Research Article

Noetherian and Artinian Lattices

Derya Keskin Tütüncü,1 Sultan Eylem Toksoy,2 and Rachid Tribak3

1 Department of Mathematics, Hacettepe University, Beytepe 06800, Ankara, Turkey
2 Department of Mathematics, Izmir Institute of Technology, Urla 35430, Izmir, Turkey
3 Centre Pédagogique Régional (CPR) Tanger, Avenue My Abdelaziz Souani, BP 3117, Tangier 90000, Morocco

Correspondence should be addressed to Derya Keskin Tütüncü, keskin@hacettepe.edu.tr

Received 30 March 2012; Accepted 12 April 2012

Academic Editor: Palle E. Jorgensen

Copyright © 2012 Derya Keskin Tütüncü et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is proved that if $L$ is a complete modular lattice which is compactly generated, then $\text{Rad}(L)/0$ is Artinian if, and only if for every small element $a$ of $L$, the sublattice $a/0$ is Artinian if, and only if $L$ satisfies DCC on small elements.

1. Introduction

By a lattice we mean a partially ordered set $(L, \leq)$ such that every pair of elements $a, b$ in $L$ has a greatest lower bound (or a meet) $a \wedge b$ and a least upper bound (or a join) $a \vee b$; that is,

(i) $a \wedge b \leq a, a \wedge b \leq b$, and $c \leq a \wedge b$ for all $c \in L$ with $c \leq a, c \leq b$,

(ii) $a \leq a \vee b, b \leq a \vee b, and a \vee b \leq d$ for all $d \in L$ with $a \leq d, b \leq d$.

Note that, for given $a, b \in L$, $a \wedge b$ and $a \vee b$ are unique, and

$$a \leq b \iff a = a \wedge b \iff b = a \vee b. \quad (1.1)$$

Let $(L, \leq, \wedge, \vee)$ (or just $L$) be any lattice. Given $a, b \in L$, we set

$$a \leq b \iff b \leq a. \quad (1.2)$$
Then \((L, \leq')\) is a partially ordered set; moreover, for any \(a, b \in L\), \(a, b\) have greatest lower bound \(a \lor b\) and least upper bound \(a \land b\). We call \((L, \leq', \lor, \land)\) the opposite lattice of \(L\), and denote it by \(L^o\).

Let \((L, \leq, \land, \lor)\) be any lattice. Let \(a \leq b\) in \(L\). We define

\[
\frac{b}{a} = \{x \in L : a \leq x \leq b\}. \tag{1.3}
\]

(Sometimes \(b \text{ frac } a\) is denoted by \(b/\text{a}\).)

A lattice \((L, \leq, \land, \lor)\) has a least element if there exists \(z \in L\) such that \(z \leq a(a \in L)\). In this case, \(z\) is uniquely defined and is usually denoted by 0. The lattice \(L\) has a greatest element if there exists \(u \in L\) such that \(a \leq u(a \in L)\). In this case, \(u\) is uniquely defined and is usually denoted by 1. A lattice \(L\) is called complete if every subset of \(L\) has a meet and a join, and it is called modular if \(a \land (b \lor c) = b \lor (a \land c)\) for all \(a, b, c\) in \(L\) with \(b \leq a\). For more information about lattice theory, refer to [1-3].

Throughout this paper \((L, \leq, \lor, \land, 0, 1)\) will be a complete modular lattice. An element \(e \in L\) is called an essential element if \(e \land x \neq 0\) for every nonzero element \(x \in L\). An element \(s \in S\) is said to be small if \(s\) is an essential element of the opposite lattice \(L^o\). Let \(E(L)\) denote the set of all essential elements of \(L\). The set of all small elements of \(L\) will be denoted by \(S(L)\).

