Research Article

Some Identities on the Twisted $(h, q)$-Genocchi Numbers and Polynomials Associated with $q$-Bernstein Polynomials

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We give some interesting identities on the twisted $(h, q)$-Genocchi numbers and polynomials associated with $q$-Bernstein polynomials.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, we always make use of the following notations: $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integer, $\mathbb{Q}_p$ denotes the ring of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\mathbb{C}_p^n = \{ \zeta | \zeta^{p^n} = 1 \}$ be the cyclic group of order $p^n$ and let

$$ T_p = \bigcup_{n \geq 1} \mathbb{C}_p^n = \lim_{n \to \infty} \mathbb{C}_p^n = \mathbb{C}_p^\infty. \quad (1.1) $$

The $p$-adic absolute value is defined by $|x| = 1/p^r$, where $x = p^r(s/t) \ (r \in \mathbb{Q}$ and $s, t \in \mathbb{Z}$ with $(s, t) = (p, s) = (p, t) = 1)$. In this paper we assume that $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$ as an indeterminate.
The $q$-number is defined by

$$[x]_q = \frac{1 - q^x}{1 - q} \quad (1.2)$$

(see [1–15]). Note that $\lim_{q \to 1} [x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, Kim defined the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{pN-1} f(x) (-q)^x \quad (1.3)$$

(see [2–6, 8–15]). From (1.3), we note that

$$q^n I_q(f_n) = (-1)^n I_q(f) + [2]_q \sum_{\ell=0}^{n-1} (-1)^{n-\ell} q^{\ell} f(\ell) \quad (1.4)$$

(see [4–6, 8–12]), where $f_n(x) = f(x + n)$ for $n \in \mathbb{N}$. For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$, Kim defined the $q$-Bernstein polynomials of the degree $n$ as follows:

$$B_{k,n}(x, q) = \binom{n}{k} [x]^k [1 - x]^{n-k}_q, \quad (1.5)$$

(see [13–15]). For $h \in \mathbb{Z}$ and $\zeta \in T_p$, let us consider the twisted $(h, q)$-Genocchi polynomials as follows:

$$t \int_{\mathbb{Z}_p} e^{[x + y]_q t} \zeta^y q^{(h-1)y} d\mu_{-q}(y) = \sum_{n=0}^{\infty} C_n^{(h)}(x) \frac{t^n}{n!}. \quad (1.6)$$

Then, $C_n^{(h)}_{n,q,h}(x)$ is called $n$th twisted $(h, q)$-Genocchi polynomials.

In the special case, $x = 0$ and $C_n^{(h)}_{n,q,h}(0) = C_n^{(h)}{n,q,h}$ are called the $n$th twisted $(h, q)$-Genocchi numbers.

In this paper, we give the fermionic $p$-adic integral representation of $q$-Bernstein polynomial, which are defined by Kim [13], associated with twisted $(h, q)$-Genocchi numbers and polynomials. And we construct some interesting properties of $q$-Bernstein polynomials associated with twisted $(h, q)$-Genocchi numbers and polynomials.
2. On the Twisted \((h, q)\)-Genocchi Numbers and Polynomials

From (1.6), we note that

\[
\frac{G^{(h)}_{n+1,q,\xi}(x)}{n+1} = \int_{\mathbb{Z}_p} [x + y]^n q^y q^{(h-1)y} d\mu_q(y)
\]

\[
= \int_{\mathbb{Z}_p} (\left[x\right]_q + q^x [y])_q^n q^y q^{(h-1)y} d\mu_q(y)
\]

\[
= \sum_{\ell=0}^n \left(\binom{n}{\ell}\right)_q [x]_q^{n-\ell} q^x q^{(h-1)y} \int_{\mathbb{Z}_p} [y]_q^\ell q^{(h-1)y} d\mu_q(y)
\]

\[
= \sum_{\ell=0}^n \left(\binom{n}{\ell}\right)_q [x]_q^{n-\ell} q^x \frac{G^{(h)}_{\ell+1,q,\xi}}{q^{\ell+1}}.
\]

We also have

\[
G^{(h)}_{n,q,\xi}(x) = q^{-x} \sum_{\ell=0}^n \left(\binom{n}{\ell}\right)_q [x]_q^{n-\ell} q^x G^{(h)}_{\ell,q,\xi}.
\]

Therefore, we obtain the following theorem.

