Research Article

$W_\theta g$-Closed and $W_\delta g$-Closed in $[0,1]$-Topological Spaces

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We investigate various classes of generalized closed fuzzy sets in $[0,1]$-topological spaces, namely, $W_\theta g$-closed fuzzy sets and $W_\delta g$-closed fuzzy sets. Also, we introduce a new separation axiom $FT_{3/4}$ of the $[0,1]$-topological spaces, and we prove that every $FT_{3/4}$-space is a $FT_{3/4}$-space. Furthermore, we use the new generalized closed fuzzy sets to construct new types of fuzzy mappings.

1. Introduction

In 1970, Levine [1] introduced the notion of generalized closed sets in topological spaces as a generalization of closed sets. Since then, many concepts related to generalized closed sets were defined and investigated. In 1997, Balasubramanian and Sundaram [2] introduced the concepts of generalized closed sets in fuzzy setting. Also, they studied various generalizations fuzzy continuous mappings.

Recently, El-Shafei and Zakari [3–5] introduced new types of generalized closed fuzzy sets in $[0,1]$-topological spaces and studied many of their properties. Also, they studied various generalizations fuzzy continuous mappings.

In the present paper, we introduce the concepts of $W_\theta g$-closed fuzzy sets and $W_\delta g$-closed fuzzy sets and study some of their properties. Also, we introduce the concept of $FT_{3/4}$-space. Moreover, we introduce and study the concepts of two new classes of fuzzy mappings, namely, fuzzy $W_\theta g$-continuous mappings and fuzzy $W_\delta g$-irresolute mappings.

2. Preliminaries

Let $X$ be a set and $I$ the unit interval. A fuzzy set in $X$ is an element of the set of all functions from $X$ into $I$. The family of all fuzzy sets in $X$ is denoted by $I^X$. A fuzzy singleton $x_i$ is a
fuzzy set in $X$ defined by $x_t(x) = t$, $x_t(y) = 0$ for all $y \neq x$, $t \in (0,1]$. The set of all fuzzy singletons in $X$ is denoted by $S(X)$. For every $x_t \in S(X)$ and $\mu \in I^X$, we define $x_t \in \mu$ if and only if $t \leq \mu(x)$. A fuzzy set $\mu$ is called quasicoincident with a fuzzy set $\rho$, denoted by $\mu \ q \ \rho$, if and only if there exists $x \in X$ such that $\mu(x) + \rho(x) > 1$. If $\mu$ is not quasicoincident with $\rho$, then we write $\mu \neq \rho$. By $\text{cl}(\mu)$, $\text{int}(\mu)$, $\mu^\epsilon$, $N(x_t, \tau)$, and $N_Q(x_t, \tau)$, we mean the fuzzy closure of $\mu$, the fuzzy interior of $\mu$, the complement of $\mu$, the class of all open neighborhoods of $x_t$, and the class of all open $Q$-neighborhoods of $x_t$, respectively.

**Definition 2.1** (see [6, 7]). A fuzzy subset $\mu$ of a $[0,1]$-topological space $(X, \tau)$ is called

(i) regular open if and only if $\mu = \text{int(cl}(\mu))$,

(ii) preopen if and only if $\mu \leq \text{int(cl}(\mu))$.

The complement of a regular open (resp. preopen) fuzzy set is called a regular closed (resp. preclosed).

**Definition 2.2** (see [8, 9]). Let $(X, \tau)$ be a $[0,1]$-topological space, $x_t \in S(X)$, and $\mu \in I^X$. Then,

(i) the $\delta$-closure of $\mu$, denoted by $\text{cl}_\delta(\mu)$, is defined by

$$x_t \in \text{cl}_\delta(\mu) \text{ if and only if } \text{cl}(\eta)q\mu \text{ for each } \eta \in N_Q(x_t, \tau),$$

(ii) the $\delta$-closure of $\mu$ denoted by $\text{cl}_\delta(\mu)$, is defined by

$$x_t \in \text{cl}_\delta(\mu) \text{ if and only if } \text{int(cl}(\eta))q\mu \text{ for each } \eta \in N_Q(x_t, \tau),$$

(iii) $\mu$ is called $\theta$-closed (resp. $\delta$-closed) if and only if $\mu = \text{cl}_\theta(\mu)$ (resp. $\mu = \text{cl}_\delta(\mu)$).

**Definition 2.3** (see [9]). Let $(X, \tau)$ be a $[0,1]$-topological space and $\mu \in I^X$. Then,

(i) the family $\gamma = \{\eta_j : j \in J\} \subseteq \tau$ is called an open $P$-cover of $\mu$ if and only if for every $x_t \in \mu$, there exists $j_0 \in J$ such that $x_t \in \eta_{j_0}$,

(ii) $\mu$ is called a C-set if and only if every open $P$-cover of $\mu$ has a finite subcover.