A set \(\{c_i \mid i \in I\} \subseteq L\) is called a direct set if, for all \(i, j \in I\), there exists \(k \in I\) with \(c_i \lor c_j \leq c_k\). The lattice \(L\) is said to be upper continuous if, for every direct set \(\{c_i \mid i \in I\}\) in \(L\) and element \(a \in L\), we have \(a \land \bigvee_{i \in I} c_i = \bigvee_{i \in I} (a \land c_i)\). On the other hand, \(L\) is said to be lower continuous if for every inverse set \(\{c_i \mid i \in I\}\) (i.e., for all \(i, j \in I\), there exists \(k \in I\) with \(c_k \leq c_i \land c_j\)) and element \(a \in L\), \(a \lor \bigwedge_{i \in I} c_i = \bigwedge_{i \in I} (a \lor c_i)\). We will call an element \(f\) in \(L\) finitely generated element (or compact element) if whenever \(f \leq F\), for some direct set \(S\) in \(L\), there exists \(x \in S\) such that \(f \leq x\). Note that 0 is always a finitely generated element of \(L\). It is known that an element \(f\) is finitely generated if and only if for every nonempty subset \(U\) of \(L\) with \(f \leq \bigvee U\) there exists a finite subset \(F\) of \(U\) such that \(f \leq \bigvee F\). A lattice \(L\) is said to be finitely generated (or compact) if 1 is finitely generated. We call the lattice \(L\) compactly generated if each of its elements is a join of finitely generated elements (see [2]). Note that every compactly generated lattice is upper continuous (see, e.g., [4, Proposition 2.4]). Moreover, it is shown in [4, Exercises 2.7 and 2.9] that for every element \(a\) of a compactly generated lattice \(L\), the sublattices \(a/0\) and \(1/a\) are again compactly generated. A lattice \(L\) is called a finitely cogenerated (or cocompact) lattice, if for every subset \(X\) of \(L\) such that \(\land X = 0\) there is a finite subset \(F\) of \(X\) such that \(\land F = 0\). An element \(g \in L\) is said to be finitely cogenerated (or cocompact) if the sublattice \(g/0\) is a finitely cogenerated lattice. If \(a < b\) and \(a \leq c < b\) imply \(c = a\), then we say that \(a\) is covered by \(b\) (or \(b\) covers \(a\)). If 0 is covered by an element \(a\) of \(L\), then \(a\) is called an atom element of \(L\). A lattice \(L\) is said to be semiatomic if 1 is a join of atoms in \(L\) (see [4]). The meet of all maximal elements (different from 1) in \(L\) is denoted by \(\text{Rad}(L)\), and it is called the radical of \(L\) (see [2]). If \(L\) is compactly generated, then \(\text{Rad}(L)\) is the join of all small elements of \(L\) (see [2, Theorem 8]). The join of all atoms of \(L\), denoted by \(\text{Soc}(L)\), is called the socle of \(L\). The socle of a compactly generated lattice is equal to the meet of all essential elements (see [4, Theorem 5.1]).

A non-empty subset \(S\) of \(L\) is called an independent set if, for every \(x \in S\) and finite subset \(T = \{t_1, \ldots, t_n\}\) of \(S\) with \(x \notin T\), \(x \land (t_1 \lor \cdots \lor t_n) = 0\). We say that a nonzero lattice \(L\) has finite uniform (or Goldie) dimension if \(L\) contains no infinite independent sets; equivalently, \(\sup\{\{k \mid L\text{ contains an independent subset of cardinality equal to } k\} = n < \infty\). In this case \(L\) is said to have uniform (or Goldie) dimension \(n\) and this is denoted by \(u(L)\). We shall say
that $L$ has hollow (or dual Goldie) dimension $n$, provided the opposite lattice $L^\circ$ has uniform dimension $n$. The lattice $L$ is said to be Artinian (noetherian) if $L$ satisfies the descending (ascending) chain condition on its elements. A lattice $L$ will be called an $E$-complemented lattice if, for each $a \in L$, there exists $b \in L$ such that $a \land b = 0$ and $a \lor b \in E(L)$.

In Section 2 we mainly prove that a lattice $L$ is noetherian if and only if $L$ is $E$-complemented and every essential element of $L$ is finitely generated (Corollary 2.4). In Section 3 we generalize Theorem 5 in [5] to lattice theory (Theorem 3.7).

