Lightlike Hypersurfaces of a Semi-Riemannian Product Manifold and Quarter-Symmetric Nonmetric Connections

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We study lightlike hypersurfaces of a semi-Riemannian product manifold. We introduce a class of lightlike hypersurfaces called screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces. We consider lightlike hypersurfaces with respect to a quarter-symmetric nonmetric connection which is determined by the product structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifolds and the quarter-symmetric nonmetric connection, and we obtain some results.

1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of important topics of differential geometry. The geometry of lightlike submanifolds of a semi-Riemannian manifold, was presented in [1] (see also [2, 3]) by Duggal and Bejancu. In [4], Atçeken and Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [5], Kılıç and Şahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold and gave some examples and results for lightlike submanifolds. The lightlike hypersurfaces have been studied by many authors in various spaces (for example [6, 7]).

In [8], Hayden introduced a metric connection with nonzero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semisymmetric (symmetric) and nonmetric connection have been studied by many authors [9–14]. In [15], Yaşar et al. have studied lightlike hypersurfaces in semi-Riemannian manifolds with semisymmetric nonmetric connection. The idea of quarter-symmetric linear connections in a differential
A linear connection is said to be a quarter-symmetric connection if its torsion tensor $T$ is of the form:

$$T(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \quad (1.1)$$

for any vector fields $X, Y$ on a manifold, where $u$ is a 1-form and $\varphi$ is a tensor of type $(1,1)$. In this paper, we study lightlike hypersurfaces of a semi-Riemannian product manifold. As a first step, in Section 3, we introduce screen semi-invariant lightlike hypersurfaces and radical anti-invariant lightlike hypersurfaces of a semi-Riemannian product manifold. We give some examples and study their geometric properties. In Section 4, we consider lightlike hypersurfaces of a semi-Riemannian product manifold with quarter-symmetric nonmetric connection determined by the product structure. We compute the Riemannian curvature tensor with respect to the quarter-symmetric nonmetric connection and give some results.

2. Lightlike Hypersurfaces

Let $(\overline{M}, \overline{g})$ be an $(m+2)$-dimensional semi-Riemannian manifold with index $(\overline{g}) = q \geq 1$ and let $(M, g)$ be a hypersurface of $\overline{M}$, with $g = \overline{g}|_M$. If the induced metric $g$ on $M$ is degenerate, then $M$ is called a lightlike (null or degenerate) hypersurface [1] (see also [2, 3]). Then there exists a null vector field $\xi \neq 0$ on $M$ such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(TM). \quad (2.1)$$

The radical or the null space of $T_xM$, at each point $x \in M$, is a subspace $\text{Rad} T_xM$ defined by

$$\text{Rad} T_xM = \{ \xi \in T_xM | g_x(\xi, X) = 0, \forall X \in \Gamma(TM) \}, \quad (2.2)$$

whose dimension is called the nullity degree of $g$. We recall that the nullity degree of $g$ for a lightlike hypersurface of $\overline{M}$ is 1. Since $g$ is degenerate and any null vector being perpendicular to itself, $T_xM$ is also null and

$$\text{Rad} T_xM = T_xM \cap T_xM^\perp. \quad (2.3)$$

Since $\dim T_xM^\perp = 1$ and $\dim \text{Rad} T_xM = 1$, we have $\text{Rad} T_xM = T_xM^\perp$. We call $\text{Rad} TM$ a radical distribution and it is spanned by the null vector field $\xi$. The complementary vector bundle $S(TM)$ of $\text{Rad} TM$ in $TM$ is called the screen bundle of $M$. We note that any screen bundle is nondegenerate. This means that

$$TM = \text{Rad} TM \perp S(TM). \quad (2.4)$$
Here $\perp$ denotes the orthogonal-direct sum. The complementary vector bundle $S(TM)\perp$ of $S(TM)$ in $T\overline{M}$ is called screen transversal bundle and it has rank 2. Since $\text{Rad} \ TM$ is a lightlike subbundle of $S(TM)\perp$ there exists a unique local section $N$ of $S(TM)\perp$ such that

$$\overline{g}(N, N) = 0, \quad \overline{g}(\xi, N) = 1.$$  \hfill (2.5)

Note that $N$ is transversal to $M$ and $\{\xi, N\}$ is a local frame field of $S(TM)\perp$ and there exists a line subbundle $\text{ltr}(TM)$ of $T\overline{M}$, and it is called the lightlike transversal bundle, locally spanned by $N$. Hence we have the following decomposition:

$$T\overline{M} = TM \oplus \text{ltr}(TM) = S(TM) \perp \text{Rad} \ TM \oplus \text{ltr}(TM),$$  \hfill (2.6)

where $\oplus$ is the direct sum but not orthogonal [1, 3]. From the above decomposition of a semi-Riemannian manifold $\overline{M}$ along a lightlike hypersurface $M$, we can consider the following local quasiothronormal field of frames of $\overline{M}$ along $M$:

$$\{X_1, \ldots, X_m, \xi, N\},$$  \hfill (2.7)

where $\{X_1, \ldots, X_m\}$ is an orthonormal basis of $\Gamma(S(TM))$. According to the splitting (2.6), we have the following Gauss and Weingarten formulas, respectively:

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X N = -A_N X + \nabla^t_X N,$$  \hfill (2.8)

for any $X, Y \in \Gamma(TM)$, where $\nabla_X Y, A_N X \in \Gamma(TM)$ and $h(X, Y), \nabla^t_X N \in \Gamma(\text{ltr}(TM))$. If we set $B(X, Y) = \overline{g}(h(X, Y), \xi)$ and $\tau(X) = \overline{g}(\nabla^t_X N, \xi)$, then (2.8) become

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) N,$$  \hfill (2.9)

$$\overline{\nabla}_X N = -A_N X + \tau(X) N.$$  \hfill (2.10)

$B$ and $A$ are called the second fundamental form and the shape operator of the lightlike hypersurface $M$, respectively [1]. Let $P$ be the projection of $S(TM)$ on $M$. Then, for any $X \in \Gamma(TM)$, we can write

$$X = PX + \eta(X)\xi,$$  \hfill (2.11)

where $\eta$ is a 1-form given by

$$\eta(X) = \overline{g}(X, N).$$  \hfill (2.12)

From (2.9), we get

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \quad \forall X, Y, Z \in \Gamma(TM),$$  \hfill (2.13)
and the induced connection $\nabla$ is a nonmetric connection on $M$. From (2.4), we have

$$\nabla_X W = \nabla^*_X W + h^*(X, W)$$

$$= \nabla^*_X W + C(X, W)\xi, \quad X \in \Gamma(TM), W \in \Gamma(S(TM)), \quad (2.14)$$

$$\nabla_X \xi = - A^*_X X - \tau(X) \xi,$$

where $\nabla^*_X W$ and $A^*_X X$ belong to $\Gamma(S(TM))$. $C$, $A^*_X X$ and $\nabla^*$ are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$, respectively. Also, we have the following identities:

$$g\left(A^*_X X, W\right) = B(X, W), \quad g\left(A^*_X X, N\right) = 0,$$

$$B(X, \xi) = 0, \quad g(A_N X, N) = 0. \quad (2.15)$$

Moreover, from the first and third equations of (2.15) we have

$$A^*_X \xi = 0. \quad (2.16)$$

Now, we will denote $\overline{R}$ and $R$ the curvature tensors of the Levi-Civita connection $\nabla$ on $\overline{M}$ and the induced connection $\nabla$ on $M$. Then the Gauss equation of $M$ is given by

$$\overline{R}(X, Y) Z = R(X, Y) Z + A_{h(X,Z)} Y - A_{h(Y,Z)} X$$

$$+ (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \quad \forall X, Y, Z \in \Gamma(TM), \quad (2.17)$$

where $(\nabla_X h)(Y, Z) = \nabla^*_X (h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. Then the Gauss-Codazzi equations of a lightlike hypersurface are given by

$$\overline{g}\left(\overline{R}(X, Y) Z, PW\right) = g(R(X, Y) Z, PW)$$

$$+ B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW),$$

$$\overline{g}\left(\overline{R}(X, Y) Z, \xi\right) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z)$$

$$+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y), \quad (2.18)$$

$$\overline{g}\left(\overline{R}(X, Y) Z, N\right) = g(R(X, Y) Z, N),$$

$$\overline{g}\left(\overline{R}(X, Y) \xi, N\right) = g(R(X, Y) \xi, N)$$

$$= C\left(Y, A^*_X X\right) - C\left(X, A^*_Y Y\right) - 2d\tau(X, Y),$$

for any $X, Y, Z, W \in \Gamma(TM), \xi \in \Gamma(Rad TM)$.

For geometries of lightlike submanifolds, hypersurfaces and curves, we refer to [1–3].
2.1. Product Manifolds

Let $\bar{M}$ be an $n$-dimensional differentiable manifold with a tensor field $F$ of type (1,1) on $\bar{M}$ such that

$$F^2 = I.$$  \hfill (2.19)

Then $\bar{M}$ is called an almost product manifold with almost product structure $F$. If we put

$$\pi = \frac{1}{2}(I + F), \quad \sigma = \frac{1}{2}(I - F),$$ \hfill (2.20)

then we have

$$\pi + \sigma = I, \quad \pi^2 = \pi, \quad \sigma^2 = \sigma, \quad \sigma \pi = \pi \sigma = 0, \quad F = \pi - \sigma.$$ \hfill (2.21)

Thus $\pi$ and $\sigma$ define two complementary distributions and $F$ has the eigenvalue of $+1$ or $-1$. If an almost product manifold $\bar{M}$ admits a semi-Riemannian metric $\bar{g}$ such that

$$\bar{g}(FX, FY) = \bar{g}(X, Y),$$ \hfill (2.22)

for any vector fields $X, Y$ on $\bar{M}$, then $\bar{M}$ is called a semi-Riemannian almost product manifold. From (2.19) and (2.22), we have

$$\bar{g}(FX, Y) = \bar{g}(X, FY).$$ \hfill (2.23)

If, for any vector fields $X, Y$ on $\bar{M}$,

$$\nabla F = 0,$$ \hfill (2.24)

that is $\nabla_X FY = F\nabla_X Y$,

then $\bar{M}$ is called a semi-Riemannian product manifold, where $\nabla$ is the Levi-Civita connection on $\bar{M}$.