**Definition 2.4** (see [2–4]). Let $(X, \tau)$ be a $[0,1]$-topological space. A fuzzy set $\mu \in I^X$ is called

(i) a generalized closed ($g$-closed, for short) if and only if $\text{cl}(\mu) \leq \eta$ whenever $\mu \leq \eta$ and $\eta$ is open fuzzy set,

(ii) a $\theta$-generalized closed ($\theta g$-closed, for short) if and only if $\text{cl}_\theta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and $\eta$ is open fuzzy set,

(iii) a $\delta$-generalized closed ($\delta g$-closed, for short) if and only if $\text{cl}_\delta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and $\eta$ is open fuzzy set.

**Definition 2.5** (see [2, 4, 6, 10]). A $[0,1]$-topological space $(X, \tau)$ is called

(i) $FR_1$ if and only if $x_t \eta \| \text{cl}(y_r)$ implies that there exist $\eta \in N(x_t, \tau)$ and $v \in N(y_r, \tau)$ such that $\eta \| \nu$,

(ii) $FR_2$ or $F$-regular if and only if $x_t \| \lambda$ is closed fuzzy set implies that there exist $\eta \in N(x_t, \tau)$ and $v \in \tau$, $\lambda \leq v$ such that $\eta \| \nu$,

(iii) $FT_{1/2}$ if and only if every $g$-closed fuzzy set in $X$ is closed,

(iv) $FT_{3/4}$ if and only if every $\delta g$-closed fuzzy set in $X$ is $\delta$-closed,
(v) fuzzy weakly Hausdorff (FWT₂, for short) if \( x_i \tilde{\in} y_r \) implies that there exists regular open fuzzy set \( \eta \in N(x_i, \tau) \) such that \( y_r \tilde{\in} \eta \),

(vi) fuzzy semiregular if and only if the collection of all regular open fuzzy sets in \( X \) forms a base for the \([0,1]\)-topology \( \tau \),

(vii) a fuzzy partition space if and only if every open fuzzy subset is closed.

**Definition 2.6** (see [2-4, 11]). A fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is called

(i) fuzzy generalized continuous (fuzzy \( g \)-continuous, for short) if and only if \( f^{-1}(\eta) \) is \( g \)-closed in \( X \) for any closed fuzzy set \( \eta \) in \( Y \),

(ii) fuzzy \( \theta \)-generalized continuous (fuzzy \( \theta g \)-continuous, for short) if and only if \( f^{-1}(\eta) \) is \( \theta g \)-closed in \( X \) for any closed fuzzy set \( \eta \) in \( Y \),

(iii) fuzzy \( \delta \)-generalized continuous (fuzzy \( \delta g \)-continuous, for short) if and only if \( f^{-1}(\eta) \) is \( \delta g \)-closed in \( X \) for any closed fuzzy set \( \eta \) in \( Y \),

(iv) fuzzy \( \delta \)-continuous if the inverse image of every \( \delta \)-open fuzzy set in \( Y \) is \( \delta \)-open in \( X \),

(v) fuzzy \( \delta \)-open (fuzzy \( \delta \)-open, for short) if and only if \( f(\eta) \) is \( \delta \)-open in \( Y \) for any \( \delta \)-open fuzzy set \( \eta \) in \( X \),

(vi) fuzzy \( \delta \)-closed (fuzzy \( \delta \)-closed, for short) if and only if \( f(\eta) \) is \( \delta \)-closed in \( Y \) for any \( \delta \)-closed fuzzy set \( \eta \) in \( X \).

**Theorem 2.7** (see [3]). A fuzzy subset \( \mu \) of an \( FR_2 \)-fts \( (X, \tau) \) is \( \theta g \)-closed if and only if it is \( g \)-closed.

**Theorem 2.8** (see [3]). Let \( (X, \tau) \) be a \([0,1]\)-topological space. Then, the following conditions are equivalent:

(i) \( (X, \tau) \) is \( FR_1 \)-space,

(ii) for each \( C \)-set \( \mu \in I^X \), \( cl(\mu) = cl_\theta(\mu) \),

(iii) for each \( x_i \in S(X) \), \( cl(x_i) = cl_\theta(x_i) \).

**Theorem 2.9** (see [3, 4]). Let \( (X, \tau) \) be a \([0,1]\)-topological space and \( \mu \in I^X \) be a preopen. Then, \( \mu \) is \( \theta g \)-closed (resp. \( \delta g \)-closed) if and only if it is \( g \)-closed.

