Research Article

Characteristic Lightlike Submanifolds of an Indefinite $S$-Manifold

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Received 12 April 2011; Revised 12 August 2011; Accepted 23 August 2011

Academic Editor: Christian Corda

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We study characteristic $r$-lightlike submanifolds $M$ tangent to the characteristic vector fields in an indefinite metric $S$-manifold, and we also discuss the existence of characteristic lightlike submanifolds of an indefinite $S$-space form under suitable hypotheses: (1) $M$ is totally umbilical or (2) its screen distribution $S(TM)$ is totally umbilical in $M$.

1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds, it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is nontrivial, making it interesting and remarkably different from the study of nondegenerate submanifolds. In particular, many authors study lightlike submanifolds on indefinite Sasakian manifolds (e.g., [1–4]).

Similar to Riemannian geometry, it is natural that indefinite $S$-manifolds are generalizations of indefinite Sasakian manifolds. Brunetti and Pastore analyzed some properties of indefinite $S$-manifolds and gave some characterizations in terms of the Levi-Civita connection and of the characteristic vector fields [5]. After then, they studied the geometry of lightlike hypersurfaces of indefinite $S$-manifold [6]. As Jin’s generalizations of lightlike submanifolds of the Sasakian manifolds with the general codimension [3, 4, 7], Lee and Jin recently extended lightlike hypersurfaces on indefinite $S$-manifold to lightlike submanifolds with codimension 2 on an indefinite $S$-manifold, called characteristic half lightlike submanifolds [8]. However, a general notion of characteristic lightlike submanifolds of an indefinite $S$-manifold have not been introduced as yet.

The objective of this paper is to study characteristic $r$-lightlike submanifolds $M$ of an indefinite $S$-manifold $\overline{M}$ subject to the conditions: (1) $M$ is totally umbilical, or (2) $S(TM)$
is totally umbilcal in $M$. In Section 2, we begin with some fundamental formulae in the theory of $r$-lightlike submanifolds. In Section 3, for an indefinite metric $g_{ij}$ manifold we consider a lightlike submanifold $M$ tangent to the characteristic vector fields, we recall some basic information about indefinite $S$-manifolds and deal with the existence of irrotational characteristic submanifolds of an indefinite $S$-space form. Afterwards, we study characteristic $r$-lightlike submanifolds of $\overline{M}$ in Sections 4 and 5.

2. Preliminaries

Let $(M, g)$ be an $m$-dimensional lightlike submanifold of an $(m + n)$-dimensional semi-Riemannian manifold $(\overline{M}, g)$. Then the radical distribution $\text{Rad}(TM) = TM \cap TM^\perp$ is a vector subbundle of the tangent bundle $TM$ and the normal bundle $TM^\perp$, of rank $r$ ($1 \leq r \leq \min\{m, n\}$). In general, there exist two complementary nondegenerate distributions $S(TM)$ and $S(TM^\perp)$ of $\text{Rad}(TM)$ in $TM$ and $TM^\perp$, respectively, called the screen and coscreen distributions on $M$, such that

$$TM = \text{Rad}(TM) \oplus_{\text{orth}} S(TM), \quad TM^\perp = \text{Rad}(TM) \oplus_{\text{orth}} S(TM^\perp),$$

(2.1)

where the symbol $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. We use the same notation for any other vector bundle. We use the following range of indices:

$$i, j, k, \ldots \in \{1, \ldots, r\}, \quad \alpha, \beta, \gamma, \ldots \in \{r + 1, \ldots, n\}.$$  

(2.2)

Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $\overline{TM}_M$ and $TM^\perp$ in $S(TM^\perp)$, respectively, and let $\{N_1, \ldots, N_r\}$ be a lightlike basis of $\Gamma(\text{ltr}(TM))$ consisting of smooth sections of $S(TM^\perp)_M$, where $\mathcal{M}$ is a coordinate neighborhood of $M$, such that

$$\nabla(N_i, \xi_j) = \delta_{ij}, \quad \nabla(N_i, N_j) = 0,$$

(2.3)

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad}(TM))$. Then we have

$$TM = TM \oplus \text{tr}(TM) = \{\text{Rad}(TM) \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM)$$

$$= \{\text{Rad}(TM) \oplus \text{ltr}(TM)\} \oplus_{\text{orth}} S(TM) \oplus_{\text{orth}} S(TM^\perp).$$

(2.4)

