Research Article

Commutative Pseudo Valuations on BCK-Algebras

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Received 26 September 2010; Accepted 10 November 2010

Academic Editor: Young Bae Jun

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The notion of a commutative pseudo valuation on a BCK-algebra is introduced, and its characterizations are investigated. The relationship between a pseudo valuation and a commutative pseudo valuation is examined.

1. Introduction


In this paper, we introduce the notion of a commutative pseudo valuation on a BCK-algebra, and investigate its characterizations. We discuss the relationship between a pseudo valuation and a commutative pseudo valuation. We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

2. Preliminaries

A BCK-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.
An algebra \((X; *, 0)\) of type \((2,0)\) is called a \textit{BCI-algebra} if it satisfies the following axioms:

(i) \((\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),\)

(ii) \((\forall x, y \in X) ((x * (x * y)) * y = 0),\)

(iii) \((\forall x \in X) (x * x = 0),\)

(iv) \((\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y).\)

If a BCI-algebra \(X\) satisfies the following identity:

(v) \((\forall x \in X) (0 * x = 0),\)

then \(X\) is called a \textit{BCK-algebra}. Any BCK/BCI-algebra \(X\) satisfies the following conditions:

(a1) \((\forall x \in X) (x * 0 = x),\)

(a2) \((\forall x, y, z \in X) (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0),\)

(a3) \((\forall x, y, z \in X) ((x * y) * z = (x * z) * y),\)

(a4) \((\forall x, y, z \in X) (((x * z) * (y * z)) * (x * y) = 0).\)

We can define a partial ordering \(\leq\) by \(x \leq y\) if and only if \(x * y = 0\).

A BCK-algebra \(X\) is said to be \textit{commutative} if \(x \land y = y \land x\) for all \(x, y \in X\) where \(x \land y = y * (y * x)\).

A subset \(A\) of a BCK/BCI-algebra \(X\) is called an \textit{ideal} of \(X\) if it satisfies the following conditions:

(b1) \(0 \in A,\)

(b2) \((\forall x, y \in X) (x * y \in A, y \in A \Rightarrow x \in A).\)

A subset \(A\) of a BCK-algebra \(X\) is called a \textit{commutative ideal} of \(X\) (see [6]) if it satisfies (b1) and

(b3) \((\forall x, y, z \in X) ((x * y) * z \in A, z \in A \Rightarrow x * (y \land z) \in A).\)

We refer the reader to the book in [7] for further information regarding BCK-algebras.

#### 3. Commutative Pseudo Valuations on BCK-Algebras

In what follows let \(X\) denote a BCK-algebra unless otherwise specified.

\textbf{Definition 3.1} (see [4]). A real-valued function \(\varphi\) on \(X\) is called a \textit{weak pseudo valuation} on \(X\) if it satisfies the following condition:

\[(c1) (\forall x, y \in X)(\varphi(x * y) \leq \varphi(x) + \varphi(y)).\]

\textbf{Definition 3.2} (see [4]). A real-valued function \(\varphi\) on \(X\) is called a \textit{pseudo valuation} on \(X\) if it satisfies the following two conditions:

\[(c2) \varphi(0) = 0,\]

\[(c3) (\forall x, y \in X)(\varphi(x) \leq \varphi(x * y) + \varphi(y)).\]
Proposition 3.3 (see [4]). For any pseudo valuation \( \varphi \) on \( X \), one has the following assertions:

1. \( \varphi(x) \geq 0 \) for all \( x \in X \).
2. \( \varphi \) is order preserving;
3. \( \varphi(x \ast y) \leq \varphi(x \ast z) + \varphi(z \ast y) \) for all \( x, y, z \in X \).

Definition 3.4. A real-valued function \( \varphi \) on \( X \) is called a \textit{commutative pseudo valuation} on \( X \) if it satisfies (c2) and

\[(c4) \ (\forall x, y, z \in X) \ (\varphi(x \ast (y \land x)) \leq \varphi((x \ast y) \ast z) + \varphi(z)).\]

Example 3.5. Let \( X = \{0, a, b, c\} \) be a BCK-algebra with the \( \ast \)-operation given by Table 1. Let \( \vartheta \) be a real-valued function on \( X \) defined by

\[
\vartheta = \begin{pmatrix}
0 & a & b & c \\
0 & 7 & 9 & 9
\end{pmatrix}
\] (3.1)

Routine calculations give that \( \vartheta \) is a commutative pseudo valuation on \( X \).

Theorem 3.6. In a BCK-algebra, every commutative pseudo valuation is a pseudo valuation.