**Theorem 2.10** (see [3]). Let \( (X, \tau) \) be a \([0,1]\)-topological space and \( \mu, \eta \in I^X \). Then,

(i) \( cl_\theta(\mu \lor \eta) = cl_\theta(\mu) \lor cl_\theta(\eta) \),

(ii) \( cl_\theta(\mu \land \eta) \leq cl_\theta(\mu) \land cl_\theta(\eta) \).

**Theorem 2.11** (see [4]). Let \( (X, \tau) \) be a fuzzy semiregular space and \( \mu \in I^X \). Then,

(i) \( \mu \) is \( \delta g \)-closed if and only if \( \mu \) is \( g \)-closed,

(ii) If, in addition, \( (X, \tau) \) is \( FIT_{1/2} \), then \( \mu \) is \( \delta g \)-closed if and only if \( \mu \) is closed.

**Theorem 2.12** (see [4]). Let \( (X, \tau) \) be an \( FR_1 \)-space and \( \mu \in I^X \) be a \( C \)-set. Then, \( \mu \) is \( \delta g \)-closed if and only if it is \( g \)-closed.
Theorem 2.13 (see [4]). Let \((X, \tau)\) be a fuzzy partition space and \(\mu \in I^X\). Then, \(\mu\) is \(\delta g\)-closed if and only if it is \(g\)-closed.

Theorem 2.14 (see [4]). Let \((X, \tau)\) be a \([0,1]\)-topological space and \(\mu, \eta \in I^X\). Then,

(i) \(\text{cl}_\delta(\mu \lor \eta) = \text{cl}_\delta(\mu) \lor \text{cl}_\delta(\eta)\),

(ii) \(\text{cl}_\delta(\mu \land \eta) \leq \text{cl}_\delta(\mu) \land \text{cl}_\delta(\eta)\).

Theorem 2.15 (see [4]). A \([0,1]\)-topological space \((X, \tau)\) is \(FT_{3/4}\)-space if for every \(x_i \in S(X)\) either \(x_i\) is \(\delta\)-open or \(x_i\) is closed.

Theorem 2.16 (see [4]). Let \((X, \tau)\) be a \([0,1]\)-topological space. Then, the following conditions are equivalent:

(i) \(X\) is an \(FWT_2\)-space,

(ii) \(x_i = \text{cl}_\delta(x_i)\) for each \(x_i \in S(X)\).

3. \(W\theta g\)-Closed Fuzzy Sets

In this section, we introduce the concept of weakly \(\theta\)-generalized closed fuzzy sets, and we study some of their properties.

Definition 3.1. A fuzzy subset \(\mu\) of a \([0,1]\)-topological space \((X, \tau)\) is said to be weakly \(\theta\)-generalized closed (\(W\theta g\)-closed, for short) if and only if \(\text{cl}_\delta(\mu) \land \eta \leq \eta\) whenever \(\mu \leq \eta\) and \(\eta\) is \(\theta\)-open fuzzy set.

The complement of a \(W\theta g\)-closed fuzzy set is called \(W\theta g\)-open.

Theorem 3.2. Let \((X, \tau)\) be a \([0,1]\)-topological space. Then,

(i) Every \(\theta\)-closed fuzzy set is \(W\theta g\)-closed,

(ii) Every \(\theta g\)-closed fuzzy set is \(W\theta g\)-closed.

Proof. Obvious. \(\square\)

From the above discussion, we introduce the following diagram.

\[
\begin{array}{ccc}
W\theta g\text{-closed} & \longrightarrow & \theta g\text{-closed} \\
\text{\(\theta\)-closed} & \longrightarrow & \\
\end{array}
\]

(3.1)

None of these implications is reversible as the following examples show.

Example 3.3. Let \(X = \{x, y\}\) and \(\tau = \{0_X, y_{0,7}, 1_X\}\). If \(\mu = x_{0,5} \lor y_{0,6}\), then \(\mu\) is \(W\theta g\)-closed fuzzy set but not \(\theta\)-closed.

Example 3.4. Let \(X = \{x\}\) and \(\tau = \{0_X, x_{0,6}, 1_X\}\). If \(\mu = x_{0,5}\), then \(\mu\) is \(W\theta g\)-closed, since the only \(\theta\)-open superset of \(\mu\) is \(1_X\). But \(\mu\) is not \(\theta g\)-closed, since \(\mu \leq x_{0,6}\) and \(\text{cl}_\delta(\mu) = 1_X \nsubseteq x_{0,6}\).
Theorem 3.5. A fuzzy subset \( \mu \) of a \([0,1]\)-topological space \((X, \tau)\) is \(W\theta g\)-closed if for every \( x_i \in S(X) \) such that \( x_i \text{cl}_\theta(\mu) \), one has \( \text{cl}_\theta(x_i) \mu \).