We say that a lightlike submanifolds $(M, g, S(TM), S(TM^\perp))$ of $\overline{M}$ are characterized as follows:

1. $r$-lightlike if $1 \leq r < \min\{m, n\}$;
2. cosotropic if $1 \leq r = n < m$;
3. isotropic if $1 \leq r = m < n$;
4. totally lightlike if $1 \leq r = m = n$.
The above three classes (2)–(4) are particular cases of the class (1) as follows: \( S(TM^1) = \{0\} \), \( S(TM) = \{0\} \), and \( S(TM) = S(TM^1) = \{0\} \), respectively. The geometry of \( r \)-lightlike submanifolds is more general form than that of the other three type submanifolds. For this reason, in this paper we consider only \( r \)-lightlike submanifolds \( M = (M,g,S(TM),S(TM^1)) \), with the following local quasiorthonormal field of frames on \( \overline{M} \):

\[
\{ \xi_1, \ldots, \xi_r, N_1, \ldots, N_r, F_{r+1}, \ldots, F_m, W_{r+1}, \ldots, W_n \},
\]

where the sets \( \{ F_{r+1}, \ldots, F_m \} \) and \( \{ W_{r+1}, \ldots, W_n \} \) are orthonormal basis of \( \Gamma(S(TM)) \) and \( \Gamma(S(TM^1)) \), respectively.

Let \( \overline{\nabla} \) be the Levi-Civita connection of \( \overline{M} \) and \( P \) the projection morphism of \( \Gamma(TM) \) on \( \Gamma(S(TM)) \) with respect to (2.1). For an \( r \)-lightlike submanifold, the local Gauss-Weingarten formulas are given by

\[
\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^e(X, Y) N_i + \sum_{a=r+1}^{n} h_a^e(X, Y) W_a,
\]

(2.6)

\[
\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) W_a,
\]

(2.7)

\[
\overline{\nabla}_X W_a = -A_{W_a} X + \sum_{i=1}^{r} \phi_{ai}(X) N_i + \sum_{\beta=r+1}^{n} \sigma_{\alpha\beta}(X) W_\beta,
\]

(2.8)

\[
\nabla_X PY = \nabla'_X PY + \sum_{i=1}^{r} h_i^s(X, PY) \xi_i,
\]

(2.9)

\[
\nabla_X \xi_i = -A_{\xi_i}^s X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j,
\]

(2.10)

for any \( X, Y \in \Gamma(TM) \), where \( \nabla \) and \( \nabla^* \) are induced linear connections on \( TM \) and \( S(TM) \), respectively, the bilinear forms \( h_i^e \) and \( h_i^s \) on \( M \) are called the local lightlike and screen second fundamental forms on \( TM \), respectively, \( h_a^e \) are called the local radical second fundamental forms on \( S(TM) \). \( A_{N_i}, A_{\xi_i}^e, \) and \( A_{W_a} \) are linear operators on \( \Gamma(TM) \) and \( \tau_{ij}, \rho_{ia}, \phi_{ai}, \) and \( \sigma_{\alpha\beta} \) are 1-forms on \( TM \). Since \( \overline{\nabla} \) is torsion-free, \( \nabla \) is also torsion-free and both \( h_i^e \) and \( h_i^s \) are symmetric. From the fact \( h_i^e(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i) \), we know that \( h_i^s \) are independent of the choice of a screen distribution. We say that

\[
h(X, Y) = \sum_{i=1}^{r} h_i^e(X, Y) N_i + \sum_{a=r+1}^{n} h_a^e(X, Y) W_a
\]

(2.11)

is the second fundamental tensor of \( M \).

The induced connection \( \nabla \) on \( TM \) is not metric and satisfies

\[
(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \left\{ h_i^e(X, Y) \eta_i(Z) + h_i^s(X, Z) \eta_i(Y) \right\},
\]

(2.12)
for all \(X, Y \in \Gamma(TM)\), where \(\eta_i\)s are the 1-forms such that
\[
\eta_i(X) = \overline{g}(X, N_i), \quad \forall X \in \Gamma(TM).
\tag{2.13}
\]

