Research Article

The Generalized Janowski Starlike and Close-to-Starlike Log-Harmonic Mappings

Maisarah Haji Mohd and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor D. Ehsan, Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

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Motivated by the success of the Janowski starlike function, we consider here closely related functions for log-harmonic mappings of the form \( f(z) = zh(z)g(z) \) defined on the open unit disc \( U \). The functions are in the class of the generalized Janowski starlike log-harmonic mapping, \( S^*_lh(A,B,\alpha) \), with the functional \( zh(z) \) in the class of the generalized Janowski starlike functions, \( S^*(A,B,\alpha) \). By means of these functions, we obtained results on the generalized Janowski close-to-starlike log-harmonic mappings, \( CSTlh(A,B,\alpha) \).

1. Introduction

The class \( S^*(A,B) \) was investigated by Janowski [1] in early 1970, and since then various other subclasses in relation with this Janowski class have been introduced and studied. In that direction, the log-harmonic mappings which have been studied extensively for the past 3 decades, (see [2–10]) were also associated with the Janowski class. Perhaps, the Janowski starlike log-harmonic univalent functions were first introduced by Polatoglu and Deniz [11].

A function \( f \) is said to be log-harmonic on the open unit disc \( U = \{z : |z| < 1\} \) if it satisfies the nonlinear elliptic partial differential equation:

\[
\frac{\bar{f}z}{f} = a \frac{fz}{f},
\]

where the second dilatation function \( a \in \mathcal{A}(U) \) (set of all analytic functions defined on \( U \)) such that \( |a(z)| < 1 \) for all \( z \in U \). For analytic functions \( h \) and \( g \) in \( U \), the function \( f \) can be expressed as

\[
f(z) = h(z)\overline{g(z)}
\]
if $f$ is a nonvanishing log-harmonic mapping and

$$f(z) = z|z|^{2p}h(z) g(z)$$

(1.3)

if $f$ vanishes at $z = 0$ but is not identically zero (for $\Re{\beta} > -1/2$, $g(0) = 1$, and $h(0) \neq 0$).

Let $f(z) = z h(z) g(z)$ be a univalent log-harmonic mapping, where $0 \not\in f(U)$ or equivalently $0 \not\in hg(U)$. Then $f$ is starlike log-harmonic mapping if

$$\Re\left(\frac{zf_z - z f_z}{f}\right) > 0.$$  

(1.4)

Results on starlike log-harmonic mapping of order $\alpha$ was given in [6].

Motivated by [11], the class of the generalized Janowski log-harmonic starlike functions was introduced in [12]. For real numbers $A$ and $B$, with $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, the family of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

(1.5)

is in $P(A, B, \alpha)$ if and only if

$$p(z) = \frac{1 + [(1 - \alpha)A + AB\phi(\zeta)]}{1 + B\phi(z)},$$

(1.6)

where the function $\phi$ is analytic in $U$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. The following lemma is also essential for $p(z)$ to be in $P(A, B, \alpha)$.

**Lemma 1.1** (see [13]). The function $p(z) \in P(A, B, \alpha)$ if and only if

$$\left|p(z) - 1 - [(1 - \alpha)A + ABr^2] \right| \leq \frac{(1 - \alpha)(A - B)r}{1 - B^2 r^2}$$

(1.7)

for $|z| \leq r < 1$.

Let $S^*(A, B, \alpha)$ denote the class of the generalized Janowski starlike functions of the analytic functions $s(z) = z + s_2 z^2 + \cdots$ such that $s(z) \in S^*(A, B, \alpha)$ if and only if

$$\frac{zs'(z)}{s(z)} = p(z)$$

(1.8)

and $p(z) \in P(A, B, \alpha)$ for $z \in U$. 