Proof. Let \( \eta \) be \( \theta \)-open and \( \mu \leq \eta \). If \( x_i \text{cl}_\theta(\mu) \), then by assumption, \( \text{cl}_\theta(x_i) \mu \). Hence, there exists \( y \in X \) such that \( \text{cl}_\theta(x_i)(y) + \mu(y) > 1 \). Put \( \text{cl}_\theta(x_i)(y) = \varepsilon \). Then, \( y_x \in \text{cl}_\theta(x_i) \) and \( y_x \mu \). Thus, \( \rho q x_i \) for each \( \rho \in \mathcal{N}(\varepsilon, \tau_0) \). Since \( y_x \mu \), then \( \eta q x_i \) and so \( \text{cl}_\theta(\mu) \leq \eta \). Thus, \( \mu \) is \( W\theta g \)-closed.

Theorem 3.6. Let \((X, \tau)\) be a \([0,1]\)-topological space and \( \mu \in I^X \). Then, \( \mu \) is \( W\theta g \)-closed if there is not any \( \theta \)-closed fuzzy set \( \lambda \) such that \( \lambda \not\subseteq \mu \) and \( \lambda q \text{cl}_\theta(\mu) \).

Proof. Suppose that \( \mu \) is not \( W\theta g \)-closed. Then, there exists \( \theta \)-open fuzzy set \( \eta \) such that \( \mu \leq \eta \) and \( \text{cl}_\theta(\mu) \notin \eta \). Put \( \lambda = \eta' \). Then, there exists \( \theta \)-closed fuzzy set \( \lambda \) such that \( \lambda \not\subseteq \mu \) and \( \lambda q \text{cl}_\theta(\mu) \). This is a contradiction.

Theorem 3.7. Let \((X, \tau)\) be an \( FR_1 \)-space and \( \mu \in I^X \) be a \( C \)-set and \( g \)-closed. Then, \( \mu \) is \( W\theta g \)-closed.

Proof. Suppose that \((X, \tau)\) is an \( FR_1 \)-space and \( \mu \) is a \( C \)-set in \( X \). If \( \mu \) is \( g \)-closed, then by Theorem 2.8 \( \mu \) is \( \theta g \)-closed and hence \( W\theta g \)-closed.

Theorem 3.8. Let \((X, \tau)\) be a \([0,1]\)-topological space and \( \mu \in I^X \) be a \( \text{preopen} \) and \( g \)-closed. Then, \( \mu \) is \( W\theta g \)-closed.

Proof. It is an immediate consequence of Theorems 2.9 and 3.2.

Theorem 3.9. Let \((X, \tau)\) be an \( FR_2 \)-space and \( \mu \in I^X \) be a \( g \)-closed. Then, \( \mu \) is \( W\theta g \)-closed.

Proof. It is an immediate consequence of Theorems 2.7 and 3.2.

Theorem 3.10. A finite union of \( W\theta g \)-closed fuzzy sets, is always \( W\theta g \)-closed fuzzy set.

Proof. Suppose that \( \mu, \eta \in I^X \) are \( W\theta g \)-closed fuzzy sets and let \( v \in \tau_0 \) such that \( \mu \vee \eta \leq v \). Since \( \mu \) and \( \eta \) are \( W\theta g \)-closed, then we have \( \text{cl}_\theta(\mu) \vee \text{cl}_\theta(\eta) \leq v \) and by Theorem 2.10(i) \( \text{cl}_\theta(\mu \vee \eta) \leq v \). Hence, \( \mu \vee \eta \) is \( W\theta g \)-closed.

4. \( W\delta g \)-Closed Fuzzy Sets

In this section, we introduce the concept of weakly \( \delta \)-generalized closed fuzzy sets, and we study some of their properties. Also, we introduce the notion of \( FT^*_{3/4} \)-space, and we prove that every \( FT^*_{3/4} \)-space is a \( FT^*_{3/4} \)-space.

Definition 4.1. A fuzzy subset \( \mu \) of \([0,1]\)-topological space \((X, \tau)\) is said to be weakly \( \delta \)-generalized closed (\( W\delta g \)-closed, for short) if and only if \( \text{cl}_\delta(\mu) \leq \eta \) whenever \( \mu \leq \eta \) and \( \eta \) is \( \delta \)-open fuzzy set.

The complement of a \( W\delta g \)-closed fuzzy set is called \( W\delta g \)-open.

Theorem 4.2. Let \((X, \tau)\) be a \([0,1]\)-topological space. Then,

(i) Every \( \delta \)-closed fuzzy set is \( W\delta g \)-closed,

(ii) Every \( \delta g \)-closed fuzzy set is \( W\delta g \)-closed.
Proof. Obvious.

From the above discussion, we introduce the following diagram.

```
\[\text{W}δg\text{-closed} \quad \delta\text{-closed} \quad \delta g\text{-closed}\]
```

None of these implications is reversible as the following examples show.

