Research Article

On Quasi-\(\omega\)-Confluent Mappings

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We introduce a new class of mappings called quasi-\(\omega\)-confluent maps, and we study the relation between these mappings, and some other forms of confluent maps. Moreover, we prove several results about some operations on quasi-\(\omega\)-confluent mappings such as: composition, factorization, pullbacks, and products.

1. Introduction

A generalization of the notion of the classical open sets which has received much attention lately is the so-called \(\omega\)-open sets. These sets are characterized as follows [1]: a subset \(W\) of a topological space \((X, \tau)\) is an \(\omega\)-open set if and only if for each \(x \in W\), there exists \(U \in \tau\) such that \(x \in U\) and \(U - W\) is countable. One can then show that the family of all \(\omega\)-open subsets of a space \((X, \tau)\), denoted by \(\tau_{\omega}\), forms a topology on \(X\) finer than \(\tau\). Using this notion of \(\omega\)-open sets, one can then define notions such as \(\omega\)-compact and \(\omega\)-connected sets whose definitions follow closely the definitions of the related classical notions. For example, a space \(X\) is called \(\omega\)-connected provided that \(X\) is not the union of two disjoint nonempty \(\omega\)-open sets. And \(X\) is said to be \(\omega\)-compact if every \(\omega\)-open cover of \(X\) has a finite subcover. For more information regarding these notions and some recent related results, see [2–4].

Recall that a subset \(K\) of a space \(X\) is said to be a continuum if \(K\) is connected and compact. Using this idea of a continuum, Charatonik introduced and studied the idea of a confluent mapping between topological spaces [5] as follow: A mapping \(f : X \rightarrow Y\) is said to be confluent provided that for each continuum \(K\) of \(Y\) and for each component \(C\) of \(f^{-1}(K)\), we have \(f(C) = K\).

In [6], motivated by Charatonik’s work, we have introduced the notion of \(\omega\)-confluent mappings and studied its basic properties. In particular, we say a space \(X\) is an \(\omega\)-continuum
It is clear that every -confluent provided that for each -continuum \( K \) of \( Y \) and for each component \( C \) of \( f^{-1}(K) \), we have \( f(C) = K \).

In this paper, we are interested in the further generalizations of the work of Charatonik in the context of -open sets and the idea of quasicomponents. Recall that a quasicomponent of space \( X \) containing a point \( p \in X \) is the intersection of all nonempty clopen sets in \( X \) containing \( p \) [7]. In particular, we will introduce the notion of quasi- -confluent maps and study its relation with the classical mappings such as confluent, -confluent, and quasiconfluent maps. We also study operations on such mappings like compositions, pullback of quasi- -confluent, factorizations, and products.

### 2. Quasi- -Confluent Mappings

In this section, we introduce and study a new form of -confluent mapping, which is a quasi- -confluent mapping. Throughout this paper, all mappings are assumed to be continuous.

Now, we introduce the following notion.

**Definition 2.1.** A mapping \( f : X \to Y \) is said to be quasi- -confluent (resp., quasiconfluent) if for each -continuum (resp., continuum) \( K \) in \( Y \) and for each quasicomponent \( QC \) of \( f^{-1}(K) \), we have \( f(QC) = K \).

First, we need the following theorem.

**Theorem 2.2** (see [6]). Let \( X \) be a topological space. Then,

1. every -connected subset \( K \) of \( X \) is connected,
2. every -compact subset \( K \) of \( X \) is compact,
3. every -continuum subset \( K \) of \( X \) is a continuum.

**Proposition 2.3.** (1) Every -confluent mapping is quasi- -confluent.

(2) Every quasiconfluent mapping is quasi- -confluent.

**Proof.** (1) Suppose that \( f : X \to Y \) be an -confluent mapping, let \( K \) be any -continuum in \( Y \), and let \( x \) be any point in \( f^{-1}(K) \) and \( QC_x \) be the quasicomponent of \( x \) in \( f^{-1}(K) \). Then, any component \( C_x \) of \( x \) in \( f^{-1}(K) \) contained in the quasicomponents \( QC_x \), or \( C_x \subset QC_x \). Thus, \( f(C_x) \subset f(QC_x) \). Since \( f \) is an -confluent, then \( f(C_x) = K \). This implies, \( K \subset f(QC_x) \). But we have \( QC_x \subset f^{-1}(K) \). So, \( f(QC_x) \subset K \). Thus, \( f(QC_x) = K \). Therefore, \( f \) is quasi- -confluent mapping.