For univalent log-harmonic mapping \( f(z) = zh(z)g(z) \) with \( g(0) = 1 \) and \( h(0) \neq 0 \), \( f \) is in the class of the generalized Janowski starlike log-harmonic mapping denoted by \( S^*_lh(A,B,a) \) if

\[
\begin{aligned}
|p(z) - \frac{1 - [(1 - \alpha)A + aB]Br^2}{1 - B^2r^2}| \leq \frac{(1 - \alpha)(A - B)r}{1 - B^2r^2},
\end{aligned}
\]

where

\[
p(z) = \frac{h(z)g(z) + zh'(z)g(z) - zg'(z)h(z)}{h(z)g(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}.
\]

Also observe that if \( f \in S^*_lh(A,B,a) \), then

\[
\Re\left(\frac{zf - \overline{zf}}{f}\right) \geq \frac{1 - [(1 - \alpha)A + aB]}{1 - B}.
\]

In the present work, we consider the log-harmonic mapping \( f(z) = zh(z)g(z) \) in the generalized Janowski starlike functions with the functional \( zh(z) \in S^*(A,B,a) \). We also study the class of generalized Janowski close-to-starlike in the next section.

2. The Generalized Janowski Starlike Log-Harmonic

**Theorem 2.1.** If \( zh(z) \in S^*(A,B,a) \), then

\[
(1 - Br)^{(1-a)(A-B)/B} \leq |h(z)| \leq (1 + Br)^{(1-a)(A-B)/B} \quad \text{for } B \neq 0,
\]

\[
e^{-1(a)(A-B)r} \leq |h(z)| \leq e^{(1-a)(A-B)r} \quad \text{for } B = 0.
\]

**Proof.** Since \( zh(z) \in S^*(A,B,a) \), Lemma 1.1 yields that for \( B \neq 0 \) we have

\[
1 - \frac{[(1 - \alpha)A + aB]r}{1 - Br} \leq \Re\left(\frac{z(h(z))'}{zh(z)}\right) \leq \frac{1 + [(1 - \alpha)A + aB]r}{1 + Br} (2.2)
\]

or

\[
\frac{-(1 - \alpha)(A - B)r}{1 - Br} \leq \Re\left(\frac{zh'(z)}{h(z)}\right) \leq \frac{(1 - \alpha)(A - B)r}{1 + Br}.
\]

Simple calculations yield

\[
\frac{-(1 - \alpha)(A - B)}{-B} \log(1 - Br) \leq \log|h(z)| \leq \frac{(1 - \alpha)(A - B)}{B} \log(1 + Br),
\]

and the result follows immediately.
For $B = 0$, Lemma 1.1 yields

$$1 - (1 - \alpha)Ar \leq \text{Re}\left(\frac{zh(z)'}{zh(z)}\right) \leq 1 + (1 - \alpha)Ar,$$

and the proof is completed similarly.

**Theorem 2.2.** Let $f(z) = zh(z)g(z) \in S^*_lh(A, B, \alpha)$ with $zh(z) \in S^*(A, B, \alpha)$. Then one has

$$\frac{(1 - Br)^{(1- \alpha)(A-B)/B}}{(1 + Br)^{(1- \alpha)(A-B)/B}} \leq \left|\frac{g(z)}{f(z)}\right| \leq \frac{(1 + Br)^{(1- \alpha)(A-B)/B}}{(1 - Br)^{(1- \alpha)(A-B)/B}} \quad \text{for } B \neq 0,$$

$$e^{-2(1-\alpha)Ar} \leq \left|\frac{g(z)}{f(z)}\right| \leq e^{2(1-\alpha)Ar} \quad \text{for } B = 0.$$

**Proof.** It follows from [12] that for $f(z) = zh(z)g(z) \in S^*_lh(A, B, \alpha)$, we have

$$\frac{(1 - Br)^{(1- \alpha)(A-B)/B}}{(1 + Br)^{(1- \alpha)(A-B)/B}} \leq \left|\frac{h(z)}{g(z)}\right| \leq \frac{(1 + Br)^{(1- \alpha)(A-B)/B}}{(1 - Br)^{(1- \alpha)(A-B)/B}} \quad \text{for } B \neq 0,$$

$$e^{-r(1-\alpha)Ar} \leq \left|\frac{h(z)}{g(z)}\right| \leq e^{r(1-\alpha)Ar} \quad \text{for } B = 0.$$

With these inequalities and Theorem 2.1, we can conclude the following statement.