**Example 4.3.** Let \(X = \{x\}\) and \(\tau = \{0_X, x_{0.8}, 1_X\}\). If \(\mu = x_{0.7}\), then \(\mu\) is \(Wδg\)-closed, since the only \(δ\)-open superset of \(\mu\) is \(1_X\). But \(\mu\) is not \(δg\)-closed, since \(\mu \leq x_{0.8}\) and \(\text{cl}_g(\mu) = 1_X \neq x_{0.8}\).

**Example 4.4.** Let \(X = \{x, y\}\) and \(\tau = \{0_X, y_{0.8}, 1_X\}\). A fuzzy subset \(\mu = x_{0.2} \lor y_{0.3}\) is \(δg\)-closed and hence \(Wδg\)-closed, but it is not \(δ\)-closed.

**Theorem 4.5.** A fuzzy subset \(\mu\) of a \([0,1]\)-topological space \((X, \tau)\) is \(Wδg\)-closed if and only if for every \(x_i \in S(X)\) such that \(x_i q\text{cl}_g(\mu)\) one has \(\text{cl}_g(x_i)q\mu\).

**Proof.** Let \(x_i q\text{cl}_g(\mu)\) and suppose that \(\text{cl}_g(x_i)q\mu\). Since \(\mu\) is \(Wδg\)-closed, then it is easy to observe that \(\text{cl}_g(x_i)q\text{cl}_g(\mu)\) which implies that \(x_i q\text{cl}_g(\mu)\). This is a contradiction.

The converse is similar to the proof of Theorem 3.5.

**Theorem 4.6.** Let \((X, \tau)\) be a \([0,1]\)-topological space and \(\mu \in I^X\). Then, \(\mu\) is \(Wδg\)-closed if and only if there is not any \(δ\)-closed fuzzy set \(\lambda\) such that \(\lambda q\mu\) and \(\lambda q\text{cl}_g(\mu)\).

**Proof.** Suppose that there is a \(δ\)-closed fuzzy set \(\lambda\) such that \(\lambda q\mu\) and \(\lambda q\text{cl}_g(\mu)\). Then, there exists some \(x_i \in \lambda\) such that \(x_i q\text{cl}_g(\mu)\). Since \(\mu\) is \(Wδg\)-closed, then by using Theorem 4.5, \(\text{cl}_g(x_i)q\mu\) and hence \(\text{cl}_g(\lambda)q\mu\). Since \(\lambda\) is \(δ\)-closed, then we have \(\lambda q\mu\). This is a contradiction.

The converse is similar to the proof of Theorem 3.6.

**Theorem 4.7.** Let \((X, \tau)\) be a fuzzy semiregular space and \(\mu \in I^X\). Then,

(i) \(\mu\) is \(Wδg\)-closed if and only if it is \(δg\)-closed,

(ii) If, in addition, \((X, \tau)\) is \(FT_{1/2}\)-space, then \(\mu\) is \(Wδg\)-closed if and only if it is closed.

**Proof.** (i) Since \((X, \tau)\) is semiregular space, then \(\tau = \tau_δ\), and so \(\mu\) is \(Wδg\)-closed if and only if it is \(δg\)-closed.

(ii) From (i), Theorem 2.11, and by \(FT_{1/2}\)-ness, the result is given.

**Theorem 4.8.** Let \((X, \tau)\) be an \(FR_1\)-space and \(\mu \in I^X\) be a \(C\)-set and \(g\)-closed. Then, \(\mu\) is \(Wδg\)-closed.

**Proof.** Suppose that \((X, \tau)\) is an \(FR_1\)-space and \(\mu\) is a \(C\)-set in \(X\). If \(\mu\) is \(g\)-closed, then by Theorem 2.12 \(\mu\) is \(δg\)-closed and hence \(Wδg\)-closed.

**Theorem 4.9.** Let \((X, \tau)\) be a \([0,1]\)-topological space and \(\mu \in I^X\) be a preopen and \(g\)-closed. Then, \(\mu\) is \(Wδg\)-closed.

**Proof.** It is an immediate consequence of Theorems 2.9 and 4.2.
Theorem 4.10. Every fuzzy subset of a fuzzy partition space \((X, \tau)\) is \(W\delta g\)-closed.

Proof. Let \((X, \tau)\) be a fuzzy partition space, and let \(\mu\) be a fuzzy subset of \(X\). Then, by Theorem 2.13, \(\mu\) is \(\delta g\)-closed and hence, by Theorem 4.2, \(\mu\) is \(W\delta g\)-closed. \(\Box\)

Theorem 4.11. A finite union of \(W\delta g\)-closed fuzzy sets is always \(W\delta g\)-closed fuzzy set.