But the connection \(\nabla^*\) on \(S(TM)\) is metric. The above three local second fundamental forms are related to their shape operators by

\[
h_i^\ell(X, Y) = g\left(A_{\xi_i}^\ast X, Y\right) - \sum_{k=1}^r h_k(X, \xi_i)\eta_k(Y),
\tag{2.14}
\]

\[
h_i^\ell(X, PY) = g\left(A_{\xi_i}^\ast X, PY\right), \quad \overline{g}\left(A_{\xi_i}^\ast X, N_i\right) = 0,
\tag{2.15}
\]

\[
\epsilon_a h_a^\ell(X, Y) = g(A_{W_a}X, Y) - \sum_{i=1}^r \phi_{ai}(X)\eta_i(Y),
\tag{2.16}
\]

\[
\epsilon_a h_a^\ell(X, PY) = g(A_{W_a}X, PY), \quad \overline{g}(A_{W_a}X, N_i) = \epsilon_a \rho_{ai}(X),
\tag{2.17}
\]

\[
h_i^\ell(X, PY) = g(A_{N_i}X, PY), \quad \eta_i(A_{N_i}X) + \eta_j\left(A_{N_i}X\right) = 0,
\tag{2.18}
\]

where \(X, Y \in \Gamma(TM)\) and \(\epsilon_a\) is the sign of \(W_a\) but it is \(\pm 1\) related to the causal character of \(W_a\). From (2.18), we know that each \(A_{N_i}\) is shape operator related to the local second fundamental form \(h_i^\ell\) on \(S(TM)\). Replacing \(Y\) by \(\xi_j\) in (2.14), we have

\[
h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0,
\tag{2.19}
\]

for all \(X \in \Gamma(TM)\). It follows

\[
h_i^\ell(X, \xi_i) = 0, \quad h_i^\ell(\xi_j, \xi_k) = 0.
\tag{2.20}
\]

Also, replacing \(X\) by \(\xi_j\) in (2.14) and using (2.20), we have

\[
h_i^\ell(X, \xi_j) = g\left(X, A_{\xi_j}^\ast \xi_j\right), \quad A_{\xi_i}^\ast \xi_j + A_{\xi_j}^\ast \xi_i = 0, \quad A_{\xi_i}^\ast \xi_i = 0.
\tag{2.21}
\]

For an \(r\)-lightlike submanifold, replace \(Y\) by \(\xi_i\) in (2.16), we have

\[
h_i^\ell(X, \xi_i) = -\epsilon_a \phi_{ai}(X), \quad \forall X \in \Gamma(TM).
\tag{2.22}
\]

\textbf{Note I.} Using (2.14) and the fact that \(h_i^\ell\) are symmetric, we have

\[
g\left(A_{\xi_i}^\ast X, Y\right) - g\left(X, A_{\xi_i}^\ast Y\right) = \sum_{k=1}^r \left\{h_k(X, \xi_i)\eta_k(Y) - h_k(Y, \xi_i)\eta_k(X)\right\}.
\tag{2.23}
\]

From this, (2.20) and (2.21), we show that \(A_{\xi_i}^\ast\) are self-adjoint on \(\Gamma(TM)\) with respect to \(g\) if and only if \(h_i^\ell(X, \xi_i) = 0\) for all \(X \in \Gamma(TM)\), \(i\) and \(j\) if and only if \(A_{\xi_i}^\ast \xi_j = 0\) for all \(i, j\). We call
self-adjoint $A^*_M$ the lightlike shape operators of $M$. It follows from the above equivalence and (2.10) that the radical distribution $\text{Rad}(TM)$ of a lightlike submanifold $M$, with the lightlike shape operators $A^*_M$, is always an integrable distribution.

3. Characteristic Lightlike Submanifolds

A manifold $\overline{M}$ is called a globally framed $f$-manifold (or g.f.f-manifold) if it is endowed with a nonnull $(1,1)$-tensor field $\overline{g}$ of constant rank, such that $\ker \overline{g}$ is parallelizable, that is, there exist global vector fields $\overline{\xi}_\alpha, \alpha \in \{1, \ldots, k\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying $\overline{g}^2 = -I + \overline{\eta}^\alpha \otimes \overline{\eta}_\alpha$ and $\overline{g}(\overline{\xi}_\alpha, \overline{\xi}_\beta) = \delta^\alpha_\beta$.

