1)^{m+n} B_m G_n + 4(-1)^{m+n-1} \delta_{1,m} \delta_{1,n} \\
+ 2(-1)^{m+n} \delta_{1,m} G_n + 2(-1)^{m+n} B_m G_n.
\]

Therefore, by (2.9), we obtain the following theorem.

**Theorem 2.3.** For \( n, m \in \mathbb{Z}_+ \), one has

\[
\left(1 + (-1)^{n+m+1}\right) \sum_{k=0}^{m} \sum_{l=0}^{n} \binom{m}{k} \binom{n}{l} B_{m-k} G_{n-l} \frac{G_{k+l+1}}{n-l+1 k+l+1} \\
= 4(-1)^{m+n} B_m G_n + 4(-1)^{m+n-1} B_m \delta_{1,n} + 4(-1)^{m+n-1} \delta_{1,m} \delta_{1,n} \\
+ 2(-1)^{m+n} \delta_{1,m} G_n.
\]

**Corollary 2.4.** For \( n, m \in \mathbb{N} \), one has

\[
\sum_{k=0}^{2m} \sum_{l=0}^{2n} \binom{2m}{k} \binom{2n}{l} B_{2m-k} G_{2n-l} \frac{G_{k+l+1}}{k+l+1} = 2B_{2m} G_{2n}.
\]

Let us consider the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) for the product of the Bernoulli polynomials and the Bernstein polynomials. For \( n, k \in \mathbb{Z}_+ \), with \( 0 \leq k \leq n \), \( B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \) are called the Bernstein polynomials of degree \( n \), see [11]. It is easy to show that \( B_{k,n}(x) = B_{n-k,n}(1-x) \),

\[
I_n = \int_{\mathbb{Z}_p} B_{m}(x) B_{k,n}(x) d\mu_{-1}(x) \\
= \binom{n}{k} \sum_{l=0}^{m} \binom{m}{l} B_{m-l} \int_{\mathbb{Z}_p} x^k (1-x)^{n-k} d\mu_{-1}(x) \\
= \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \int_{\mathbb{Z}_p} x^k x^j d\mu_{-1}(x) \\
= \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1}.
\]
On the other hand, by (1.14) and (2.12), we get

\[ I_4 = (-1)^m \int_{\mathbb{Z}_p} B_m(1-x)B_{n-k,n}(1-x)d\mu_{-1}(x) \]

\[ = (-1)^m \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \int_{\mathbb{Z}_p} (1-x)^{n-k+l+j} d\mu_{-1}(x) \]

\[ = (-1)^m \binom{n}{k} \sum_{j=0}^{k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \times \left( 2 - 2\delta_{1,n-k+l+j+1} \frac{G_{n-k+l+j+1}}{n-k+l+j+1} \right) \]

\[ = 2(-1)^m \binom{n}{k} B_m(1)\delta_{0,k} + 2(-1)^{m+1} \binom{n}{k} B_m\delta_{k,n} \]

\[ + (-1)^m \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}. \]

Equating (2.12) and (2.13), we see that

\[ \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{n-k}{j} (-1)^j B_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1} \]

\[ = 2(-1)^m B_m(1)\delta_{0,k} + 2(-1)^{m+1} B_m\delta_{k,n} \]

\[ + (-1)^m \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}. \]

Thus, from (2.14), we obtain the following theorem.

**Theorem 2.5.** For \( n, m \in \mathbb{N} \), one has

\[ \sum_{l=0}^{2m} \sum_{j=0}^{m} \binom{2m}{l} \binom{n}{j} (-1)^l B_{2m-l} \frac{G_{l+j+1}}{l+j+1} = 2B_{2m}(1) + \sum_{l=0}^{2m} \binom{2m}{l} B_{2m-l} \frac{G_{n+l+1}}{n+l+1}. \]

Finally, we consider the fermionic \( p \)-adic integral on \( \mathbb{Z}_p \) for the product of the Euler polynomials and the Bernstein polynomials as follows:

\[ I_5 = \int_{\mathbb{Z}_p} G_m(x)B_{k,n}(x)d\mu_{-1}(x) \]

\[ = \binom{n}{k} \sum_{l=0}^{m} \binom{m}{l} G_{m-l} \int_{\mathbb{Z}_p} x^{k+l}(1-x)^{n-k} d\mu_{-1}(x) \]
\[
\begin{align*}
&\sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^l G_{m-l} \int_{z_p} x^{k+l+j} d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^l G_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1}.
\end{align*}
\] (2.16)

On the other hand, by (1.10) and (2.12), we get

\[
I_5 = (-1)^{m-1} \int_{z_p} G_m(1-x) B_{n-k,n}(1-x) d\mu_{-1}(x)
\]

\[
= (-1)^{m-1} \binom{n}{k} \sum_{l=0}^{m} \binom{m}{l} \frac{G_m}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_{z_p} (1-x)^{n-k+l+j} d\mu_{-1}(x)
\]

\[
= (-1)^{m-1} \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l}
\]

\[
\times \left( 2 + \frac{G_{n-k+l+j+1}}{n-k+l+j+1} - 2 \delta_{1,n-k+l+j+1} \right)
\]

\[
= 2(-1)^{m-1} \binom{n}{k} G_m(1) \delta_{0,k} + 2(-1)^m \binom{n}{k} G_m \delta_{k,n}
\]

\[
+ (-1)^{m-1} \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\] (2.17)

Equating (2.16) and (2.17), we obtain

\[
\sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l} \binom{n-k}{j} (-1)^l G_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1} = 2(-1)^{m-1} G_m(1) \delta_{0,k} + 2(-1)^m G_m \delta_{k,n}
\]

\[
+ (-1)^{m-1} \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l} \binom{k}{j} (-1)^j G_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\] (2.18)

Therefore, by (2.18), we obtain the following theorem.

**Theorem 2.6.** For \( n, m \in \mathbb{N} \), one has

\[
\sum_{l=0}^{2m} \sum_{j=0}^{n} \binom{2m}{l} \binom{n}{j} (-1)^l G_{2m-l} \frac{G_{l+j+1}}{l+j+1} = -2G_{2m}(1) - \sum_{l=0}^{2m} \binom{2m}{l} G_{2m-l} \frac{G_{n+l+1}}{n+l+1}.
\] (2.19)
Acknowledgment

This paper was supported by Kynugpook National University Research Fund, 2012.

References

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