(2) Let \( K \) be any -continuum in \( Y \) and \( QC \) be any quasicomponent of \( f^{-1}(K) \). Then, \( K \) is a continuum in \( Y \) by the Theorem 2.2(3). Since, \( f \) is quasiconfluent. So that, \( f(QC) = K \). Thus, \( f \) is quasi- -confluent mapping.

**Remark 2.4.** It is clear that every -confluent (confluent or quasiconfluent) mapping is quasi- -confluent, but the converses are not necessarily true, as shown by the following examples.

**Example 2.5.** Let \( K = \{1/n : n \text{ is a positive integer}\}, D = K \times [0,1] \).
Remark 2.7. Let \( X = D \cup \{(0,0),(0,1)\} \) subspaces of \( \mathbb{R}^2 \) under the usual topology \( \tau_u \), and \( Y = [0,1] \), with the topology \( \tau_Y = \{\phi,Y\} \). Let \( f : X \to Y \) be the mapping defined by

\[
f(x,y) = \begin{cases} 
0, & \text{for } (x,y) \in \{(0,0),(0,1)\}, \\
1, & \text{for } (x,y) \in \{k\} \times [0,1], \text{ for each } k \in K.
\end{cases}
\tag{2.1}
\]

Then, \( f \) is quasi-\( \omega \)-confluent but not quasiconfluent. Since, if we take the continuum \( K = [0,1] \) in \( Y \), then the quasicomponents of \( f^{-1}(K) \) are \( \{(0,0),(0,1)\} \) and \( D \). So, \( f(\{(0,0),(0,1)\}) \neq K \), and \( f(D) \neq K \).

(b) Let \( X = D \cup \{(0,0),(0,1)\} \cup ([0,1] \times \{0\}) \) subspaces of \( \mathbb{R}^2 \) under the usual topology \( \tau_u \), and \( Y = [0,1] \), with the topology \( \tau_Y = \{\phi,Y\} \). Let \( f : X \to Y \) be the mapping defined by

\[
f(x,y) = \begin{cases} 
0, & \text{for } (x,y) = (0,1), \\
1, & \text{otherwise}.
\end{cases}
\tag{2.2}
\]

Then, \( f \) is quasi-\( \omega \)-confluent, but not confluent. Since if we take the continuum \( K = [0,1] \) in \( Y \), then the components of \( f^{-1}(K) \) are \( \{(0,1)\} \) and \( X \setminus \{(0,1)\} \). So, \( f(\{(0,1)\}) \neq K \), and \( f(X \setminus \{(0,1)\}) \neq K \).

Example 2.6. Let \( X = \{p,q,r\} \) and \( Y = \{a,b,c\} \) with topologies \( \tau = \{\phi,X,\{p\},\{q\},\{p,q\}\} \) and \( \sigma = \{\phi,Y,\{a\}\} \) defined on \( X \) and \( Y \), respectively. Let \( f : X \to Y \) be a mapping defined by \( f(p) = a, f(q) = b, f(r) = c \). Then, \( f \) is quasi-\( \omega \)-confluent, but it is not confluent.

Remark 2.7. Quasi-\( \omega \)-confluent does not imply \( \omega \)-confluent in general, since the quasicomponent containing \( p \), \( QC(X,p) \) may be different from the component containing \( p \), \( C(X,p) \), as the following example shows.

Example 2.8. Let \( X = K \times [0,1] \cup \{(0,0),(0,1)\} \cup ([0,1] \times \{0\}) \) be a subspaces of \( \mathbb{R}^2 \) under the usual topology \( \tau_u \), where \( K \) be as in Example 2.5, and let \( Y = [0,1] \) with the topology \( \tau_{ind} = \{\phi,Y\} \). Let \( f : X \to Y \) be the mapping defined by

\[
f(x,y) = x, \quad \forall (x,y) \in X. \tag{2.3}
\]

Then, \( f \) is quasi-\( \omega \)-confluent, but \( f \) is not \( \omega \)-confluent. Note that if we take the \( \omega \)-continuum \( K = [0,1] \), then the components of \( f^{-1}(K) \) are \( C_1 = \{(0,1)\} \) and \( C_2 = X \setminus \{(0,1)\} \). Thus, \( f(C_1) \neq K \) and \( f(C_2) = K \).