The g.f.f-manifold $(\overline{M}^{2n+r}, \overline{g}, \overline{\xi}_\alpha, \overline{\eta}^\alpha), \alpha \in \{1, \ldots, k\}$, is said to be an indefinite metric g.f.f-manifold if $\overline{g}$ is a semi-Riemannian metric, with index $\nu$, $0 < \nu < 2n + k$, satisfying the following compatibility condition

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) - \sum_{\alpha=1}^{r} \epsilon_\alpha \overline{\eta}^\alpha(X) \overline{\eta}_\alpha(Y),$$

for any $X, Y \in \Gamma(T\overline{M})$, being $\epsilon_\alpha = \pm 1$ according to whether $\overline{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha \in \{1, \ldots, k\}$, one has $\overline{\eta}^\alpha(X) = \epsilon_\alpha \overline{g}(X, \overline{\xi}_\alpha)$. An indefinite metric g.f.f-manifold is called an indefinite $S$-manifold if it is normal and $d\overline{\eta}^\alpha = \Phi$, for any $\alpha \in \{1, \ldots, k\}$, where $\Phi(X, Y) = \overline{g}(X, \overline{\phi}Y)$ for any $X, Y \in \Gamma(T\overline{M})$. The normality condition is expressed by the vanishing of the tensor field $N = N_\Phi + 2d\overline{\eta}^\alpha \otimes \overline{\xi}_\alpha$, $N_\Phi$ being the Nijenhuis torsion of $\overline{g}$. Furthermore, as proved in [5], the Levi-Civita connection of an indefinite $S$-manifold satisfies:

$$\left(\nabla_X \overline{\phi}\right)Y = \overline{g}(\overline{\phi}X, \overline{\phi}Y) \overline{\xi} + \overline{\eta}(Y) \overline{\phi}^2(X),$$

where $\overline{\xi} = \sum_{\alpha=1}^{k} \epsilon_\alpha \overline{\xi}_\alpha$ and $\overline{\eta} = \sum_{\alpha=1}^{k} \epsilon_\alpha \overline{\eta}^\alpha$. We recall that $\nabla_X \overline{\xi}_\alpha = -\epsilon_\alpha \overline{\phi}X$ and $\ker \overline{g}$ is an integrable flat distribution since $\nabla_X \overline{\xi}_\alpha \overline{\eta}_\beta = 0$ (more details in [5]).

Following the notations in [9], we adopt the curvature tensor $R$, and thus we have $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$, and $R(X, Y, Z, W) = g(R(Z, W, Y), X)$, for any $X, Y, Z, W \in \Gamma(TM)$.

An indefinite $S$-manifold $(\overline{M}, \overline{g}, \overline{\xi}_\alpha, \overline{\eta}^\alpha)$ is called an indefinite $S$-space form, denoted by $\overline{M}(c)$, if it has the constant $\overline{g}$-sectional curvature $c$ [5]. The curvature tensor $\overline{R}$ of this space form $\overline{M}(c)$ is given by

$$4\overline{R}(X, Y, Z, W) = -(c + 3\epsilon) \left\{ \overline{g}(\overline{\phi}Y, \overline{\phi}Z) \overline{g}(\overline{\phi}X, \overline{\phi}W) - \overline{g}(\overline{\phi}X, \overline{\phi}Z) \overline{g}(\overline{\phi}Y, \overline{\phi}W) \right\}$$

$$- (c - \epsilon) \{ \Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z) \}$$

$$- 4 \left\{ \overline{\eta}(W)\overline{\eta}(X)\overline{g}(\overline{\phi}Z, \overline{\phi}Y) - \overline{\eta}(W)\overline{\eta}(Y)\overline{g}(\overline{\phi}Z, \overline{\phi}X) + \overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(\overline{\phi}W, \overline{\phi}X) - \overline{\eta}(Z)\overline{\eta}(X)\overline{g}(\overline{\phi}W, \overline{\phi}Y) \right\},$$

for any vector fields $X, Y, Z, W \in \Gamma(T\overline{M})$. 


Note 2. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^* = TM/\text{Rad}(TM)$ considered by Kupeli [10]. Thus all screen distributions $S(TM)$ are mutually isomorphic. For this reason, we newly define generic lightlike submanifolds of $\tilde{M}$ as follows.

**Definition 3.1.** Let $M$ be a $r$-lightlike submanifold of $\tilde{M}$ such that all the characteristic vector fields $\tilde{\xi}_a$ are tangent to $M$. A screen distribution $S(TM)$ is said to be *characteristic* if $\ker \tilde{\phi} \subset S(TM)$ and $\tilde{\phi}(S(TM)^{\perp}) \subset \Gamma(S(TM))$.