2. Noetherian Lattices

The following lemma was given us by Patrick F. Smith from his unpublished notes.

**Lemma 2.1.** Let $L$ be a lattice. Consider the following statements.

(i) $L$ is noetherian.

(ii) $L$ has finite uniform dimension.

(iii) $L$ is $E$-complemented.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

**Proof.** (i) $\Rightarrow$ (ii) Suppose $L$ is noetherian but that $L$ does not have finite uniform dimension. Then there exists an infinite independent set of nonzero elements $x_n (n \in \mathbb{N})$. Consider the ascending chain $x_1 \leq x_2 \leq \cdots$ in $L$. Because $L$ is noetherian, there exists a positive integer $n$ such that $x_1 \lor \cdots \lor x_n = x_1 \lor \cdots \lor x_n \lor x_{n+1}$. This implies that $x_{n+1} \leq (x_1 \lor \cdots \lor x_n) \land x_{n+1} = 0$, a contradiction. Therefore $L$ has finite uniform dimension.

(ii) $\Rightarrow$ (iii) Let $a \in L$. If $a \in E(L)$, we are done. If $a \notin E(L)$, then there exists $0 \neq b_1 \in L$ such that $a \land b_1 = 0$. If $a \lor b_1 \in E(L)$, we are done. Otherwise, there exists $0 \neq b_2 \in L$ such that $(a \lor b_1) \land b_2 = 0$. Repeating this argument we produce an independent set $\{a, b_1, b_2, \ldots \}$. Thus this process must stop, so there exists $k \in \mathbb{N}$ such that $a \land (b_1 \lor \cdots \lor b_k) = 0$ and $a \lor (b_1 \lor \cdots \lor b_k) \in E(L)$. 

**Remark 2.2.** Note that if $f$ is a finitely generated element of a lattice $L$, then for every nonempty set $U$ with $f = \lor U$ there exists a finite subset $F$ of $U$ such that $f = \lor F$.

**Proposition 2.3.** Let $L$ be a lattice such that $x$ is finitely generated for every $x \in E(L)$.

Then the following are equivalent.

(i) $L$ is noetherian.

(ii) $L$ has finite uniform dimension.

(iii) $L$ is $E$-complemented.

**Proof.** We only need to prove (iii) $\Rightarrow$ (i) by Lemma 2.1. Let $a$ be a nonzero element in $L$. By (iii), there exists an element $b$ of $L$ such that $a \land b = 0$ and $a \lor b \in E(L)$. By hypothesis, $a \lor b$ is finitely generated. Let $a = \lor S$ for a nonempty set $S$ in $L$. Then $a \lor b = \lor (S \cup \{b\})$. Since $a \lor b$ is finitely generated, $a \lor b = \lor F \lor b$ for a finite subset $F$ of $S$. Since $L$ is modular, we have $a = \lor F$. Therefore every element in $L$ is finitely generated. Hence $L$ is noetherian by [4, Proposition 2.3].
Corollary 2.4. A lattice \( L \) is noetherian if and only if \( L \) is \( E \)-complemented and every essential element of \( L \) is finitely generated.

Lemma 2.5. Every upper continuous lattice \( L \) is \( E \)-complemented.

Proof. Let \( a \in L \). Let \( S = \{ b \in L \mid a \land b = 0 \} \). Clearly, \( 0 \in S \). Let \( \{ c_i \mid i \in I \} \) be a chain in \( S \) and let \( c = \bigvee_{i \in I} c_i \). Then \( a \land c = a \land (\bigvee_{i \in I} c_i) = \bigvee_{i \in I} (a \land c_i) = 0 \). By Zorn’s lemma, \( S \) contains a maximal member \( u \). Then \( a \land u = 0 \). Suppose that \( (a \lor u) \land x = 0 \) for some \( x \in L \). Then \( a \land (u \lor x) = 0 \), and hence \( u \lor x \in S \). Since \( u \leq u \lor x \), we have \( u = u \lor x \) and \( x \leq u \). Thus \( x = (a \lor u) \land x = 0 \). It follows that \( a \lor u \in E(L) \). Therefore \( L \) is \( E \)-complemented.