The following diagram summarizes the relations between confluent mapping, quasicontuent mapping, and \( \omega \)-confluent mapping with quasi-\( \omega \)-confluent mapping.

\[
\text{confluent} \quad \longrightarrow \quad \text{quasi-confluent}
\]

\[
\text{\( \omega \)-confluent} \quad \longrightarrow \quad \text{quasi-\( \omega \)-confluent}
\]

The following theorem shows that under certain conditions, quasi-\( \omega \)-confluent mapping will be \( \omega \)-confluent.
Theorem 2.9. Every quasi-ω-confluent mapping \( f : X \to Y \) of a compact Hausdorff space \( X \) into a Hausdorff space \( Y \) is ω-confluent.

Proof. Let \( K \) be any ω-continuum in \( Y \) and \( C \) any component of \( f^{-1}(K) \). Then, by the Theorem 2.2, \( K \) is continuum subset of \( Y \). Since \( Y \) is Hausdorff, then \( K \) is closed in \( Y \) and since \( f \) is continuous, then \( f^{-1}(K) \) is closed in \( X \), since \( X \) is compact Hausdorff space, so that \( f^{-1}(K) \) is compact Hausdorff space. Thus, the quasicomponents are connected and coincide with components of \( f^{-1}(K) \). Thus, \( f(C) = K \). Therefore, \( f \) is ω-confluent. □

Proposition 2.10. If \( X \) is hereditarily locally connected, then any quasi-ω-confluent mapping \( f : X \to Y \) is ω-confluent.

Proof. It follows that from the fact that in locally connected space, the components and quasicomponents are the same. □

Definition 2.11 (see [2]). A space \((X, \tau)\) is said to be ω-space if every ω-open set is open in \( X \).

It is easy to see that in an ω-space that the continuum and ω-continuum sets coincide.

Proposition 2.12. If \( Y \) is an ω-space and if \( f : X \to Y \) is a mapping of a compact Hausdorff space \( X \) into a Hausdorff space \( Y \), then the following are equivalent:

1. \( f \) is confluent,
2. \( f \) is ω-confluent,
3. \( f \) is quasiconfluent,
4. \( f \) is quasi-ω-confluent.

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (3). Let \( f \) be an ω-confluent mapping, \( K \) any continuum in \( Y \), and \( QC \) any quasicomponent of \( f^{-1}(K) \). Since \( Y \) is an ω-space, then \( K \) is an ω-continuum, since \( Y \) is Hausdorff and \( X \) is compact Hausdorff, so that the components and quasicomponents of \( f^{-1}(K) \) are the same. Hence, \( f(QC) = K \) by assumption. Thus, \( f \) is quasiconfluent mapping.

(3) ⇒ (4). It follows from Proposition 2.3(2).

(4) ⇒ (1). Let \( f \) be quasi-ω-confluent mapping, \( K \subseteq Y \) any continuum, and \( C \) be an arbitrary component of \( f^{-1}(K) \), since \( Y \) is an ω-space, then \( K \) is an ω-continuum in \( Y \), since \( X \) is a compact Hausdorff and \( Y \) is a Hausdorff. Then, \( C \) is a quasicomponent of \( f^{-1}(K) \). Thus, \( f(C) = K \). Therefore, \( f \) is confluent mapping. □

Theorem 2.13. Let \( f : X \to Y \) be a mapping of zero-dimensional space \( X \) into space \( Y \). Then, the following are equivalent:

1. \( f \) is an ω-confluent,
2. \( f \) is quasi-ω-confluent.

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (1). Let \( f \) be quasi-ω-confluent mapping, \( K \subseteq Y \) any ω-continuum, and \( C \) any component of \( f^{-1}(K) \). Since \( X \) is a zero-dimensional space, then it is totally disconnected. Then the components of \( f^{-1}(K) \) are coincide with quasicomponents. Thus, \( C \) is a quasicomponent of \( f^{-1}(K) \). Then, \( f(C) = K \), by the assumption. Therefore, \( f \) is an ω-confluent. □
Proposition 2.14. Let \( f : X \to Y \) be a mapping of space \( X \) into zero-dimensional space \( Y \). Then, the following are equivalent:

1. \( f \) is quasiconfluent,
2. \( f \) is quasi-\( \omega \)-confluent.