**Definition 3.2.** A $r$-lightlike submanifold $M$ of $\tilde{M}$ is said to be *characteristic* if $\ker \tilde{\phi} \subset TM$ and a characteristic screen distribution $(S(TM))$ is chosen.

**Proposition 3.3** (see [6]). Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite $S$-manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}_a, \tilde{\Pi}^i, \tilde{g})$ such that the characteristic vector fields are tangent to $M$. Then there exists a screen distribution such that $\ker \tilde{\phi} \subset TM$ and $\tilde{\phi}(E) \subset \Gamma(S(TM))$, where $E$ is a nonzero section of $\text{Rad}(TM)$.

**Proposition 3.4** (see [8]). Let $(M, g, S(TM))$ be a $1$-lightlike submanifold of codimension 2 of an indefinite $S$-manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}_a, \tilde{\Pi}^i, \tilde{g})$. Then $M$ is a characteristic lightlike submanifold of $\tilde{M}$.

**Definition 3.5.** A lightlike submanifold $M$ is said to be irrotational [10] if $\nabla_{X} \xi_i \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi_i \in \Gamma(\text{Rad}(TM))$ for all $i$.

**Note 3.** For an $r$-lightlike $M$, the above definition is equivalent to

\[ h^a_j(X, \xi_i) = 0, \quad h^a_i(X, \xi_i) = \phi_{ai}(X) = 0, \quad \forall X \in \Gamma(TM). \quad (3.4) \]

The extrinsic geometry of lightlike hypersurfaces depends on a choice of screen distribution, or equivalently, normalization. Since the screen distribution is not uniquely determined, a well-defined concept of $S$-manifold is not possible for an arbitrary lightlike submanifold of a semi-Riemannian manifold, then one must look for a class of normalization for which the induced Riemannian curvature has the desired symmetries. Let $(M, g)$ be a semi-Riemannian manifold, $p \in M$. $F \otimes^T p M$ is said to be an *algebraic curvature tensor* [11] on $T_p M$ if it satisfies the following symmetries:

\begin{align*}
F(X, Y, Z, W) + F(Y, Z, X, W) + F(Z, X, Y, W) &= 0. \quad (3.5)
\end{align*}

**Definition 3.6.** A screen distribution $S(TM)$ is said to be *admissible* if the associated induced Riemannian curvature is an algebraic curvature tensor.

**Theorem 3.7.** Let $(M, g, S(TM))$ be an irrotational generic characteristic lightlike submanifold of an indefinite $S$-space form $(\tilde{M}(c), \tilde{\phi}, \tilde{\xi}_a, \tilde{\Pi}^i, \tilde{g})$ with an admissible screen distribution $S(TM)$. Then one has $c = e$. 

\[ \text{Note 4.} \]
Proof. Denote by $\overline{R}$ and $R$ the curvature tensors of $\overline{\nabla}$ and $\nabla$, respectively. Using the local Gauss-Weingarten formulas for $M$, we obtain

$$
\overline{R}(X,Y)Z = R(X,Y)Z + \sum_{i=1}^{r} \left\{ h^i(X,Z)A_{N_i}Y - h^i(Y,Z)A_{N_i}X \right\} 
+ \sum_{a=r+1}^{n} \left\{ (\nabla_X h^a)(Y,Z) - (\nabla_Y h^a)(X,Z) \right\} 
+ \sum_{j=1}^{r} \left[ \tau_{ji}(X) h^j(Y,Z) - \tau_{ji}(Y) h^j(X,Z) \right] 
+ \sum_{a=r+1}^{n} \left\{ \phi_{ai}(X) h^a(Y,Z) - \phi_{ai}(Y) h^a(X,Z) \right\} \right\} N_i
+ \sum_{a=r+1}^{n} \left\{ (\nabla_X h^a)(Y,Z) - (\nabla_Y h^a)(X,Z) \right\} 
+ \sum_{j=1}^{r} \left[ \rho_{ja}(X) h^j(Y,Z) - \rho_{ja}(Y) h^j(X,Z) \right] 
+ \sum_{b=a+1}^{n} \left[ \sigma_{ja}(X) h^a(Y,Z) - \sigma_{ja}(Y) h^a(X,Z) \right] \right\} W_a,
$$