Proof. Similar to the proof of Theorem 3.10. \(\Box\)

The following example shows that the finite intersection of \(W\delta g\)-closed fuzzy set may fail to be \(W\delta g\)-closed fuzzy set.

Example 4.12. Let \(X = \{a, b, c, d, e\}\). Define \(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 : X \to [0, 1]\) as follows:

\[
\begin{align*}
\lambda_1(a) &= 1, & \lambda_1(b) &= 1, & \lambda_1(c) &= 0, & \lambda_1(d) &= 0, & \lambda_1(e) &= 0, \\
\lambda_2(a) &= 0, & \lambda_2(b) &= 0, & \lambda_2(c) &= 1, & \lambda_2(d) &= 0, & \lambda_2(e) &= 0, \\
\lambda_3(a) &= 1, & \lambda_3(b) &= 1, & \lambda_3(c) &= 1, & \lambda_3(d) &= 0, & \lambda_3(e) &= 0, \\
\lambda_4(a) &= 1, & \lambda_4(b) &= 0, & \lambda_4(c) &= 1, & \lambda_4(d) &= 1, & \lambda_4(e) &= 0, \\
\lambda_5(a) &= 0, & \lambda_5(b) &= 1, & \lambda_5(c) &= 1, & \lambda_5(d) &= 0, & \lambda_5(e) &= 1.
\end{align*}
\]

Consider the \([0, 1]\)-topology \(\tau = \{0_X, \lambda_1, \lambda_2, \lambda_3, 1_X\}\). It is clear that \(\lambda_4\) and \(\lambda_5\) are \(W\delta g\)-closed fuzzy sets. But \(\lambda_4 \land \lambda_5 = \lambda_2\) is not \(W\delta g\)-closed.

Definition 4.13. A \([0, 1]\)-topological space \((X, \tau)\) is called \(FT^*_3/4\)-space if and only if every \(W\delta g\)-closed fuzzy set is \(\delta\)-closed.

Theorem 4.14. Every \(FT^*_3/4\)-space is \(FT_3/4\)-space.

Proof. It is an immediate consequence of Theorem 4.2(ii). \(\Box\)

Theorem 4.15. A \([0, 1]\)-topological space \((X, \tau)\) is \(FT^*_3/4\)-space if for every \(x_i \in S(X)\) either \(x_i\) is \(\delta\)-open or \(\delta\)-closed.

Proof. Let \(\mu \in F^X\) be \(W\delta g\)-closed, and let \(x_i \bar{\nu} \mu\). We consider the following two cases.

Case 1. \(x_i\) is \(\delta\)-open. Then, \(x_i^c\) is \(\delta\)-closed. Since \(x_i \bar{\nu} \mu\), then \(\mu \leq x_i^c\). But \(x_i^c\) is \(\delta\)-closed. Then, \(cl_\delta(\mu) \leq x_i^c\). This shows that \(x_i \bar{\nu} cl_\delta(\mu)\).

Case 2. \(x_i\) is \(\delta\)-closed. Then, \(x_i^c\) is \(\delta\)-open. Since \(x_i \bar{\nu} \mu\), then \(\mu \leq x_i^c\). But \(\mu\) is \(W\delta g\)-closed. Then, \(cl_\delta(\mu) \leq x_i^c\) and hence \(x_i \bar{\nu} cl_\delta(\mu)\). \(\Box\)

Corollary 4.16. Every \(FWT_2\)-space is \(FT^*_3/4\)-space.

Proof. This is an immediate consequence of Theorems 2.16 and 4.15.

The converse of Corollary 4.16 need not be true, in general, and as a sample, we give the following example.
Example 4.17. Let $X = \{a, b, c\}$. Define $\lambda_1, \lambda_2, \lambda_3 : X \to [0, 1]$ as follows:

\[
\begin{align*}
\lambda_1(a) &= 1, & \lambda_1(b) &= 0, & \lambda_1(0) &= 0, \\
\lambda_2(a) &= 0, & \lambda_2(b) &= 1, & \lambda_2(0) &= 0, \\
\lambda_3(a) &= 1, & \lambda_3(b) &= 1, & \lambda_3(0) &= 0.
\end{align*}
\]

Consider the $[0, 1]$-topology $\tau = \{0_X, \lambda_1, \lambda_2, \lambda_3, 1_X\}$. Then, $X$ is $FT_{3/4}$-space but not $FWT_2$-space.

Theorem 4.18. Let $(X, \tau)$ be a $[0, 1]$-topological space. Then, the following conditions are equivalent:

(i) $X$ is $FWT_2$-space,

(ii) $X$ is $FT_{3/4}^*$ and each $x_i \in S(X)$ is $W\delta g$-closed.