Proof. Let \( \varphi \) be a commutative pseudo valuation on \( X \). For any \( x, y, z \in X \), we have

\[
\varphi(x) = \varphi(x \ast (0 \land x)) \leq \varphi((x \ast 0) \ast z) + \varphi(z) = \varphi(x \ast z) + \varphi(z).
\] (3.2)

This completes the proof. \( \square \)

Combining Theorem 3.6 and [4, Theorem 3.9], we have the following corollary.

Corollary 3.7. In a BCK-algebra, every commutative pseudo valuation is a weak pseudo valuation.

The converse of Theorem 3.6 may not be true as seen in the following example.

Example 3.8. Let \( X = \{0, a, b, c, d\} \) be a BCK-algebra with the \( \ast \)-operation given by Table 2. Let \( \vartheta \) be a real-valued function on \( X \) defined by

\[
\vartheta = \begin{pmatrix}
0 & a & b & c & d \\
0 & 5 & 8 & 8 & 8
\end{pmatrix}
\] (3.3)
Then $\vartheta$ is a pseudo valuation on $X$. Since

$$\vartheta(b \ast (c \wedge b)) = 8 \neq 0 = \vartheta((b \ast c) \ast 0) + \vartheta(0), \quad (3.4)$$

$\vartheta$ is not a commutative pseudo valuation on $X$.

We provide conditions for a pseudo valuation to be a commutative pseudo valuation.

**Theorem 3.9.** For a real-valued function $\varphi$ on $X$, the following are equivalent:

1. $\varphi$ is a commutative pseudo valuation on $X$.
2. $\varphi$ is a pseudo valuation on $X$ that satisfies the following condition:

$$\forall x, y \in X \quad (\varphi(x \ast (y \wedge x)) \leq \varphi(x \ast y)). \quad (3.5)$$

**Proof.** Assume that $\varphi$ is a commutative pseudo valuation on $X$. Then $\varphi$ is a pseudo valuation on $X$ by Theorem 3.6. Taking $z = 0$ in (c4) and using (a1) and (c2) induce the condition (3.5).

Conversely let $\varphi$ be a pseudo valuation on $X$ satisfying the condition (3.5). Then $\varphi(x \ast y) \leq \varphi((x \ast y) \ast z) + \varphi(z)$ for all $x, y, z \in X$. It follows from (3.5) that

$$\varphi(x \ast (y \wedge x)) \leq \varphi(x \ast y) \leq \varphi((x \ast y) \ast z) + \varphi(z) \quad (3.6)$$

for all $x, y, z \in X$ so that $\varphi$ is a commutative pseudo valuation on $X$. \qed

**Lemma 3.10** (see [8]). Every pseudo valuation $\varphi$ on $X$ satisfies the following implication:

$$\forall x, y, z \in X \quad ((x \ast y) \ast z = 0 \Rightarrow \varphi(x) \leq \varphi(y) + \varphi(z)). \quad (3.7)$$

**Theorem 3.11.** In a commutative BCK-algebra, every pseudo valuation is a commutative pseudo valuation.

**Proof.** Let $\varphi$ be a pseudo valuation on a commutative BCK-algebra $X$. Note that

$$((x \ast (y \wedge x)) \ast ((x \ast y) \ast z)) \ast z = ((x \ast (y \wedge x)) \ast z) \ast ((x \ast y) \ast z)$$

$$\leq (x \ast (y \wedge x)) \ast (x \ast y) \quad (3.8)$$

$$= (x \wedge y) \ast (y \wedge x) = 0$$

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International Journal of Mathematics and Mathematical Sciences

for all \(x, y, z \in X\). Hence \(((x \ast (y \land x)) \ast ((x \ast y) \ast z)) \ast z = 0\) for all \(x, y, z \in X\). It follows from Lemma 3.10 that \(\varphi((x \ast (y \land x)) \leq \varphi(x \ast y) + \varphi(z)\) for all \(x, y, z \in X\). Therefore \(\varphi\) is a commutative pseudo valuation on \(X\).

For any real-valued function \(\varphi\) on \(X\), we consider the set

\[I_\varphi := \{x \in X \mid \varphi(x) = 0\}, \tag{3.9}\]

Lemma 3.12 (see [4]). If \(\varphi\) is a pseudo valuation on \(X\), then the set \(I_\varphi\) is an ideal of \(X\).

Lemma 3.13 (see [7]). For any nonempty subset \(I\) of \(X\), the following are equivalent:

1. \(I\) is a commutative ideal of \(X\).
2. \(I\) is an ideal of \(X\) that satisfies the following condition:

\[(\forall x, y \in X) \ (x \ast y \in I \implies x \ast (y \land x) \in I). \tag{3.10}\]

Theorem 3.14. If \(\varphi\) is a commutative pseudo valuation on \(X\), then the set \(I_\varphi\) is a commutative ideal of \(X\).