**Theorem 2.3.** Let $f(z) = zh(z)g(z) \in S^*_lh(A, B, \alpha)$ with $zh(z) \in S^*(A, B, \alpha)$. Then one has

$$\frac{r(1 - Br)^{2(1- \alpha)(A-B)/B}}{(1 + Br)^{2(1- \alpha)(A-B)/B}} \leq \left|f(z)\right| \leq \frac{r(1 + Br)^{2(1- \alpha)(A-B)/B}}{(1 - Br)^{2(1- \alpha)(A-B)/B}} \quad \text{for } B \neq 0,$$

$$re^{-3(1-\alpha)Ar} \leq \left|f(z)\right| \leq re^{3(1-\alpha)Ar} \quad \text{for } B = 0.$$

**Proof.** For $f(z) = zh(z)g(z)$ and $|z| = r$, it is easy to see that

$$|f(z)| = |zh(z)g(z)| = |zh(z)||g(z)| = r|h(z)||g(z)|.$$

Thus, we can obtain the results from Theorems 2.1 and 2.2.

### 3. The Generalized Janowski Close-to-Starlike Log-Harmonic

Let $P_{lh}$ be mapping the set of all log-harmonic mappings, and let $R$ be defined on $U$ which are of the form $R(z) = K(z)\overline{J}(z)$, where $K$ and $J$ are in $\mathcal{E}(U)$, $K(0) = J(0) = 1$ and such that $\text{Re}\ R(z) > 0$ for all $z \in U$. These log-harmonic mappings with positive real part were studied in [5]. Other interesting studies in the same paper were on the close-to starlike log-harmonic mappings. The author then extended the results to close-to starlike of order $\alpha$ log-harmonic mappings [2].
In that direction, we say that \( F(z) = zH(z) \overline{G(z)} \) is the generalized Janowski close-to-starlike log-harmonic mapping if there exist a log-harmonic mapping \( f(z) = zh(z)g(z) \in ST_{lh}^+(A, B, a) \) \((-1 \leq B < A \leq 1\) and \(0 \leq a < 1\)), with respect to the second dilatation function \( a \in \mathcal{K}(U) \) and a log-harmonic mapping with positive real part \( R \in P_{lh} \) where its second dilatation function is the same such that

\[
F(z) = f(z)R(z) \tag{3.1}
\]

or equivalently

\[
\text{Re} \left( \frac{F(z)}{f(z)} \right) > 0. \tag{3.2}
\]

We could also easily derive from (3.1) that

\[
\text{Re} \left( \frac{zf_z - \overline{f}z}{F} \right) = \text{Re} \left( \frac{zf_z - \overline{f}z}{f} \right) + \text{Re} \left( \frac{zR_z - \overline{R}z}{R} \right). \tag{3.3}
\]

The geometrical interpretation is that under a generalized Janowski close-to-starlike log-harmonic mapping, the radius vector of the image of \(|z| = r < 1\) never turns back by the amount more than \((1 - \alpha)(A - B)/(1 - B))\pi\). As special cases, we see that

(i) for \( \alpha = 0 \) or under the Janowski close-to-starlike log-harmonic mappings, the radius vector of the image of \(|z| = r < 1\) never turns back by an amount more than \((A - B)/(1 - B))\pi\),

(ii) for when \( A = 1, B = -1 \) or under the close-to-starlike of order \( \alpha \) log-harmonic mappings, the radius vector of the image of \(|z| = r < 1\) never turns back by an amount more than \((1 - \alpha)\pi\),

(iii) for \( \alpha = 0, A = 1, B = -1 \) or under the close-to-starlike log-harmonic mappings, the radius vector of the image of \(|z| = r < 1\) never turns back by an amount more than \(\pi\).

The following theorem gives us the radius of starlikeness for \( F(z) = zH(z) \overline{G(z)} \in CST_{lh}(A, B, \alpha)\).