Proof. Obvious. □

5. $W\theta g$-Continuous and $W\delta g$-Continuous Mappings

Definition 5.1. A fuzzy mapping $f : (X, \tau) \to (Y, \Delta)$ is called

(i) fuzzy $W\theta g$-continuous if the inverse image of every closed fuzzy set in $Y$ is $W\theta g$-closed fuzzy set in $X$,

(ii) fuzzy $W\delta g$-continuous if the inverse image of every closed fuzzy set in $Y$ is $W\delta g$-closed fuzzy set in $X$.

Theorem 5.2. Every fuzzy $\theta g$-continuous (resp. $\delta g$-continuous) mapping is fuzzy $W\theta g$-continuous (resp. $W\delta g$-continuous).

Proof. Obvious. □

The converse of the above Theorem may not be true, in general, by the following example.

Example 5.3. Let $X = Y = \{x\}$ and consider a $[0, 1]$-topology $\tau$ of Example 4.3, $\Delta = \{0_Y, x_{0.3}, 1_Y\}$. If $f : (X, \tau) \to (Y, \Delta)$ is the identity fuzzy mapping, then $f$ is fuzzy $W\theta g$-continuous but not fuzzy $\theta g$-continuous, since $x_{0.7} \in \Delta'$ and $f^{-1}(x_{0.7}) = x_{0.7} \leq x_{0.8} \in \tau$ but $cl_\theta(x_{0.7}) = 1X \not\in x_{0.8}$. Also, $f$ is fuzzy $W\theta g$-continuous but not fuzzy $\delta g$-continuous.

Theorem 5.4. Let $f : (X, \tau) \to (Y, \Delta)$ be fuzzy mapping and $(X, \tau)$ be fuzzy semiregular space. Then, the following conditions are equivalent:

(i) $f$ is fuzzy $W\delta g$-continuous,

(ii) $f$ is fuzzy $\delta g$-continuous,

(iii) $f$ is fuzzy $g$-continuous.

Proof. It follows directly from Theorems 2.11 and 4.7(i). □
Definition 5.5. A fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is called

(i) fuzzy \( W{\theta}g \)-irresolute if the inverse image of every \( W{\theta}g \)-closed fuzzy set in \( Y \) is
\( W{\theta}g \)-closed fuzzy set in \( X \),

(ii) fuzzy \( W{\delta}g \)-irresolute if the inverse image of every \( W{\delta}g \)-closed fuzzy set in \( Y \) is
\( W{\delta}g \)-closed fuzzy set in \( X \).

Theorem 5.6. Let \( f : (X, \tau) \rightarrow (Y, \Delta) \) and \( g : (Y, \Delta) \rightarrow (Z, \Omega) \) be two fuzzy mappings. Then,

(i) \( g \circ f \) is fuzzy \( W{\theta}g \)-continuous if \( g \) is fuzzy continuous and \( f \) is fuzzy \( W{\theta}g \)-continuous,

(ii) \( g \circ f \) is fuzzy \( W{\delta}g \)-irresolute if \( g \) is fuzzy \( W{\delta}g \)-irresolute and \( f \) is fuzzy \( W{\delta}g \)-irresolute,

(iii) \( g \circ f \) is fuzzy \( W{\theta}g \)-continuous if \( g \) is fuzzy \( W{\theta}g \)-continuous and \( f \) is fuzzy \( W{\delta}g \)-irresolute.

Proof. Obvious.

Theorem 5.7. Let \( f : (X, \tau) \rightarrow (Y, \Delta) \) and \( g : (Y, \Delta) \rightarrow (Z, \Omega) \) be two fuzzy mappings. Then,

(i) \( g \circ f \) is fuzzy \( W{\delta}g \)-continuous if \( g \) is fuzzy continuous and \( f \) is fuzzy \( W{\delta}g \)-continuous,

(ii) \( g \circ f \) is fuzzy \( W{\delta}g \)-irresolute if \( g \) is fuzzy \( W{\delta}g \)-irresolute and \( f \) is fuzzy \( W{\delta}g \)-irresolute,

(iii) \( g \circ f \) is fuzzy \( W{\delta}g \)-continuous if \( g \) is fuzzy \( W{\delta}g \)-continuous and \( f \) is fuzzy \( W{\delta}g \)-irresolute,

(iv) Let \((Y, \Delta)\) be \( FT_{3/4}^*\) space. Then, \( g \circ f \) is fuzzy \( W{\delta}g \)-continuous if \( g \) is fuzzy \( W{\delta}g \)-continuous and \( f \) is fuzzy \( W{\delta}g \)-continuous,

(v) Let \((Y, \Delta)\) be a fuzzy semiregular space. Then, \( g \circ f \) is fuzzy \( W{\delta}g \)-continuous if \( g \) is fuzzy \( g \)-continuous and \( f \) is fuzzy \( W{\delta}g \)-irresolute.