3. Lightlike Hypersurfaces of Semi-Riemannian Product Manifolds

Let $M$ be a lightlike hypersurface of a semi-Riemannian product manifold $(\bar{M}, \bar{g})$. For any $X \in \Gamma(TM)$ we can write

$$FX = fX + w(X)N,$$ \hfill (3.1)

where $f$ is a (1,1) tensor field and $w$ is a 1-form on $M$ given by $w(X) = \bar{g}(FX, \xi) = \bar{g}(X, F\xi)$. 
Definition 3.1. Let $M$ be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$:

(i) if $F \text{Rad} \ TM \subset S(\text{TM})$ and $F \text{litr} \ TM \subset S(\text{TM})$ then we say that $M$ is a screen semi-invariant lightlike hypersurface;

(ii) if $FS(\text{TM}) = S(\text{TM})$ then we say that $M$ is a screen invariant lightlike hypersurface;

(iii) if $F \text{Rad} \ TM = \text{litr} \ (\text{TM})$ then we say that $M$ is a radical anti-invariant lightlike hypersurface.

We note that a radical anti-invariant lightlike hypersurface is a screen invariant lightlike hypersurface.

Remark 3.2. We recall that there are some lightlike hypersurfaces of a semi-Riemannian product manifold which differ from the above definition, that is, this definition does not cover all lightlike hypersurfaces of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$. In this paper we will study the hypersurfaces determined above.

Now, let $M$ be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold. If we set $D_1 = F \text{Rad} \ TM, D_2 = F \text{litr} \ TM$ then we can write

$$S(\text{TM}) = D \perp \{D_1 \oplus D_2\},$$

(3.2)

where $D$ is a $(m-2)$-dimensional distribution. Hence we have the following decomposition:

$$T \text{M} = D \perp \{D_1 \oplus D_2\} \perp \text{Rad} \ TM,$$

$$T \overline{\text{M}} = D \perp \{D_1 \oplus D_2\} \perp \{\text{Rad} \ TM \oplus \text{litr}(TM)\}.$$  

(3.3)

Proposition 3.3. The distribution $D$ is an invariant distribution with respect to $F$.

Proof. For any $X \in \Gamma(D)$ and $U \in \Gamma(D_1), V \in \Gamma(D_2)$ we obtain

$$g(FX, U) = g(X, FU) = 0,$$

$$g(FX, V) = g(X, FV) = 0.$$  

(3.4)

Thus there are no components of $FX$ in $D_1$ and $D_2$. Furthermore, we have

$$g(FX, \xi) = g(X, F\xi) = 0,$$

$$g(FX, N) = g(X, FN) = 0.$$  

(3.5)

Proof is completed. $\square$
If we set $D = D \perp \text{Rad } TM \perp F \text{ Rad } TM$, we can write

$$TM = \overline{D} \oplus D_2. \quad (3.6)$$

From the above proposition we have the following corollary.

**Corollary 3.4.** The distribution $\overline{D}$ is invariant with respect to $F$.

**Example 3.5.** Let $(\overline{M} = R^5_2, \overline{g})$ be a 5-dimensional semi-Euclidean space with signature $(-,+,−,+,+)$ and $(x, y, z, s, t)$ be the standard coordinate system of $R^5_2$. If we set $F(x, y, z, s, t) = (x, y, z, -s, -t)$, then $F^2 = I$ and $F$ is a product structure on $R^5_2$. Consider a hypersurface $M$ in $\overline{M}$ by the equation:

$$t = x + y + z. \quad (3.7)$$

Then $TM = \text{Span}\{U_1, U_2, U_3, U_4\}$, where

$$U_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad U_2 = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \quad U_3 = \frac{\partial}{\partial z} + \frac{\partial}{\partial t}, \quad U_4 = \frac{\partial}{\partial s}. \quad (3.8)$$

It is easy to check that $M$ is a lightlike hypersurface and

$$TM^\perp = \text{Span}\{\xi = U_1 - U_2 + U_3\}. \quad (3.9)$$

Then take a lightlike transversal vector bundle as follow:

$$\text{ltr}(TM) = \text{Span}\left\{ N = -\frac{1}{4} \left\{ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right\} \right\}. \quad (3.10)$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = U_4, W_2 = U_1 - U_2 - U_3, W_3 = U_1 + U_2 - U_3\}. \quad (3.11)$$

If we set $\mathbb{D} = \text{Span}\{W_1\}$, $\mathbb{D}_1 = \text{Span}\{W_2\}$ and $\mathbb{D}_2 = \text{Span}\{W_3\}$, then it can be easily checked that $M$ is a screen semi-invariant lightlike hypersurface of $\overline{M}$.