Proof. Let \(\varphi\) be a commutative pseudo valuation on a BCK-algebra \(X\). Using Theorem 3.6 and Lemma 3.12, we conclude that \(I_\varphi\) is an ideal of \(X\). Let \(x, y \in X\) be such that \(x \ast y \in I_\varphi\). Then \(\varphi(x \ast y) = 0\). It follows from (3.5) that \(\varphi((x \ast (y \land x)) \leq \varphi(x \ast y) = 0\) so that \(\varphi((x \ast (y \land x)) = 0\). Hence \(x \ast (y \land x) \in I_\varphi\). Therefore \(I_\varphi\) is a commutative ideal of \(X\) by Lemma 3.13.

The following example shows that the converse of Theorem 3.14 is not true.

Example 3.15. Consider a BCK-algebra \(X = \{0, a, b, c\}\) with the \(\ast\)-operation given by Table 3. Let \(\varphi\) be a real-valued function on \(X\) defined by

\[\varphi = \begin{pmatrix} 0 & a & b & c \\ 0 & 3 & 7 & 0 \end{pmatrix}. \tag{3.11}\]

Then \(I_\varphi = \{0, c\}\) is a commutative ideal of \(X\). Since

\[\varphi(b) = 7 > 6 = \varphi(b \ast a) + \varphi(a), \tag{3.12}\]

\(\varphi\) is not a pseudo valuation on \(X\) and so \(\varphi\) is not a commutative pseudo valuation on \(X\).

Using an ideal, we establish a pseudo valuation.

Theorem 3.16. For any ideal \(I\) of \(X\), we define a real-valued function \(\varphi_I\) on \(X\) by

\[\varphi_I(x) = \begin{cases} 0 & \text{if } x = 0, \\ t_1 & \text{if } x \in I \setminus \{0\}, \\ t_2 & \text{if } x \in X \setminus I \end{cases} \tag{3.13}\]
for all \( x \in X \) where \( 0 < t_1 < t_2 \). Then \( \varphi_I \) is a pseudo valuation on \( X \).

Proof. Let \( x, y \in X \). If \( x = 0 \), then \( \varphi_I(x) \leq \varphi_I(x \ast y) + \varphi_I(y) \). Assume that \( x \neq 0 \). If \( y = 0 \), then \( \varphi_I(x) \leq \varphi_I(x \ast y) + \varphi_I(y) \). If \( y \neq 0 \), we consider the following four cases:

(i) \( x \ast y \in I \) and \( y \in I \),
(ii) \( x \ast y \notin I \) and \( y \notin I \),
(iii) \( x \ast y \in I \) and \( y \notin I \),
(iv) \( x \ast y \notin I \) and \( y \in I \).

Case (i) implies that \( x \in I \) because \( I \) is an ideal of \( X \). If \( x \ast y = 0 \), then \( \varphi_I(x \ast y) = 0 \) and so \( \varphi_I(x) = t_1 = \varphi_I(x \ast y) + \varphi_I(y) \). If \( x \ast y \neq 0 \), then \( \varphi_I(x \ast y) = t_1 \) and thus \( \varphi_I(x) = t_1 \leq \varphi_I(x \ast y) + \varphi_I(y) \). The second case implies that \( \varphi_I(x \ast y) = t_2 \) and \( \varphi_I(y) = t_2 \). Hence \( \varphi_I(x) \leq t_2 < \varphi_I(x \ast y) + \varphi_I(y) \). Let us consider the third case. If \( x \ast y = 0 \), then \( \varphi_I(x \ast y) = 0 \) and thus \( \varphi_I(x) \leq t_2 = \varphi_I(x \ast y) + \varphi_I(y) \). If \( x \ast y \neq 0 \), then \( \varphi_I(x \ast y) = t_1 \) and so \( \varphi_I(x) \leq t_2 < t_1 + t_2 = \varphi_I(x \ast y) + \varphi_I(y) \). For the final case, the proof is similar to the third case. Therefore \( \varphi_I \) is a pseudo valuation on \( X \).

Before ending our discussion, we pose a question.

Question 1. If \( I \) is commutative ideal of \( X \), then is the function \( \varphi_I \) in Theorem 3.16 a commutative pseudo valuation on \( X \)?

Acknowledgment

The authors wish to thank the anonymous reviewers for their valuable suggestions.

References