(3.6)

for all $X,Y,Z \in \Gamma(TM)$. Replace $Z$ by $\xi_k$ in (3.6) and use (2.10), (2.15), (2.17), and (3.4), we have

$$
\overline{R}(X,Y)\xi_k = R(X,Y)\xi_k + \sum_{i=1}^{r} \left\{ g\left(A^*_i Y, A^*_i X\right) - g\left(A^*_i X, A^*_i Y\right) \right\} N_i
+ \sum_{a=r+1}^{n} \left\{ g\left(A_{W_a} Y, A^*_a X\right) - g\left(A_{W_a} X, A^*_a Y\right) \right\} W_a.
$$

(3.7)

Using (3.7), the fact $R(X,Y)Z \in \Gamma(TM)$ for $X,Y,Z \in \Gamma(TM)$, and a screen distribution $S(TM)$ is admissible, we get

$$
\overline{g}\left(\overline{R}(X,Y)Z, \xi_k\right) = -\overline{g}\left(\overline{R}(X,Y)\xi_k, Z\right)
= -g(R(X,Y)\xi_k, Z) + \sum_{i=1}^{r} \left\{ g\left(A^*_i Y, A^*_i X\right) - g\left(A^*_i X, A^*_i Y\right) \right\} \eta_i(Z)
$$
\[
\begin{align*}
&= g(R(X, Y)Z, \xi) + \sum_{i=1}^{r} \left\{ g \left( A^*_i X, A^*_i Y \right) - g \left( A^*_i Y, A^*_i X \right) \right\} \eta_i(Z) \\
&= \sum_{i=1}^{r} \left\{ g \left( A^*_i X, A^*_i Y \right) - g \left( A^*_i Y, A^*_i X \right) \right\} \eta_i(Z), \quad \forall X, Y, Z \in \Gamma(TM).
\end{align*}
\]

(3.8)

On the other hand, since \( \eta(\xi) = 0 \) and \( g(\Phi^{\xi \alpha}, \Phi X) = 0 \) for any \( X \in \Gamma(TM) \), \( \overline{M}(c) \) is an indefinite \( S \)-space form implies the Riemannian curvature \( R \) in (3.3) is given by

\[
\begin{align*}
4R(X, Y, Z, \xi) &= (c - e) \{ \Phi(\xi, X) \Phi(Z, Y) - \Phi(Z, X) \Phi(\xi, Y) + 2 \Phi(X, Y) \Phi(\xi, Z) \} \\
&= -(c - e) \left\{ g\left( \Phi^{\xi \alpha}, X \right) \Phi(Z, Y) - \Phi(Z, X) g\left( \Phi^{\xi \alpha}, Y \right) + 2 \Phi(X, Y) g\left( \Phi^{\xi \alpha}, Z \right) \right\},
\end{align*}
\]

(3.9)

for any \( X, Y, Z, \xi \in \Gamma(TM) \). So, replacing \( X, Y, Z \) by \( PX, \xi, PZ \) in (3.9), we find

\[
\begin{align*}
4R(X, Y, Z, \xi) &= -(c - e) \left\{ -g\left( \Phi^{\xi \alpha}, PX \right) g\left( PZ, \Phi^{\xi \alpha} \right) - 2g\left( X, \Phi^{\xi \alpha} \right) g\left( \Phi^{\xi \alpha}, Z \right) \right\} \\
&= 3(c - e) g\left( \Phi^{\xi \alpha}, PX \right) g\left( \Phi^{\xi \alpha}, PZ \right).
\end{align*}
\]

(3.10)

Then, using (3.3), (3.8), and (3.9), we get

\[
\begin{align*}
4 \sum_{i=1}^{r} \left\{ g \left( A^*_i X, A^*_i Y \right) - g \left( A^*_i Y, A^*_i X \right) \right\} \eta_i(Z) \\
&= -3(c - e) g\left( \Phi^{\xi \alpha}, PX \right) g\left( \Phi^{\xi \alpha}, PZ \right), \quad \forall X, Y, Z \in \Gamma(TM).
\end{align*}
\]

(3.11)

Choosing \( X = Z = \Phi N_\alpha \in \Gamma(S(TM)) \), we obtain \( c = e \).

Corollary 3.8. There exist no irrotational characteristic \( r \)-lightlike submanifolds \((M, g, S(TM))\) of an indefinite \( S \)-space form \((\overline{M}(c), \Phi, \overline{\xi}^{\alpha}, \overline{\eta}^{\alpha}, \overline{g})\) with \( c \neq e \) such that the screen distribution \( S(TM) \) is admissible.