Proof. (1) \( \Rightarrow \) (2). It follows immediately from the Proposition 2.3(2).

(2) \( \Rightarrow \) (1). Let \( K \) be any \( \omega \)-continuum, and let \( QC \) be any quasicomponent of \( f^{-1}(K) \). Since \( Y \) is zero-dimensional space. Then, the connected subsets of \( Y \) are precisely the singleton sets. Thus, the \( \omega \)-continuum are coincide with continuum sets in \( Y \), therefore, \( K \) is a continuum in \( Y \), so that \( f(QC) = K \). Hence, \( f \) is quasiconfluent mapping.

Proposition 2.15. Let \( f : X \to Y \) be any mapping. If \( X \) is a hereditarily locally connected space, then the following conditions (1) and (2) are equivalent, and the conditions (3) and (4) are equivalent:

1. \( f \) is \( \omega \)-confluent mapping,
2. \( f \) quasi-\( \omega \)-confluent mapping,
3. \( f \) is confluent mapping,
4. \( f \) quasi-confluent mapping.

Proof. Similar to the proof of Proposition 2.10.

3. Composition and Factorization of Quasi-\( \omega \)-Confluent Mappings

In this section, we study the composition and factorization of quasi-\( \omega \)-confluent mapping. So, we need to recall the following theorem.

Theorem 3.1 (see [6]). Let \( f : X \to Y \) and \( g : Y \to Z \) be two \( \omega \)-confluent mappings, where \( f \) is a surjective. Then, \( h = g \circ f \) is an \( \omega \)-confluent mapping.

Theorem 3.2. Let \( f : X \to Y \) be a surjective quasi-\( \omega \)-confluent of compact Hausdorff space \( X \) into space \( Y \) and \( g : Y \to Z \) a quasi-\( \omega \)-confluent of space \( Y \) into Hausdorff space \( Z \). Then, \( h = g \circ f \) is quasi-\( \omega \)-confluent mapping.

Proof. Since \( X \) and \( Y \) are two compact Hausdorff spaces and since \( f \) and \( g \) are two quasi-\( \omega \)-confluent mappings, then \( f \) and \( g \) are \( \omega \)-confluent mappings by Theorem 2.9. Therefore, \( h = g \circ f \) is an \( \omega \)-confluent mapping by Theorem 3.1. Then, from Proposition 2.3, \( h = g \circ f \) is quasi-\( \omega \)-confluent mapping.

Proposition 3.3. If \( X \) is hereditarily locally connected space and if \( f : X \to Y \) and \( g : Y \to Z \) are two quasi-\( \omega \)-confluent mapping such that \( f \) is onto closed or open map, then \( h = g \circ f \) is quasi-\( \omega \)-confluent mapping.

Proof. The proof follows immediately from Proposition 2.10 and Theorem 3.1.

Theorem 3.4. Let \( f : X \to Y \) be a mapping of strongly connected space \( X \) into Hausdorff space \( Y \), and let \( f \) be a canonical decomposition \( (f = \text{inc} \circ f' \circ \text{pr}_R) \) of the following mappings:

\[
f' : \frac{X}{R_f} \to f(X), \quad \text{inc}: f(X) \to Y, \quad \text{and} \quad \text{pr}_R : X \to \frac{X}{R_f}, \tag{3.1}
\]
where \( p_{R_f} \) is the quotient surjection map, inc is the inclusion map, and \( f' \) is the bijection mapping, where \( X/R_f \) denote to quotients space over the kernel relation \( R_f = \{(x, x'): f(x) = f(x')\} \). Then, \( f \) is a canonical decomposition of \( \omega \)-confluent mappings.