Corollary 2.6. Let \( L \) be an upper continuous lattice. Then \( L \) is noetherian if and only if every essential element in \( L \) is finitely generated.

Lemma 2.7 (see [4, Lemmas 7.3 and 7.5]). Let \( L \) be a lattice and \( k \) a positive integer. Then

(i) if \( t \in S(L) \), then \( s \in S(L) \) for every \( s \leq t \);

(ii) if \( s_1, s_2, \ldots, s_k \in S(L) \), then \( s_1 \lor s_2 \lor \cdots \lor s_k \in S(L) \).

As an easy observation of Lemma 2.7, we can give the following two results.

Proposition 2.8 (see cf. [5, Proposition 2]). Let \( L \) be a compactly generated lattice. Then \( \text{Rad}(L)/0 \) is noetherian if and only if \( L \) satisfies ACC on small elements.

Proof. (\( \Rightarrow \)) By [2, Theorem 8].

(\( \Leftarrow \)) By assumption, \( L \) contains a maximal small element \( x \). Since \( x \) is small in \( L \), \( x \leq \text{Rad}(L) \). Suppose that \( x \neq \text{Rad}(L) \). Then there exists a small element \( s \) of \( L \) such that \( s \notin x/0 \). On the other hand, \( s \lor x \) is a small element of \( L \) by Lemma 2.7(ii). By the maximality of \( x \), we have \( s \lor x = x \). This gives \( s \in x/0 \), a contradiction. Thus \( x = \text{Rad}(L) \). By Lemma 2.7(i), \( \text{Rad}(L)/0 \subseteq S(L) \). Consequently, \( \text{Rad}(L)/0 \) is noetherian.

Proposition 2.9 (see cf. [5, Proposition 3]). Let \( L \) be a compactly generated lattice. Then the following are equivalent.

(i) \( \text{Rad}(L)/0 \) has finite uniform dimension.

(ii) There exists a positive integer \( k \) such that for every small element \( s \) of \( L \) we have \( u(s/0) \leq k \).

(iii) \( L \) does not contain an infinite independent set of nonzero small elements.

Proof. \( (i) \Rightarrow (ii) \) Let \( s \) be a small element of \( L \). By [2, Theorem 8], \( s \leq \text{Rad}(L) \). Since \( u(s/0) \leq u(\text{Rad}(L)/0) \), \( s/0 \) has finite uniform dimension. The rest is clear.

\( (ii) \Rightarrow (iii) \) Let \( \{ s_1, s_2, \ldots \} \) be an infinite independent set of nonzero small elements of \( L \). By Lemma 2.7(ii), \( s_1 \lor s_2 \lor \cdots \lor s_{k+1} \in S(L) \), and \( u((s_1 \lor s_2 \lor \cdots \lor s_{k+1})/0) \geq k + 1 \), a contradiction.

\( (iii) \Rightarrow (i) \) Suppose that \( \text{Rad}(L)/0 \) does not have finite uniform dimension. Then there exists an infinite independent set of nonzero elements \( \{ x_1, x_2, \ldots \} \) of \( \text{Rad}(L)/0 \). Let \( i \geq 1 \). Since \( \text{Rad}(L)/0 \) is compactly generated, there exists a nonzero finitely generated element \( k_i \) of \( \text{Rad}(L)/0 \) such that \( k_i \leq x_i \). So by Lemma 2.7, \( k_i \in S(L) \). Therefore \( \{ k_1, k_2, \ldots \} \) is an infinite independent set of nonzero small elements of \( L \), a contradiction. Thus \( \text{Rad}(L)/0 \) has finite uniform dimension.
3. Artinian Lattices

Lemma 3.1. Let \( L \) be a compactly generated semiatomic lattice. Then the following are equivalent.

(i) \( L \) is finitely generated.