**Example 3.6.** Let $(x, y, z, t)$ be the standard coordinate system of $R^4$ and $ds^2 = -dx^2 - dy^2 + dz^2 + dt^2$ be a semi-Riemannian metric on $R^4$ with 2-index. Let $F$ be a product structure on $R^4$ given
by \( F(x, y, z, t) = (z, t, x, y) \). We consider the hypersurface \( M \) given by \( t = x + (1/2)(y + z)^2 \) [1]. One can easily see that \( M \) is a lightlike hypersurface and

\[
\text{Rad } TM = \text{Span} \left\{ \xi = \frac{\partial}{\partial x} + (y + z) \frac{\partial}{\partial y} - (y + z) \frac{\partial}{\partial z} + \frac{\partial}{\partial t} \right\},
\]

\[
\text{litr}(TM) = \text{Span} \left\{ N = -\frac{1}{2(1 + (y + z)^2)} \left( \frac{\partial}{\partial x} + (y + z) \frac{\partial}{\partial y} + (y + z) \frac{\partial}{\partial z} - \frac{\partial}{\partial t} \right) \right\},
\]

\[
S(TM) = \text{Span} \left\{ W_1 = -(y + z) \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, W_2 = \frac{\partial}{\partial z} + (y + z) \frac{\partial}{\partial t} \right\}.
\]

We can easily check that

\[
F_\xi = W_1 + W_2, \quad FN = \frac{1}{2(1 + (y + z)^2)} [W_1 - W_2].
\]

Thus \( M \) is a screen semi-invariant lightlike hypersurface with \( \mathbb{D} = \{0\} \), \( \mathbb{D}_1 = \text{Span}\{F_\xi\} \) and \( \mathbb{D}_2 = \text{Span}\{FN\} \).

**Example 3.7.** Let \( (R^4_2, g) \) be a 4-dimensional semi-Euclidean space with signature \((-,-,+,+)\) and \((x_1, x_2, x_3, x_4)\) be the standard coordinate system of \( R^4_2 \). Consider a Monge hypersurface \( M \) of \( R^4_2 \) given by

\[
x_4 = Ax_1 + Bx_2 + Cx_3, \quad A^2 + B^2 - C^2 = 1, \quad A, B, C \in R.
\]

Then the tangent bundle \( TM \) of the hypersurface \( M \) is spanned by

\[
\left\{ U_1 = \frac{\partial}{\partial x_1} + A \frac{\partial}{\partial x_4}, U_2 = \frac{\partial}{\partial x_2} + B \frac{\partial}{\partial x_4}, U_3 = \frac{\partial}{\partial x_3} + C \frac{\partial}{\partial x_4} \right\}.
\]

It is easy to check that \( M \) is a lightlike hypersurface (p.196, Ex.1, [3]) whose radical distribution \( \text{Rad } TM \) is spanned by

\[
\xi = AU_1 + BU_2 - CU_3 = A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial x_2} - C \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}.
\]

Furthermore, the lightlike transversal vector bundle is given by

\[
\text{litr}(TM) = \text{Span} \left\{ N = -\frac{1}{2(C^2 + 1)} \left( A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial x_2} + C \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right) \right\}.
\]

It follows that the corresponding screen distribution \( S(TM) \) is spanned by

\[
\left\{ W_1 = \frac{1}{A^2 + B^2} \left( B \frac{\partial}{\partial x_1} - A \frac{\partial}{\partial x_2} \right), W_2 = \frac{1}{A^2 + B^2} \left( \frac{\partial}{\partial x_3} + C \frac{\partial}{\partial x_4} \right) \right\}.
\]
If we define a mapping $F$ by $F(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$ then $F^2 = I$ and $F$ is a product structure on $R^4$. One can easily check that $F\mathcal{S}(TM) = \mathcal{S}(TM)$ and $F\text{ Rad } TM = \text{ ltr}(TM)$. Thus $M$ is a radical anti-invariant lightlike hypersurface of $R^4$. Furthermore, this lightlike hypersurface is a screen invariant lightlike hypersurface.

**Theorem 3.8.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and $M$ be a screen semi-invariant lightlike hypersurface of $\overline{M}$. Then the following assertions are equivalent.

(i) The distribution $\overline{D}$ is integrable with respect to the induced connection $\overline{\nabla}$ of $M$.

(ii) $g(A_x^* P, Y) = g(A_y^* P, X)$, for any $X, Y \in \Gamma(\overline{D})$.

(iii) $g(A_x^* X, P Y) = g(A_y^* Y, P X)$, for any $X, Y \in \Gamma(\overline{D})$.