**Theorem 3.1.** The radius of starlikeness for \( F(z) = zH(z) \overline{G(z)} \in CST_{lh}(A, B, \alpha) \) is the largest positive root, \( r \in (0, 1) \), such that

\[
(1 - [(1 - \alpha)A + aB]r)(1 - r)(1 + r) - 2r(1 - Br) > 0. \tag{3.4}
\]

**Proof.** For \( F(z) = zH(z) \overline{G(z)} \in CST_{lh}(A, B, \alpha) \), we have

\[
\text{Re} \left( \frac{zf_z - \overline{f}z}{F} \right) = \text{Re} \left( \frac{zf_z - \overline{f}z}{f} \right) + \text{Re} \left( \frac{zR_z - \overline{R}z}{R} \right). \tag{3.5}
\]
and since \( f \in S^*_lh(A,B,\alpha) \) and \( R \in P_{lh} \), (3.5) becomes

\[
\text{Re} \left( \frac{zf_z - \overline{zf_z}}{F} \right) \geq 1 - \frac{[(1 - \alpha)A + \alpha B]r}{1 - Br} + \frac{-2r}{1 - r^2}.
\] (3.6)

Hence,

\[
\text{Re} \left( \frac{zf_z - \overline{zf_z}}{F} \right) > 0
\] (3.7)

if

\[
\frac{1 - [(1 - \alpha)A + \alpha B]r}{1 - Br} - \frac{2r}{1 - r^2} > 0.
\] (3.8)

\begin{corollary}[see [2]]
The radius of starlikeness for \( F(z) = zH(z) \overline{G(z)} \in CST_{lh} \) is

\[
r < 2 - \sqrt{3}.
\] (3.9)
\end{corollary}

\begin{corollary}[see [2]]
The radius of starlikeness for \( F(z) = zH(z) \overline{G(z)} \in CST_{lh}(\alpha) \) is

\[
r < 2 - \alpha - \sqrt{\alpha^2 - 2\alpha + 3}
\] (1 - 2\alpha)
\] (3.10)

\begin{corollary}
The radius of starlikeness for \( F(z) = zH(z) \overline{G(z)} \in CST_{lh}(A,B) \) is the largest positive root, \( r \in (0,1) \), such that

\[
(1 - Ar)(1 - r)(1 + r) - 2r(1 - Br) > 0.
\] (3.11)
\end{corollary}

\begin{proof}
The proof is completed by taking \( \alpha = 0 \) in (3.4).
\end{proof}

We need the following theorem from [5] to prove our next result.

**Theorem**

Let \( R(z) \in P_{lh} \), and suppose that \( a(0) = 0 \). Then, for \( z \in U \), we have

\[
e^{-2|z|/(1-|z|)} \leq |R(z)| \leq e^{2|z|/(1-|z|)}.
\] (3.12)
Theorem 3.5. For \( F(z) = zH(z)G(z) \in \text{CST}_h(A, B, \alpha) \) and \( f(z) = zh(z)g(z) \) with \( zh(z) \in S^*(A, B, \alpha) \), one has

\[
\frac{r(1 - Br)^{2(1-\alpha)(A-B)/B}e^{-2r/(1-r)}}{(1 + Br)^{(1-\alpha)(A-B)/B}} \leq |F(z)| \leq \frac{r(1 - Br)^{2(1-\alpha)(A-B)/B}e^{2r/(1-r)}}{(1 + Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0,
\]

\[
re^{-3(1-\alpha)A(2r/(1-r))} \leq |F(z)| \leq re^{3(1-\alpha)A(2r/(1-r))} \quad \text{for } B = 0.
\]

Proof. From (3.12) and Theorem 2.3, we have

\[
e^{-2r/(1-r)} \leq |R(z)|e^{2r/(1-r)}, \quad |z| = r < 1,
\]

\[
\frac{r(1 - Br)^{2(1-\alpha)(A-B)/B}}{(1 + Br)^{(1-\alpha)(A-B)/B}} \leq |f(z)| \leq \frac{r(1 + Br)^{2(1-\alpha)(A-B)/B}}{(1 - Br)^{(1-\alpha)(A-B)/B}} \quad \text{for } B \neq 0,
\]

\[
re^{-3(1-\alpha)A} \leq |f(z)| \leq re^{3(1-\alpha)A} \quad \text{for } B = 0,
\]

respectively. Also, we know that for \( F \in \text{CST}_h(A, B, \alpha) \), we have \( F(z) = f(z)R(z) \) which then leads to the desired result.

\( \square \)

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References


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