(ii) \( L \) is finitely cogenerated.

(iii) \( 1 \) is a finite independent join of atoms.

(iv) \( L \) is Artinian.

Proof. (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii) By [4, Theorem 11.1].

(iv) \( \Rightarrow \) (i) By [4, Proposition 11.2].

(iii) \( \Rightarrow \) (iv) Note that if \( a \) is an atom in \( L \), then \( a/0 \) is Artinian. Assume that \( 1 = a_1 \lor a_2 \lor \cdots \lor a_n \) such that the join is independent and each \( a_i \) is atom in \( L \). Since each \( a_i/0 \) is Artinian, \( (a_1 \lor a_2 \lor \cdots \lor a_n)/0 \) is Artinian, and hence \( L \) is Artinian. \( \square \)

Lemma 3.2. Let \( L \) be a compactly generated lattice which satisfies DCC on small elements. If \( f \) is a finitely generated element of \( \text{Rad}(L)/0 \), then \( f/0 \) is Artinian.

Proof. Let \( f \) be a finitely generated element of \( \text{Rad}(L)/0 \). Then \( f \leq \text{Rad}(L) = \bigvee_{i \in I} \{ s_i \mid s_i \in S(L) \} \) implies that \( f \leq \bigvee_{i \in I} \{ s_i \mid s_i \in S(L) \} \) for some finite subset \( I \) of \( L \). By Lemma 2.7, \( f \in S(L) \). By assumption and Lemma 2.7(i), \( f/0 \) is Artinian. \( \square \)

Lemma 3.3. Let \( L \) be a compactly generated lattice which satisfies DCC on small elements. Then, for every \( k \leq \text{Rad}(L) \), \( \text{Soc}(\text{Rad}(L)/k) \) is an essential element of \( \text{Rad}(L)/k \).

Proof. Let \( k \leq \text{Rad}(L) \), and let \( \text{Soc}(\text{Rad}(L)/k) = t \). Let \( k \leq h \leq \text{Rad}(L) \) such that \( t \land h = k \). Assume that \( k < h \). Since \( \text{Rad}(L)/0 \) is compactly generated, there exists a nonzero finitely generated element \( x \) in \( \text{Rad}(L)/0 \) such that \( x \leq h \) but \( x \notin k/0 \). By Lemma 3.2, \( x/0 \) is Artinian. Then \( x/(x \land k) \cong (k \lor x)/k \) implies that \( (k \lor x)/k \) is a nonzero Artinian sublattice. By [4, Proposition 1.4], \( (k \lor x)/k \) has an atom element \( p' \). Note that \( k < p' \leq x \lor k \leq h \). Since \( p' \) is atom in \( \text{Rad}(L)/k \), we have \( p' \leq t \). Thus \( k < p' \leq t \land h \). This contradicts the fact that \( t \land h = k \). Therefore \( k = h \) and \( t \in E(\text{Rad}(L)/k) \). This completes the proof. \( \square \)

Lemma 3.4. Let \( a \) be an element of a compactly generated lattice \( L \). If \( a \) is a finitely generated element of \( a/0 \), then \( a \) is a finitely generated element of \( L \).

Proof. Since \( L \) is compactly generated, \( a = \bigvee U \) where \( U \) is a set of finitely generated elements in \( L \). Since \( a \) is a finitely generated element of \( a/0 \), \( a = \bigvee_{1 \leq i \leq n} a_i \) for some elements \( a_i (1 \leq i \leq n) \) of \( U \). Therefore \( a \) is a finitely generated element of \( L \). \( \square \)

Lemma 3.5. Let \( L \) be a compactly generated lattice which satisfies DCC on small elements. Suppose that the set

\[
\Omega = \left\{ a_i \mid 0 \leq a_i \leq \text{Rad}(L) \text{ and } \frac{\text{Rad}(L)}{a_i} \text{ is not finitely cogenerated} \right\}
\]

is nonempty. Then:

1. the set \( \Omega \) has a minimal member \( p \) which is a small element of \( L \);