**Proof.** For any $X, Y \in \Gamma(\overline{D})$, from (2.9), (2.24), and (3.1), we obtain

$$f \nabla_X Y + w(\nabla_X Y) N + B(X, Y)FN = \nabla_X f Y + B(X, f Y)N.$$  \hspace{1cm} (3.19)

Interchanging role of $X$ and $Y$ we have

$$f \nabla_Y X + w(\nabla_Y X) N + B(Y, X)FN = \nabla_Y f X + B(Y, f X)N.$$  \hspace{1cm} (3.20)

From (3.19), (3.20) we get

$$w(\{X, Y\}) = B(X, f Y) - B(Y, f X)$$  \hspace{1cm} (3.21)

and this is (i) $\iff$ (ii). From the first equation of (2.15), we conclude (ii) $\iff$ (iii). Thus we have our assertion. \hfill $\square$

From the decomposition (3.6), we can give the following definition.

**Definition 3.9.** Let $M$ be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. If $B(X, Y) = 0$, for any $X \in \Gamma(\overline{D}), Y \in \Gamma(\overline{D}_2)$, then we say that $M$ is a mixed geodesic lightlike hypersurface.

**Theorem 3.10.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian product manifold and $M$ be a screen semi-invariant lightlike hypersurface of $\overline{M}$. Then the following assertions are equivalent.

(i) $M$ is mixed geodesic.

(ii) There is no $\mathcal{D}_2$-component of $A_N$.

(iii) There is no $\mathcal{D}_1$-component of $A_x^*$.

**Proof.** Suppose that $M$ is mixed geodesic screen semi-invariant lightlike hypersurface of $\overline{M}$ with respect to the Levi-Civita connection $\overline{\nabla}$. From (2.24), (2.9), (2.10), and (3.1), we obtain

$$\nabla_X FN + B(X, FN)N = -f A_N X + \tau(X) FN - w(A_N X) N,$$  \hspace{1cm} (3.22)
for any $X \in \Gamma(\overline{\mathbb{D}})$. If we take tangential and transversal parts of this last equation we have

\[
\nabla_X FN = -f A_N X + \tau(X) FN, \\
B(X, FN) = -w(A_N X).
\]

Furthermore, since $w(A_N X) = g(A_N X, F\xi)$, we get (i) $\Leftrightarrow$ (ii). Since $\overline{g}(FN, \xi) = \overline{g}(N, F\xi) = 0$, we obtain

\[
g(A_N X, F\xi) = -g\left(A_\xi^* X, FN\right).
\] (3.24)

This is (ii) $\Leftrightarrow$ (iii).

From the decomposition (3.6), we have the following theorem.

**Theorem 3.11.** Let $M$ be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then $M$ is a locally product manifold according to the decomposition (3.6) if and only if $f$ is parallel with respect to induced connection $\nabla$, that is $\nabla f = 0$.

**Proof.** Let $M$ be a locally product manifold. Then the leaves of distributions $\overline{\mathbb{D}}$ and $\mathbb{D}_2$ are both totally geodesic in $M$. Since the distribution $\overline{\mathbb{D}}$ is invariant with respect to $F$ then, for any $Y \in \Gamma(\overline{\mathbb{D}})$, $FY \in \Gamma(\overline{\mathbb{D}})$. Thus $\nabla_X Y$ and $\nabla_X fY$ belong to $\Gamma(\overline{\mathbb{D}})$, for any $X \in \Gamma(TM)$. From the Gauss formula, we obtain

\[
\nabla_X fY + B(X, fY) N = f \nabla_X Y + w(\nabla_X Y) N + B(X, Y) FN.
\] (3.25)

Comparing the tangential and normal parts with respect to $\overline{\mathbb{D}}$ of (3.25), we have

\[
\nabla_X fY = f \nabla_X Y, \quad \text{that is } (\nabla_X f) Y = 0,
\] (3.26)

\[
B(X, Y) = 0.
\] (3.27)

Since $fZ = 0$, for any $Z \in \Gamma(\mathbb{D}_2)$, we get $\nabla_X fZ = 0$ and $f \nabla_X Z = 0$, that is $(\nabla_X f)Z = 0$. Thus we have $\nabla f = 0$ on $M$.

Conversely, we assume that $\nabla f = 0$ on $M$. Then we have $\nabla_X fY = f \nabla_X Y$, for any $X, Y \in \Gamma(\overline{\mathbb{D}})$ and $\nabla_U fW = f \nabla_U W = 0$, for any $U, W \in \Gamma(\mathbb{D}_2)$. Thus it follows that $\nabla_X fY \in \Gamma(\overline{\mathbb{D}})$ and $\nabla_U W \in \Gamma(\mathbb{D}_2)$. Hence, the leaves of the distributions $\overline{\mathbb{D}}$ and $\mathbb{D}_2$ are totally geodesic in $M$. \hfill \Box

From Theorem 3.11 and (3.27) we have the following corollary.