**Theorem 2.1.** For \(n \in \mathbb{Z}_+\) and \(\xi \in T_p\), one has

\[
G^{(h)}_{n,q,\xi}(x) = q^{-x} \left([x]_q + q^x G^{(h)}_{q,\xi}\right)^n
\]

with usual convention about replacing \((G^{(h)}_{q,\xi})^n\) by \(G^{(h)}_{n,q,\xi}\).

By (1.6) and (2.1) one gets

\[
\frac{G^{(h)}_{n+1,q,\xi}(1-x)}{n+1} = \int_{\mathbb{Z}_p} [1-x + y]_q^n q^y q^{(h-1)y} d\mu_q^{-1}(y)
\]

\[
= \frac{[2]_q}{(1-q^{-1})^n} \sum_{\ell=0}^n \left(\binom{n}{\ell}\right)_q (-1)^n q^{h-1} q^{\ell q^{x}} 1 + q^{h+\ell q^{x}}
\]

\[
= (-1)^n q^{n+h-1} q^{x} \left(\frac{[2]_q}{(1-q)} \sum_{\ell=0}^n \left(\binom{n}{\ell}\right) (-1)^\ell q^{\ell q^{x}} 1 + q^{h+\ell q^{x}}\right)
\]

\[
= (-1)^n q^{n+h-1} q^{x} \frac{G^{(h)}_{n+1,q,\xi}(x)}{n+1}.
\]

Therefore, we obtain the following theorem.
Theorem 2.2. For \( n \in \mathbb{Z} \) and \( \zeta \in T_p \), one has

\[
G_{n, q^{-1}, \zeta^{-1}}(1 - x) = (-1)^{n-1} q^{n-h-2} G_{n, q, \zeta}^{(h)}.
\]

From (1.5), one gets the following recurrence formula:

\[
q^h \zeta G_{n, q, \zeta}^{(h)}(1) + G_{n, q, \zeta}^{(h)} = \begin{cases} [2]_q & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}
\]

Therefore, we obtain the following theorem.

Theorem 2.3. For \( n \in \mathbb{Z} \) and \( \zeta \in T_p \), one has

\[
G_{0, q, \zeta} = 0, \quad q^{h-1} \zeta \left( q G_{q, \zeta}^{(h)} + 1 \right)^n + G_{n, q, \zeta}^{(h)} = \begin{cases} [2]_q & \text{if } n = 1, \\ 0 & \text{if } n > 1 \end{cases}
\]

with usual convention about replacing \((G_{q, \zeta}^{(h)})^n\) by \(G_{n, q, \zeta}^{(h)}\).

From Theorem 2.3, we note that

\[
q^{2h} \zeta^2 G_{n, q, \zeta}^{(h)}(2) - q^h \zeta [2]_q = -q^{h-1} \zeta \sum_{\ell=0}^{n} \binom{n}{\ell} q^\ell G_{\ell, q, \zeta}^{(h)}
\]

\[
= -q^{h-1} \zeta \left( q G_{q, \zeta}^{(h)} + 1 \right)^n
\]

\[
= G_{n, q, \zeta}^{(h)} \quad \text{if } n > 1.
\]

Therefore, we obtain the following theorem.

Theorem 2.4. For \( n \in \mathbb{Z} \) and \( \zeta \in T_p \), one has

\[
q^{2h} \zeta^2 G_{n, q, \zeta}^{(h)}(2) = G_{n, q, \zeta}^{(h)} + nq^h \zeta [2]_q.
\]

Remark 2.5. We note that Theorem 2.4 also can be proved by using fermionic integral equation (1.4) in case of \( n = 2 \).
By (2.4) and Theorem 2.2, we get

\[
\frac{G_{n+1,q^{-1}}^{(h)}(2)}{n+1} = (-1)^{n} q^{n+h-1} \xi \frac{G_{n+1,q^{-1}}^{(h)}(-1)}{n+1}
\]
\[
= (-1)^{n} q^{n+h-1} \xi \int_{\mathbb{Z}_p} [x-1]_{q}^{n} \xi^{x} q^{(h-1)x} \, d\mu_{-q}(x)
\]
\[
= q^{h-1} \xi \int_{\mathbb{Z}_p} [1-x]_{q}^{n} \xi^{x} q^{(h-1)x} \, d\mu_{-q}(x). \tag{2.10}
\]

Therefore, we obtain the following theorem.