4. Totally Umbilical Characteristic Lightlike Submanifolds

Definition 4.1. An \( r \)-lightlike submanifold \( M \) of \( \overline{M} \) is said to be totally umbilical [1] if there is a smooth vector field \( \mathcal{L} \in \Gamma(\text{tr}(TM)) \) such that

\[
h(X, Y) = \mathcal{L} g(X, Y),
\]

(4.1)

for all \( X, Y \in \Gamma(TM) \). In case \( \mathcal{L} = 0 \), we say that \( M \) is totally geodesic.
It is easy to see that $M$ is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions $A_i$ and $B_a$ such that

$$h_i^r(X, Y) = A_i g(X, Y), \quad h_a^r(X, Y) = B_a g(X, Y), \quad (4.2)$$

for any $X, Y \in \Gamma(TM)$. From (4.2) we show that any totally umbilical $r$-lightlike submanifold of $\overline{M}$ is irrotational. Thus, by Theorem 3.7, we have the following.

**Theorem 4.2.** Let $(M, g, S(TM))$ be a totally umbilical characteristic $r$-lightlike submanifold of an indefinite $S$-space form $(\overline{M}(c), \overline{\phi}, \overline{\xi}_a, \overline{\eta}^i, \overline{\xi}^i, \overline{g})$. Then one has $c = \epsilon$.

**Theorem 4.3.** Let $(M, g, S(TM))$ be a totally umbilical characteristic $r$-lightlike submanifold of an indefinite $S$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi}_a, \overline{\eta}^i, \overline{g})$. Then $M$ is totally geodesic.

**Proof.** Apply $\overline{\nabla}_X \overline{\phi} \xi_a W_a = 0$ with $X \in \Gamma(TM)$, for all $i$ and $a$, and use (2.8), (2.10), (2.15), (2.17), (2.22), and (3.2), we have

$$h_i^r(X, \overline{\phi} W_a) = \epsilon_a h_a^r(X, \overline{\phi} \xi_a), \quad \forall X \in \Gamma(TM).$$

Assume that $M$ is totally umbilical. Then we have

$$A_i g(X, \overline{\phi} W_a) = \epsilon_a B_a g(X, \overline{\phi} \xi_a), \quad \forall X \in \Gamma(TM). \quad (4.4)$$

Replace $X$ by $\overline{\phi} N_i$ and $X$ by $\overline{\phi} W_a$ by turns, we get $A_i = 0$ for all $i$ and $B_a = 0$ for all $a$. Thus we show that $\mathcal{L} = \sum_{i=1}^n A_i N_i + \sum_{a=n+1}^n B_a W_a = 0$ and $M$ is totally geodesic.

**Corollary 4.4** (see [1]). Let $(M, g, S(TM))$ be a totally umbilical characteristic $r$-lightlike submanifold of an indefinite $S$-manifold $(\overline{M}, \overline{\phi}, \overline{\xi}_a, \overline{\eta}^i, \overline{g})$. Then there exists a unique torsion-free metric connection $\overline{\nabla}$ on $M$ induced by the connection $\nabla$ on $\overline{M}$.

**Proof.** From (4.2) and Theorem 4.3, we have $h_i^r(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$ and $i$. Thus, using (2.12), we obtain our assertion.

### 5. Totally Umbilical Screen Distributions

**Definition 5.1.** A screen distribution $S(TM)$ of $M$ is said to be totally umbilical [1] in $M$ if, for each locally second fundamental form $h_i^r$, there exist smooth functions $C_i$ on any coordinate neighborhood $\mathcal{U}$ in $M$ such that

$$h_i^r(X, PY) = C_i g(X, Y), \quad \forall X, Y \in \Gamma(TM). \quad (5.1)$$

In case $C_i = 0$ for all $i$, we say that $S(TM)$ is totally geodesic in $M$. 

Due to (2.18) and (5.1), we know that \( S(TM) \) is totally umbilical in \( M \) if and only if each shape operators \( A_{N_i} \) of \( S(TM) \) satisfies

\[
g(A_{N_i}X, PY) = C_i g(X, PY), \quad \forall X, Y \in \Gamma(TM),
\]

for some smooth functions \( C_i \) on \( \mathcal{H} \subseteq M \).