**Corollary 3.12.** Let $M$ be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. If $M$ has a local product structure, then it is a mixed geodesic lightlike hypersurface.
Let $M$ be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then we have the following decomposition:

$$T\overline{M} = S(TM) \perp \{\text{Rad } TM \oplus F \text{ Rad } TM\}. \quad (3.28)$$

**Theorem 3.13.** Let $M$ be a radical anti-invariant lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then the screen distribution $S(TM)$ of $M$ is an integrable distribution if and only if $B(X, FY) = B(Y, FX)$.

**Proof.** If a vector field $X$ on $M$ belongs to $S(TM)$ if and only if $\eta(X) = 0$. Since $M$ is a radical anti-invariant lightlike hypersurface, for any $X \in \Gamma(S(TM))$, $FX \in \Gamma(S(TM))$. For any $X,Y \in \Gamma(S(TM))$, we can write

$$\nabla_X FY = \nabla_X FY + B(X, FY)N. \quad (3.29)$$

In this last equation interchanging role of $X$ and $Y$, we obtain

$$F[X, Y] = \nabla_X FY - \nabla_Y FX + (B(X, FY) - B(Y, FX))N. \quad (3.30)$$

Since $\eta([X, Y]) = \overline{g}([X, Y], N) = \overline{g}(F[X, Y], FN)$, we get

$$\eta([X, Y]) = (B(X, FY) - B(Y, FX))\overline{g}(N, FN). \quad (3.31)$$

Since $\overline{g}(N, FN) \neq 0$, $\eta([X, Y]) = 0$ if and only if $B(X, FY) = B(Y, FX)$. This is our assertion. \qed

### 4. Quarter-Symmetric Nonmetric Connections

Let $(\overline{M}, \overline{g}, F)$ be a semi-Riemannian product manifold and $\nabla$ be the Levi-Civita connection on $\overline{M}$. If we set

$$\overline{D}_X Y = \nabla_X Y + u(Y)FX, \quad (4.1)$$

for any $X, Y \in \Gamma(T\overline{M})$, then $\overline{D}$ is a linear connection on $\overline{M}$, where $u$ is a 1-form on $\overline{M}$ with $U$ as associated vector field, that is

$$u(X) = \overline{g}(X, U). \quad (4.2)$$

The torsion tensor of $\overline{D}$ on $\overline{M}$ denoted by $\overline{T}$. Then we obtain

$$\overline{T}(X, Y) = u(Y)FX - u(X)FY, \quad (4.3)$$

$$\left(\overline{D}_X \overline{g}\right)(Y, Z) = -u(Y)\overline{g}(FX, Z) - u(Z)\overline{g}(FX, Y), \quad (4.4)$$
for any \( X, Y \in \Gamma(T\overline{M}) \). Thus \( \overline{D} \) is a quarter-symmetric nonmetric connection on \( \overline{M} \). From (2.24) and (4.1) we have
\[
\left( \overline{D}_X F \right)Y = u(FY)FX - u(Y)X. \tag{4.5}
\]
Replacing \( X \) by \( FX \) and \( Y \) by \( FY \) in (4.5) and using (2.19) we obtain
\[
\left( \overline{D}_{FX} F \right)FY = u(Y)X - u(FY)FX. \tag{4.6}
\]
Thus we have
\[
\left( \overline{D}_X F \right)Y + \left( \overline{D}_{FX} F \right)FY = 0. \tag{4.7}
\]
If we set
\[
'F(X, Y) = \overline{g}(FX, Y), \tag{4.8}
\]
for any \( X, Y \in \Gamma(T\overline{M}) \), from (4.1) we get
\[
\left( \overline{D}_X 'F \right)(Y, Z) = \left( \overline{\nabla}_X 'F \right)(Y, Z) - u(Y)\overline{g}(X, Z) - u(Z)\overline{g}(X, Y). \tag{4.9}
\]
From (4.1) the curvature tensor \( \overline{R}^\overline{D} \) of the quarter-symmetric nonmetric connection \( \overline{D} \) is given by
\[
\overline{R}^\overline{D}(X, Y)Z = \overline{R}(X, Y)Z + \overline{\lambda}(X, Z)FY - \overline{\lambda}(Y, Z)FX, \tag{4.10}
\]
for any \( X, Y, Z \in \Gamma(T\overline{M}) \), where \( \overline{\lambda} \) is a \( (0, 2) \)-tensor given by \( \overline{\lambda}(X, Z) = \left( \overline{\nabla}_X u \right)(Z) - u(Z)u(FX) \).
If we set \( \overline{R}^\overline{D}(X, Y, Z, W) = \overline{g}(\overline{R}^\overline{D}(X, Y)Z, W) \), then, from (4.10), we obtain
\[
\overline{R}^\overline{D}(X, Y, Z, W) = -\overline{R}^\overline{D}(Y, X, Z, W). \tag{4.11}
\]
We note that the Riemannian curvature tensor \( \overline{R}^\overline{D} \) of \( \overline{D} \) does not satisfy the other curvature-like properties. But, from (4.10), we have
\[
\overline{R}^\overline{D}(X, Y)Z + \overline{R}^\overline{D}(Y, Z)X + \overline{R}^\overline{D}(Z, X)Y = \left( \overline{\lambda}(Z, Y) - \overline{\lambda}(Y, Z) \right)FX
+ \left( \overline{\lambda}(X, Z) - \overline{\lambda}(Z, X) \right)FY
+ \left( \overline{\lambda}(Y, X) - \overline{\lambda}(X, Y) \right)FZ. \tag{4.12}
\]
Thus we have the following proposition.
Proposition 4.1. Let $M$ be a lightlike hypersurface of a semi-Riemannian product manifold $\overline{M}$. Then the first Bianchi identity of the quarter-symmetric nonmetric connection $\overline{D}$ on $M$ is provided if and only if $\overline{\lambda}$ is symmetric.