**Theorem 2.6.** For \( n \in \mathbb{Z}_+ \) and \( \xi \in T_p \), one has

\[
(n+1)q^{h-1} \xi \int_{\mathbb{Z}_p} [1-x]_{q}^{n} \xi^{x} q^{(h-1)x} \, d\mu_{-q}(x) = G_{n+1,q^{-1},\xi^{-1}}^{(h)}(2). \tag{2.11}
\]

Let \( n \in \mathbb{N} \). By Theorems 2.4 and 2.6, we get

\[
(n+1)q^{h-1} \xi \int_{\mathbb{Z}_p} [1-x]_{q}^{n} \xi^{x} q^{(h-1)x} \, d\mu_{-q}(x) = q^{2h} \xi^{2} G_{n+1,q^{-1},\xi^{-1}}^{(h)} + (n+1)q^{h-1} \xi [2]_q. \tag{2.12}
\]

Therefore, we obtain the following corollary.

**Corollary 2.7.** For \( n \in \mathbb{Z}_+ \) and \( \xi \in T_p \), one has

\[
\int_{\mathbb{Z}_p} [1-x]_{q}^{n} \xi^{x} q^{(h-1)x} \, d\mu_{-q}(x) = q^{h-1} \xi \frac{G_{n+1,q^{-1},\xi^{-1}}^{(h)}}{n+1} + [2]_q. \tag{2.13}
\]

By (1.5), we get the symmetry of \( q \)-Bernstein polynomials as follows:

\[
B_{k,n}(x, q) = B_{n-k,n}\left(1 - x, q^{-1}\right) \tag{2.14}
\]

(see [11]).
Thus, by Corollary 2.7 and (2.14), we get

\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) q^{(h-1)x} \xi^x d\mu_{-q}(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x, q^{-1}) q^{(h-1)x} \xi^x d\mu_{-q}(x)
\]

\[
= \binom{n}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \int_{\mathbb{Z}_p} [1-x]^{n-\ell} q^{(h-1)x} \xi^x d\mu_{-1}(x)
\]

\[
= \binom{n}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \left( q^{h+1} \xi \frac{G(n-\ell+1, q^{-1})}{n-\ell+1} + [2]_q \right)
\]

(2.15)

\[
= \begin{cases} 
q^{h+1} \xi \frac{G(n+1, q^{-1})}{n+1} + [2]_q & \text{if } k = 0, \\
q^{h+1} \xi \binom{n}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \frac{G(n-\ell+1, q^{-1})}{n-\ell+1} & \text{if } k > 0.
\end{cases}
\]

From (2.15), we have the following theorem.

**Theorem 2.8.** For \( n \in \mathbb{Z}_+ \) and \( \xi \in T_p \), one has

\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) q^{(h-1)x} \xi^x d\mu_{-q}(x) = \begin{cases} 
q^{h+1} \xi \frac{G(n+1, q^{-1})}{n+1} + [2]_q & \text{if } k = 0, \\
q^{h+1} \xi \binom{n}{k} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \frac{G(n-\ell+1, q^{-1})}{n-\ell+1} & \text{if } k > 0.
\end{cases}
\]

(2.16)

For \( n, k \in \mathbb{Z}_+ \) with \( n > k \), fermionic \( p \)-adic invariant integral for multiplication of two \( q \)-Bernstein polynomials on \( \mathbb{Z}_p \) can be given by the following:

\[
\int_{\mathbb{Z}_p} B_{k,n}(x, q) q^{(h-1)x} \xi^x d\mu_{-q}(x) = \int_{\mathbb{Z}_p} \binom{n}{k} [x]_q^k [1-x]_q^{n-k} q^{(h-1)x} \xi^x d\mu_{-q}(x)
\]

\[
= \int_{\mathbb{Z}_p} \binom{n}{k} [x]_q^k \left( 1 - [x]_q \right)^{n-k} q^{(h-1)x} \xi^x d\mu_{-1}(x)
\]