**Proof.** We have to prove that these mappings \( p_{R_f}, i, \) and \( f' \) are \( \omega \)-confluent mappings. Let \( K \) be any arbitrary \( \omega \)-continuum in the quotients space \( X/R_f \) and \( C \) any component of \( p_{R_f}^{-1}(K) \). Since \( p_{R_f} \) is continuous mapping, then \( X/R_f \) is a Hausdorff, so that \( K \) is closed in \( X/R_f \). Then, by the continuity of \( p_{R_f} \), we have \( p_{R_f}^{-1}(K) \) is closed in \( X \). But \( X \) is strongly connected. Therefore, \( p_{R_f}^{-1}(K) \) is connected. This means \( p_{R_f}^{-1}(K) = C \). So, \( p_{R_f}(C) = K \). Thus, \( p_{R_f} \) is an \( \omega \)-confluent mapping.

It is clearly that \( f' \) and inc are \( \omega \)-confluent mappings, since \( Y \) is a Hausdorff, then the subspace \( f(X) \) is Hausdorff, and since \( X \) is strongly connected, then \( X/R_f \) is strongly connected and also Hausdorff. Thus, \( f' \) and the inclusion map inc are \( \omega \)-confluent. Hence, \( f \) is canonical decomposition of \( \omega \)-confluent mappings. ☐

**Remark 3.5.** In the above theorem, if \( X \) is strongly connected compact Hausdorff space, then the mapping \( f \) is the canonical decomposition of quasi-\( \omega \)-confluent mappings.

**Corollary 3.6.** If \( X, Y, \) and \( Z \) are Hausdorff spaces, \( X \) is a compact space, and if \( f: X \to Y \) is a surjective \( \omega \)-confluent mapping and \( g: Y \to Z \) is a quasi-\( \omega \)-confluent mapping, then \( h = g \circ f \) is \( \omega \)-confluent mapping.

**Corollary 3.7.** If \( X, Y, \) and \( Z \) are Hausdorff spaces, \( X \) is a compact space, and if \( f: X \to Y \) is a surjective quasi-\( \omega \)-confluent mapping and \( g: Y \to Z \) is \( \omega \)-confluent mapping, then \( h = g \circ f \) is a quasi-\( \omega \)-confluent mapping.

Now, we study Whyburn’s factorization theorem for quasi-\( \omega \)-confluent mappings.

Thus, we recall the definition of a factorable mapping.

**Definition 3.8 (see [8]).** If \( f: X \to Y \) be a mapping, any representation of \( f \) in the form \( f = f_2 \circ f_1 \), where \( f_1: X \to Z \) and \( f_2: Z \to Y \) are two mappings and \( Z \) is a certain space, will said to be factorization of \( f \), and \( f \) is said be a factorable mapping and \( Z \) a middle space.

Before we study the factorization property, we state the following theorem.

**Theorem 3.9 (see [6]).** If \( f: X \to Y \) is an \( \omega \)-confluent of strongly connected compact space \( X \) into Hausdorff space \( Y \), then there exists a unique factorization for \( f \) into two \( \omega \)-confluent mappings

\[
f(x) = f_2 \circ f_1(x), \quad \forall x \in X,
\]

(3.2)

such that \( f_1 \) is confluent mapping.

Now, we can get the factorization of a quasi-\( \omega \)-confluent mapping in the following proposition.

**Proposition 3.10.** If \( f: X \to Y \) be a quasi-\( \omega \)-confluent of strongly connected compact Hausdorff space \( X \) into Hausdorff space \( Y \), then there exists a unique factorization for \( f \) into two quasi-\( \omega \)-confluent mappings in the form \( f = f_2 \circ f_1 \).
Proof. Since $f$ and $g$ are two quasi-$\omega$-confluent mappings and since $X$ is strongly connected compact Hausdorff space and $Y$ is a Hausdorff space, then from Theorem 2.9 $f$ and $g$ are $\omega$-confluent mappings. Thus, $f$ has unique factorization in the form $f = f_2 \circ f_1$ by Theorem 3.9.

Next, we study the product property of quasi-confluent mappings.

Let $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ be any two families of topological spaces. The product space of $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ is denoted by $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$, respectively. Let $f_i : X_i \to Y_i$ be a mapping for each $i \in I$. Let $f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ be the product mappings as follows: $f((x_i)) = (f_i(x_i))$ for each $(x_i) \in \prod_{i \in I} X_i$. The projection of $\prod_{i \in I} X_i$ and $\prod_{i \in I} Y_i$ onto $X_i$ and $Y_i$, respectively, is denoted by $p_i$ and $q_i$. Before we get the following result, we need to state the following theorem.