Let $M$ be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$ with quarter-symmetric nonmetric connection $\overline{D}$. Then the Gauss and Weingarten formulas with respect to $\overline{D}$ are given by, respectively,

$$\overline{D}_X Y = D_X Y + \overline{B}(X, Y) N$$

(4.13)

$$\overline{D}_X N = -\overline{A}_N X + \overline{\tau}(X) N$$

(4.14)

for any $X, Y \in \Gamma(TM)$, where $D_X Y, \overline{A}_N X \in \Gamma(TM)$, $\overline{B}(X, Y) = \overline{g}(\overline{D}_X Y, \xi)$, $\overline{\tau}(X) = \overline{g}(\overline{D}_X N, \xi)$. Here, $D, B$ and $\overline{A}_N$ are called the induced connection on $M$, the second fundamental form, and the Weingarten mapping with respect to $\overline{D}$. From (2.9), (2.10), (3.1), (4.1), (4.13), and (4.14) we obtain

$$D_X Y = \nabla_X Y + u(Y)fX,$$

(4.15)

$$\overline{B}(X, Y) = B(X, Y) + u(Y)\omega(X),$$

(4.16)

$$\overline{A}_N X = A_N X - u(N)fX,$$

(4.17)

$$\overline{\tau}(X) = \tau(X) + u(N)\omega(X),$$

for any $X, Y \in \Gamma(TM)$. From (4.1), (4.4), (4.13), and (4.16) we get

$$(D_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y)$$

$$- u(Y)g(fX, Z) - u(Z)g(fX, Y).$$

(4.18)

On the other hand, the torsion tensor of the induced connection $D$ is

$$T^D(X, Y) = u(Y)fX - u(X)fY.$$ (4.19)

From last two equations we have the following proposition.

Proposition 4.2. Let $M$ be a lightlike hypersurface of a semi-Riemannian product manifold $(\overline{M}, \overline{g})$ with quarter-symmetric nonmetric connection $\overline{D}$. Then the induced connection $D$ is a quarter-symmetric nonmetric connection on the lightlike hypersurface $M$.

For any $X, Y \in \Gamma(TM)$, we can write

$$D_X PY = D_X^\overline{P} PY + \overline{C}(X, PY)\xi,$$

(4.20)

$$D_X \xi = -\overline{A}_\xi^\ast X + \epsilon(X)\xi,$$
where \( D^*_1 \) is the distribution given by (4.1). Since \( \omega(X) = g(FX, \xi) \), for any \( X \in \Gamma(D) \), \( \omega(X) = 0 \). Thus we have the following propositions.

**Proposition 4.3.** Let \( M \) be a screen semi-invariant lightlike hypersurface of a semi-Riemannian product manifold \( \overline{M} \) with respect to the quarter symmetric connection \( \overline{D} \) given by (4.1). Since \( \omega(X) = g(FX, \xi) \), for any \( X \in \Gamma(D) \), \( \omega(X) = 0 \). Thus we have the following propositions.

**Proposition 4.4.** Let \( (\overline{M}, \overline{g}) \) be a semi-Riemannian product manifold and \( M \) be a screen semi-invariant lightlike hypersurfaces of \( \overline{M} \). If \( M \) is \( \mathbb{D} \) totally geodesic with respect to \( \overline{\nabla} \), then \( M \) is \( \mathbb{D} \) totally geodesic with respect to quarter symmetric nonmetric connection \( \overline{D} \) is degenerate.

**Theorem 4.5.** Let \( (\overline{M}, \overline{g}) \) be a semi-Riemannian product manifold and \( M \) be a screen semi-invariant lightlike hypersurfaces of \( \overline{M} \). Then the following assertions are equivalent.

(i) The distribution \( \overline{\mathbb{D}} \) is integrable with respect to the quarter symmetric nonmetric connection \( \overline{D} \).

(ii) \( \overline{B}(X, fY) = \overline{B}(Y, fX) \), for any \( X, Y \in \Gamma(\overline{\mathbb{D}}) \).