(2.17)

From Theorem 2.8 and (2.17), we have the following corollary.
Corollary 2.9. For \( n \in \mathbb{Z}_+ \) and \( \zeta \in T_p \), one has

\[
\sum_{\ell=0}^{n-k} \binom{n-k}{\ell} (-1)^{\ell} \frac{G_{n+1-k, \ell+1}^{(h)}}{n+1} = \begin{cases} 
q^{h+1} \xi_{n+1-k, \ell+1}^{(h)} + [2]_q & \text{if } k = 0, \\
q^{h+1} \ell \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell} \frac{G_{n-\ell+1-k, \ell+1}^{(h)}}{n-\ell+1} & \text{if } k > 0.
\end{cases}
\] (2.18)

Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \). Then we get

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} \xi_x d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \int_{\mathbb{Z}_p} \sum_{\ell=0}^{2k} \binom{2k}{\ell} (-1)^{2k-\ell} \left[ 1 - x \right]^{n_1+n_2-\ell} \frac{q^{(h-1)x}}{\xi_x} d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{\ell=0}^{2k} \binom{2k}{\ell} (-1)^{2k-\ell} \left( \frac{G_{n_1+n_2-\ell+1-k, \ell+1}^{(h)}}{n_1+n_2-\ell+1} \right) q^{(h-1)x} [2]_q.
\] (2.19)

From (2.19), we have the following theorem.

Theorem 2.10. For \( n \in \mathbb{Z}_+ \) and \( \zeta \in T_p \), one has

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} \xi_x d\mu_q(x)
\]

\[
= \begin{cases} 
q^{h+1} \xi_{n_1+n_2+1-k, \ell+1}^{(h)} + [2]_q & \text{if } k = 0, \\
\left( \binom{n_1}{k} \binom{n_2}{k} \sum_{\ell=0}^{2k} \binom{2k}{\ell} \right) (-1)^{2k-\ell} \left( \frac{G_{n_1+n_2-\ell+1-k, \ell+1}^{(h)}}{n_1+n_2-\ell+1} \right) & \text{if } k > 0.
\end{cases}
\] (2.20)

Let \( n_1, n_2, k \in \mathbb{Z}_+ \) with \( n_1 + n_2 > 2k \), fermionic \( p \)-adic invariant integral for multiplication of two \( q \)-Bernstein polynomials on \( \mathbb{Z}_p \) can be given by the following:

\[
\int_{\mathbb{Z}_p} B_{k,n_1}(x,q) B_{k,n_2}(x,q) q^{(h-1)x} \xi_x d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{\ell=0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{\ell} (-1)^{\ell} \left[ n_1+n_2-2k \right]^{\ell} \left( \frac{q^{(h-1)x}}{\xi_x} \right) d\mu_q(x)
\]

\[
= \binom{n_1}{k} \binom{n_2}{k} \sum_{\ell=0}^{n_1+n_2-2k} (-1)^{\ell} \left( \frac{G_{2k+\ell+1-k, \ell+1}^{(h)}}{2k+\ell+1} \right) q^{(h-1)x} y^{x}.
\] (2.21)

From Theorem 2.10 and (2.21), we have the following corollary.
Corollary 2.11. For \( n_1, n_2, k \in \mathbb{Z}_+ \) and \( n_1 + n_2 > 2k \), one has
\[
\sum_{\ell = 0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{\ell} (-1)^\ell \frac{G^{(h)}_{2k+\ell+1,q,h}}{2k+\ell+1} = \begin{cases} 
q^{h+1} \frac{G^{(h)}_{n_1+n_2+1,q,h-1}}{n_1+n_2+1} + [2]_q & \text{if } k = 0, \\
\sum_{\ell = 0}^{n_1+n_2-2k} \binom{n_1+n_2-2k}{\ell} (-1)^{2k-\ell} \frac{G^{(h)}_{n_1+n_2-\ell+1,q,h-1}}{n_1+n_2-\ell+1} & \text{if } k > 0.
\end{cases}
\] (2.22)

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