Theorem 3.11 (see [6]). Let $f_i : X_i \to Y_i$ be an $\omega$-confluent mapping, for each $i \in I$ of space $X_i$ into Hausdorff space $Y_i$. Then,

$$f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

is an $\omega$-confluent mapping if the following equality holds:

$$\left( \prod_{i \in I} \tau_i \right)_{\omega'} = \prod_{i \in I} (\tau_i)_{\omega'}, \quad \forall i \in I.$$  

(3.4)

As immediate consequence of the above theorem, we get the following corollary.

Corollary 3.12. Let $f_i : X_i \to Y_i$ be a quasi-$\omega$-confluent mapping of compact Hausdorff space $X$ into Hausdorff space $Y$ for each $i \in I$, then

$$f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$$

(3.5)

is quasi-$\omega$-confluent mapping if the following equality holds:

$$\left( \prod_{i \in I} \tau_i \right)_{\omega'} = \prod_{i \in I} (\tau_i)_{\omega'}, \quad \forall i \in I.$$  

(3.6)

Proof. Since, $X_i$ is compact Hausdorff, then the product space $\prod_{i \in I} X_i$ is compact Hausdorff, and since $Y_i$ is Hausdorff, then the product space $\prod_{i \in I} Y_i$ is also Hausdorff. From Theorem 2.9, we infer that $f_i : X_i \to Y_i$ is an $\omega$-confluent for each $i \in I$. Then, by Theorem 3.11, $f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ is an $\omega$-confluent mapping. Therefore, $f$ is quasi-$\omega$-confluent by Proposition 2.3.

4. Pullback of Quasi-$\omega$-Confluent Mappings

In this section, we study the pullback of quasi-$\omega$-confluent mappings. So, we recall the following definitions.
Definition 4.1 (see [9]). A fiber structure is a triple \((X, f, Y)\) consisting of two spaces \(X\) and \(Y\) and a mapping \(f : X \to Y\). The space \(X\) is said to be the fibered (or, total) space, \(f\) is termed the projection, and \(Y\) is the base space. Next, we recall the definition of the pullback.

Definition 4.2 (see [9]). Let \((X, f, Y)\) be a fiber structure. Let \(Z\) be any space, and let \(g : Z \to Y\) be any mapping into the base \(Y\). Let \(E_f\) be a subspace of the cartesian product \(X \times Z\), where \(E_f = \{(x, z) : f(x) = g(z)\}\), and let \(p : E_f \to Z\) be the projection of \(E_f\) onto \(Z\) such that \(p(x, z) = z\), \(\forall (x, z) \in E_f\). The fiber structure \((E_f, p, Z)\) is said to be the fiber structure over \(Z\) induced by the mapping \(g\), and the projection \(p\) is said to be the pullback of \(f\) by \(g\).

Now, let \(γ : E_f \to X\) be the projection such that \(γ(x, z) = x, \forall (x, z) \in E_f\).

We observe that the following diagram is commutative.

\[
\begin{array}{ccc}
E_f & \xrightarrow{γ} & X \\
p \downarrow & & \downarrow f \\
Z & \xrightarrow{g} & Y
\end{array}
\]

Before we prove the main results in this section, we state the following lemma.

Lemma 4.3 (see [6]). Let \(f : X \to Y\) be a mapping, let \(Z\) be any space, and let \(g : Z \to Y\) be any mapping, and if \(K \subseteq Z\), then \(p^{-1}(K) = f^{-1}(g(K)) \times K\), where \(p\) is the pullback of \(f\) by \(g\).

Theorem 4.4. The pullback of a quasiconfluent mapping is quasi-ω-confluent.

Proof. Let \(f : X \to Y\) be a quasiconfluent mapping, let \(Z\) be any space, and let \(g : Z \to Y\) be any mapping. Let \(K \subseteq Z\) be any ω-continuum and QC any quasicomponent of \(p^{-1}(K)\). Then, QC is a quasicomponent of \(f^{-1}(g(K)) \times K\) by Lemma 4.3. Since every ω-continuum is continuum, then \(K\) is a continuum by Theorem 2.2. Thus, \(g(K)\) is continuum in \(Y\). Since \(f\) is quasiconfluent mapping, then \(f(QC) = g(K)\) for each quasicomponent \(QC\) of \(f^{-1}(g(K))\). Since \(p^{-1}(K) = f^{-1}(g(K)) \times K\), so \(K = p(f^{-1}(g(K)) \times K) = p(QC \times K)\) such that \(QC = QC' \times K\) for some quasicomponent \(QC\) of \(f^{-1}(g(K))\). Thus, \(p(QC) = P(QC' \times K) = K\). Therefore, \(p\) is quasi-ω-confluent.