(iii) \( g(\overline{A}_2^*X, PfY) = g(\overline{A}_2^*Y, PfX) \), for any \( X, Y \in \Gamma(\overline{\mathbb{D}}) \).

The proof of this theorem is similar to the proof of the Theorem 3.8.

From (4.23), for any \( X \in \Gamma(\mathbb{D}) \) and \( Y \in \Gamma(\mathbb{D}_2) \), we have \( \overline{B}(X, PY) = g(\overline{A}_2^*X, PY) \). If we set \( \mathbb{D}' = \mathbb{D} \perp \mathbb{D}_2 \), then, from Theorem 3.10, we have the following corollary.

**Corollary 4.6.** Let \( (\overline{M}, \overline{g}) \) be a semi-Riemannian product manifold and \( M \) be a screen semi-invariant lightlike hypersurface of \( \overline{M} \). Then the distribution \( \mathbb{D}' \) is a mixed geodesic foliation defined with respect to quarter symmetric nonmetric connection if and only if there is no \( \mathbb{D}_1 \) component of \( \overline{A}_2^* \).
From (4.15), we obtain

\[
R^D(X,Y)Z = R(X,Y)Z + u(Z)\left[ (\nabla_X f)Y - (\nabla_Y f)X \right] + \lambda(X,Z)fY - \lambda(Y,Z)fX,
\]

(4.24)

where \( \lambda \) is a \((0,2)\) tensor on \( M \) given by \( \lambda(X,Z) = (\nabla_X u)(Z) - u(Z)u(fX) \).

From (4.24), we have the following proposition which is similar to the Proposition 4.1.

**Proposition 4.7.** Let \( M \) be a lightlike hypersurface of a semi-Riemannian product manifold \( \overline{M} \). One assumes that \( f \) is parallel on \( M \). Then the first Bianchi identity of the quarter-symmetric nonmetric connection \( D \) on \( M \) is provided if and only if \( \lambda \) is symmetric.

Now we will compute Gauss-Codazzi equations of lightlike hypersurfaces with respect to the quarter-symmetric nonmetric connection:

\[
\bar{g}\left(\overline{R}^D(X,Y)Z,PW\right) = g(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) + \overline{\lambda}(X,Z)g(fY,PW) - \overline{\lambda}(Y,Z)g(fX,PW),
\]

\[
\bar{g}\left(\overline{R}^D(X,Y)Z,\xi\right) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \overline{\lambda}(X,Z)\eta(fY) - \overline{\lambda}(Y,Z)\eta(fX),
\]

(4.25)

\[
\bar{g}\left(\overline{R}^D(X,Y)Z,N\right) = g(R(X,Y)Z,N) + \overline{\lambda}(X,Z)\eta(fY) - \overline{\lambda}(Y,Z)\eta(fX),
\]

for any \( X, Y, Z, W \in \Gamma(TM) \).

Now, let \( M \) be a screen semi-invariant lightlike hypersurface of a \((m + 2)\)-dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection \( D \) such that the tensor field \( f \) is parallel on \( M \). We consider the local quasiorthornormal basis \( \{E_i,F_\xi,FN,\xi,N\} \), \( i = 1, \ldots m - 2 \), of \( \overline{M} \) along \( M \), where \( \{E_1, \ldots, E_{m-2}\} \) is an orthonormal basis of \( \Gamma(D) \). Then, the Ricci tensor of \( M \) with respect to \( D \) is given by

\[
R^{D[0,2]}(X,Y) = \sum_{i=1}^{m-2} e_i g\left( R^D(X,E_i)Y,E_i \right) + g\left( R^D(X,F_\xi)Y,FN \right) + g\left( R^D(X,FN)Y,F_\xi \right) + g\left( R^D(X,\xi)Y,N \right).
\]

(4.26)
From (4.24) we have
\[
R^{D(0,2)}(X, Y) = R^{(0,2)}(X, Y)
\]
\[
+ \sum_{i=1}^{m-2} e_i \{ \lambda(X, Y) g(f E_i, E_i) - \lambda(E_i, Y) g(f X, E_i) \} \quad (4.27)
\]
\[
- \lambda(F \xi, Y) \eta(X) - \lambda(\xi, Y) \eta(f X),
\]

where \( R^{(0,2)}(X, Y) \) is the Ricci tensor of \( M \). Thus we have the following corollary.

**Corollary 4.8.** Let \( M \) a screen semi-invariant lightlike hypersurface of a \( (m + 2) \)-dimensional semi-Riemannian product manifold with the quarter-symmetric nonmetric connection \( \overline{\nabla} \) such that the tensor field \( f \) is parallel on \( M \) and \( R^{(0,2)}(X, Y) \) is symmetric. Then \( R^{D(0,2)} \) is symmetric on the distribution \( \nabla \) if and only if \( \lambda \) is symmetric and \( \lambda(f X, Y) = \lambda(f Y, X) \).

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