Proof. Obvious.

Definition 5.8. A fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is called fuzzy \( \theta \)-open if and only if \( f(\eta) \) is \( \theta \)-open in \( Y \) for any \( \theta \)-open fuzzy set \( \eta \) in \( X \).

Theorem 5.9. If a fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is bijective, fuzzy \( \theta \)-open, and fuzzy \( W{\theta}g \)-continuous, then \( f \) is fuzzy \( W{\theta}g \)-irresolute.

Proof. Let \( \lambda \) be \( W{\theta}g \)-closed fuzzy set in \( Y \), and let \( f^{-1}(\lambda) \leq \eta \), where \( \eta \in \tau_{\theta} \). Clearly, \( \lambda \leq f(\eta) \). Since \( f(\eta) \in \Delta_{\theta} \) and \( \lambda \) is \( W{\theta}g \)-closed in \( Y \), then \( cl_{\theta}(\lambda) \leq f(\eta) \) and thus \( f^{-1}(cl_{\theta}(\lambda)) \leq \eta \). Since \( f \) is fuzzy \( W{\theta}g \)-continuous and \( cl_{\theta}(\lambda) \) is closed in \( Y \), then \( f^{-1}(cl_{\theta}(\lambda)) \) is \( W{\theta}g \)-closed in \( X \) and hence \( cl_{\theta}(f^{-1}(cl_{\theta}(\lambda))) \leq \eta \). Thus, \( cl_{\theta}(f^{-1}(\lambda)) \leq \eta \) and so \( f^{-1}(\lambda) \) is \( W{\theta}g \)-closed in \( X \). This shows that \( f \) is fuzzy \( W{\theta}g \)-irresolute.

Theorem 5.10. If a fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is bijective, fuzzy \( \delta \)-open, and fuzzy \( W{\delta}g \)-continuous, then \( f \) is fuzzy \( W{\delta}g \)-irresolute.

Proof. Similar to the proof of Theorem 5.9.

Theorem 5.11. If a fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \Delta) \) is fuzzy \( W{\delta}g \)-irresolute and \( (X, \tau) \) is \( FT_{3/4}^* \) space, then \( f \) is fuzzy \( \delta \)-continuous.
Proof. Let $\mu$ be a $\delta$-closed fuzzy set in $Y$. By using Theorem 4.2, $\mu$ is $W\delta g$-closed. Since $f$ is fuzzy $W\delta g$- irresolute, then $f^{-1}(\mu)$ is $W\delta g$-closed in $X$. Since $X$ is $FT^{*}_{3/4}$-space, then $f^{-1}(\mu)$ is $\delta$-closed in $X$. Thus, $f$ is fuzzy $\delta$-continuous. 

**Theorem 5.12.** If a mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is fuzzy $\delta$-continuous and fuzzy $\delta$-closed, and $\mu$ is $W\delta g$-closed fuzzy set in $X$, then $f(\mu)$ is $W\delta g$-closed in $X$.

Proof. Let $\mu$ be $W\delta g$-closed in $X$, and let $f(\mu) \leq \eta$, where $\eta$ is $\delta$-open fuzzy set in $Y$. Since $\mu \leq f^{-1}(\eta)$, $\mu$ is $W\delta g$-closed fuzzy set in $X$ and since $f^{-1}(\eta)$ is $\delta$-open in $X$, then $cl_\delta(\mu) \leq f^{-1}(\eta)$. Thus $f(cl_\delta(\mu)) \leq \eta$. Hence, $cl_\delta(f(\mu)) \leq cl_\delta(f(cl_\delta(\mu))) = f(cl_\delta(\mu)) \leq \eta$, since $f$ is fuzzy $\delta$-closed. Hence, $f(\mu)$ is $W\delta g$-closed in $Y$.

**Theorem 5.13.** Let $(X, \tau)$ be an $FT^{*}_{3/4}$-space. If a fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ be surjective, fuzzy $W\delta g$- irresolute, and fuzzy $\delta$-closed, then $(Y, \Delta)$ is also $FT^{*}_{3/4}$-space.

Proof. Let $\mu$ be $W\delta g$-closed fuzzy set in $Y$. Since $f$ is fuzzy $W\delta g$- irresolute, then $f^{-1}(\mu)$ is $W\delta g$-closed in $X$. Since $X$ is $FT^{*}_{3/4}$-space, then $f^{-1}(\mu)$ is $\delta$-closed in $X$. Thus, $\mu$ is $\delta$-closed in $Y$, since $f$ is surjective and fuzzy $\delta$-closed. Hence, $(Y, \Delta)$ is $FT^{*}_{3/4}$-space.

**References**


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