In general, \( S(TM) \) is not necessarily integrable. The following result gives equivalent conditions for the integrability of a screen \( S(TM) \).

**Theorem 5.2** (see [1]). Let \( M \) be an \( r \)-lightlike submanifold of a semi-Riemannian manifold \( (\overline{M}, \overline{g}) \). Then the following assertions are equivalent:

1. \( S(TM) \) is integrable,
2. \( h_i^* \) is symmetric on \( \Gamma(S(TM)) \), for each \( i \),
3. \( A_{N_i} \) is self-adjoint on \( \Gamma(S(TM)) \) with respect to \( g \), for each \( i \).

We know that, from (5.2), each shape operator \( A_{N_i} \) is self-adjoint on \( \Gamma(S(TM)) \) with respect to \( g \), which further follows from that above theorem that any totally umbilical screen distribution \( S(TM) \) of \( M \) is integrable.

**Theorem 5.3.** Let \( (M, g, S(TM)) \) be a characteristic \( r \)-lightlike submanifold of an indefinite \( S \)-manifold \( (\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \). If \( S(TM) \) is totally umbilical in \( M \), then \( S(TM) \) is totally geodesic in \( M \).

**Proof.** Apply the operator \( \overline{\nabla}_X \) to \( g(\overline{\varphi}N_k, N_j) = 0 \) for some \( k, j \) such that \( k \neq j \), and use (2.7) and (2.18)

\[
h_k^*(X, \overline{\varphi}N_j) = h_j^*(X, \overline{\varphi}N_k), \quad \forall X \in \Gamma(TM).
\]

Assume that \( S(TM) \) is totally umbilical in \( M \). Then we have

\[
C_k g(X, \overline{\varphi}N_j) = C_j g(X, \overline{\varphi}N_k), \quad \forall X \in \Gamma(TM).
\]

Replacing \( X \) by \( \overline{\varphi}N_k \) in (5.4) and taking \((k, j) = (1, 2), (2, 3), \ldots, (r-1, r) \) and \((r, 1) \) by turns and use the above method, we have \( C_i = 0 \) for all \( i \in \{1, \ldots, r\} \). Thus we have our assertion. \( \square \)

**Theorem 5.4.** Let \( (M, g, S(TM)) \) be a characteristic \( r \)-lightlike submanifold of an indefinite \( S \)-manifold \( (\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}) \) such that \( S(TM) \) is totally umbilical in \( M \). Then \( M \) is not irrotational.

**Proof.** Apply the operator \( \overline{\nabla}_X \) to \( g(\overline{\varphi}N_i, N_j) = 0 \) for all \( i \) and \( j \), and use (2.6), (2.10), (2.15), (2.18), and Theorem 5.3, we have

\[
h_i^*(X, \overline{\varphi}N_j) = h_j^*(X, \overline{\varphi}N_i), \quad \forall X \in \Gamma(TM).
\]
Since $S(TM)$ is totally umbilical, by Theorem 5.3, we have that $S(TM)$ is totally geodesic. Then, by (5.5), we have
\[ h^i_j \left( X, \bar{\phi} N_i \right) = 0, \quad \forall X \in \Gamma(TM). \] (5.6)

Apply the operator $\nabla_X$ to $\bar{g}(\bar{\xi}_a, \bar{\xi}_i) = 0$ and use (2.6), (2.10), (2.15), we have
\[ h^i_j \left( X, \bar{\xi}_a \right) = -\epsilon_a g \left( X, \bar{\phi} \bar{\xi}_i \right), \quad \forall X \in \Gamma(TM). \] (5.7)

Replace $X$ by $\phi N_i$ in this equation and (5.6), we have
\[ 0 = h^i_j \left( \phi N_i, \bar{\xi}_a \right) = -g \left( \phi N_i, \bar{\phi} \bar{\xi}_i \right) = -1. \] (5.8)

It is a contradiction. Thus $M$ is not irrotational. \(\square\)

Since any totally umbilical $r$-lightlike submanifold of $\overline{M}$ is irrotational, by Theorem 5.4, we have the following result.

**Corollary 5.5.** There exist no totally umbilical characteristic $r$-lightlike submanifolds $(M, g, S(TM))$ of an indefinite $S$-manifold $(\overline{M}, \bar{\phi}, \bar{\xi}_a, \bar{\eta}^i, \bar{\sigma})$ equipped with a totally umbilical screen distribution $S(TM)$ in $M$.

**References**