2. if \( \text{Soc}(\text{Rad}(L)/p) = s \), then \( s \) is not a finitely generated element of \( \text{Rad}(L)/p \) and \( s \) is a small element of \( L \).
Proof. (1) Let $\Gamma$ be any chain in $\Omega$. Let $c = \bigwedge_{i \in \Gamma} c_i$. If $c \notin \Omega$, then $\text{Rad}(L)/c$ is finitely cogenerated. Therefore $c = c_i$ for some $c_i \in \Gamma$, a contradiction. By Zorn’s Lemma, $\Omega$ has a minimal member $p$. Let $\text{Soc}(\text{Rad}(L)/p) = s$. By Lemma 3.3, $s \in E(\text{Rad}(L)/p)$. Thus $s$ is not a finitely generated element of $\text{Rad}(L)/p$ by [4, Theorem 11.2]. Let $q \in L$ with $1 = p \lor q$. Then $s = s \land 1 = s \land (p \lor q) = p \lor (s \land q)$. It follows that $s/p = [p \lor (s \land q)]/p = (s \land q)/(p \land q)$. Suppose that $p \land q \neq p$. Then $\text{Rad}(L)/(p \land q)$ is finitely cogenerated. Let $\text{Soc}(\text{Rad}(L)/(p \land q)) = \alpha$. Then $\alpha$ is finitely generated in $\text{Rad}(L)/(p \land q)$ by [4, Theorem 11.2]. Therefore $\alpha/(p \land q)$ is Artinian by Lemma 3.1. Since $\text{Rad}(L)/p$ is a sublattice of $\text{Rad}(L)/(p \land q)$, we have $s \leq \alpha$. Thus $s \land q \leq \alpha \leq \text{Rad}(L)$. Since $\alpha/(p \land q)$ is Artinian, $(s \land q)/(p \land q)$ is also Artinian by [4, Proposition 1.5]. This implies that $s/p$ is Artinian, and hence $s$ is a finitely generated element of $s/p$ by Lemma 3.1. Since $\text{Rad}(L)/p$ is compactly generated, $s$ is a finitely generated element of $\text{Rad}(L)/p$ (see Lemma 3.4), a contradiction. So $p \land q = p$ and hence $q \lor p = q = 1$. This gives $p \in S(L)$.

(2) Note that $s$ is not a finitely generated element of $\text{Rad}(L)/0$ as we prove in (1). Let $v \in L$ such that $1 = s \lor v$. Note that $s/p$ is a semiacomitative lattice. Then $s/[p \lor (s \land u)]$ is also semiacomitative by [4, Corollary 6.3]. Therefore,

$$\frac{1}{{p \lor v}} = \frac{s \lor v}{{p \lor v}} = \frac{[s \lor (p \lor v)]}{{p \lor v}} = \frac{s}{{[s \land (p \lor v)]}} = \frac{s}{{[p \lor (s \land u)]}}. \tag{3.2}$$

This implies that $1/(p \lor v)$ is semiacomitative. Suppose that $1 \neq p \lor v$. By [4, Lemma 6.12], there exists a maximal element $w$ of $1/(p \lor v)$. Clearly, $w$ is a maximal element of $L$ and $v \leq w$. Thus $1 = s \lor v \leq s \land w$. But $s \leq \text{Rad}(L) \leq w$. Then $w = 1$, a contradiction. It follows that $1 = p \lor v$. Since $p \in S(L)$, we have $v = 1$. Thus $s \in S(L)$. \hfill \square

Remark 3.6. By dualizing [6, Theorem 3.4], we have the fact that if $L$ is upper continuous and $a/0$ is Artinian for every small element $a$ of $L$, then $\forall S(L)/0$ is Artinian. Therefore for compactly generated lattices $(ii) \Rightarrow (i)$ in Theorem 3.7 holds, but our aim is to give a proof in a different way. We should call attention to the fact that $\forall S(L)$ need not to be the radical of any upper continuous lattice $L$.