The pullback of quasi-ω-confluent mapping is not necessarily quasi-ω-confluent as shown by the following example.

Example 4.5. Let \(X = \mathbb{R}\) be the real number with upper limit topology, \(Y = \{a, b\}\) with the topology \(τ_Y = \{∅, \{a\}, Y\}\), and \(Z = \mathbb{R}\) with topology \(τ_Z = \{∅, \mathbb{R} - \{1\}, \mathbb{R} - \{2\}, \mathbb{R} - \{1, 2\}\}\).

Let \(f : X \to Y\) be a mapping defined by

\[
f(x) = \begin{cases} 
a, & \text{if } x > 0, \\
b, & \text{if } x \leq 0, \end{cases}
\]

and let \(g : Z \to Y\) be a mapping defined by

\[
g(z) = \begin{cases} 
a, & \text{if } z \in \mathbb{R} - \{1, 2\}, \\
b, & \text{if } z \in \{1, 2\}. \end{cases}
\]
Let $E_f$ be a subspace of the cartesian product $X \times Z$, where

$$E_f = \{(x, z) : f(x) = g(z)\}.$$  

(4.3)

Then, the pullback of $f$ by $g$ is the projection $p : E_f \rightarrow Z$ which is defined by

$$p(x, z) = z, \quad \forall (x, z) \in E_f.$$  

(4.4)

We note that $f$ is quasi-$\omega$-confluent mapping, but $p$ is not quasi-$\omega$-confluent mapping. Since if we take the $\omega$-continuum $K = [0, \infty) \subset Z$, then by Lemma 4.3, we get $p^{-1}(K) = f^{-1}(g(K)) \times K$. But $g(K) = \{a, b\}$ is not $\omega$-continuum in $Y$.

Under certain condition, the pullback $p$ of quasi-$\omega$-confluent mapping $f$ will be quasi-$\omega$-confluent as shown by the following theorem.

**Theorem 4.6.** If $Y$ is a zero-dimensional space and if $f : X \rightarrow Y$ is a quasi-$\omega$-confluent mapping, then the pullback $p$ of $f$ is quasi-$\omega$-confluent.

*Proof.* Let $f : X \rightarrow Y$ be a quasi-$\omega$-confluent mapping, let $Z$ be any space, and let $g : Z \rightarrow Y$ be an mapping. Let $K$ be any $\omega$-continuum in $Z$, and let QC be any quasicomponent of $p^{-1}(K)$, where $p$ is the pullback of $f$ by $g$. Then QC is the quasicomponent of $f^{-1}(g(K)) \times K$ by Lemma 4.3. By Theorem 2.2, $K$ is continuum. Thus, $g(K)$ is continuum in $Y$ by the continuity of $g$. Since $Y$ is zero-dimensional space, then the quasi-confluent mapping equivalent to the quasi-$\omega$-confluent by Proposition 2.14. This implies the continuum and $\omega$-continuum sets coincide in $Y$. Thus, $g(K)$ is an $\omega$-continuum in $Y$. Since $f$ is a quasi-$\omega$-confluent, then $f(QC') = g(K)$ for each quasicomponents $QC'$ of $f^{-1}(g(K))$, and since $p^{-1}(K) = f^{-1}(g(K)) \times K = p(f^{-1}(g(K)) \times K) = p(QC' \times K)$ such that $QC = QC' \times K$ for some quasicomponent of $f^{-1}(g(K))$. Thus, $p(QC) = p(QC' \times K) = K$. Therefore, $p$ is a quasi-$\omega$-confluent. \qed

**Corollary 4.7.** If $f : X \rightarrow Y$ is a quasi-$\omega$-confluent mapping of space $X$ into $\omega$-space $Y$, then the pullback of $f$ is quasi-$\omega$-confluent mapping.

**References**


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