Theorem 3.7 (see cf. [5, Theorem 5]). Let $L$ be a compactly generated lattice. Then the following are equivalent.

(i) $\text{Rad}(L)/0$ is Artinian.

(ii) For every small element $a$ of $L$ the sublattice $a/0$ is Artinian.

(iii) $L$ satisfies DCC on small elements.

Proof. (i) $\Rightarrow$ (ii) Clear by [2, Theorem 8].

(ii) $\Rightarrow$ (iii) This is immediate.

(iii) $\Rightarrow$ (i) Suppose that $\text{Rad}(L)/0$ is not Artinian. By [4, Proposition 11.2], there exists an element $g$ in $L$ with $g \leq \text{Rad}(L)$ such that $\text{Rad}(L)/g$ is not finitely cogenerated. By Lemma 3.5, the set

$$\Omega = \left\{ a_i \mid 0 \leq a_i \leq \text{Rad}(L) \text{ and } \frac{\text{Rad}(L)}{a_i} \text{ is not finitely cogenerated} \right\} \tag{3.3}$$

International Journal of Mathematics and Mathematical Sciences
has a minimal member $p$ such that $\text{Soc}(\text{Rad}(L)/p) = s \in S(L)$ and $s$ is not a finitely generated element of $\text{Rad}(L)/p$. By (iii) and Lemma 2.7(i), $s/0$ is Artinian. By Lemma 3.1, $s/0$ is finitely generated. Therefore $s$ is a finitely generated element of $\text{Rad}(L)/p$ by Lemma 3.4. This is a contradiction. Therefore $\text{Rad}(L)/0$ is Artinian.

**Corollary 3.8.** Let $L$ be a compactly generated lattice. If $1/s$ is finitely cogenerated for every small element $s$ of $L$, then $\text{Rad}(L)/0$ is Artinian.

**Proof.** Consider the descending chain

$$x_1 \geq x_2 \geq \cdots \quad (3.4)$$

of small elements of $L$. Put $x = \bigwedge_{i \geq 1} x_i$. Thus $x$ is small in $L$. By assumption, $1/x$ is finitely cogenerated. So there exists an integer $n$ such that $x = \bigwedge_{i=1}^n x_i = x_n$. Hence $L$ has DCC on small elements. By Theorem 3.7, $\text{Rad}(L)/0$ is Artinian.

Let $a$ and $b$ be elements of $L$. Then $b$ is called a *supplement* of $a$ in $L$ if $b$ is minimal with respect to $a \lor b = 1$. Equivalently, $b$ is a supplement of $a$ if and only if $a \lor b = 1$ and $a \land b \in S(a/0)$ (see [4, Proposition 12.1]). The lattice $L$ is said to be supplemented if every element $a$ of $L$ has a supplement in $L$.

The following result may be proved in much the same way as [5, Lemma 6], and $1/\text{Rad}(L)$ is a semiatomic lattice by [4, Proposition 12.3] already.

**Lemma 3.9.** Let $L$ be a compactly generated supplemented lattice with DCC on supplement elements. Then $1/\text{Rad}(L)$ is a finitely generated semiatomic lattice.

By using Theorem 3.7 and Lemma 3.9, we get the following theorem.

**Theorem 3.10.** Let $L$ be a compactly generated lattice. Then $L$ is Artinian if and only if $L$ is supplemented and $L$ satisfies DCC on supplement elements and small elements.

**Proof.** The necessity is clear. Conversely, suppose that $L$ is a supplemented lattice which satisfies DCC on supplement elements and small elements. By Theorem 3.7, $\text{Rad}(L)/0$ is Artinian, and by Lemmas 3.1 and 3.9, $1/\text{Rad}(L)$ is Artinian. Thus $L$ is Artinian.

**Acknowledgments**

The second author has been supported by the TÜBİTAK (The Scientific and Technological Research Council of Turkey) within the program numbered 2218. She would like to thank TÜBİTAK for their support. Also the authors would like to thank Professor Patrick F. Smith (Glasgow University) for his helpful comments on the paper.

**References**


