Research Article

On Differential Subordinations of Multivalent Functions Involving a Certain Fractional Derivative Operator

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We investigate several results concerning the differential subordination of analytic and multivalent functions which is defined by using a certain fractional derivative operator. Some special cases are also considered.

1. Introduction and Definitions

Let \( A(p) \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \quad (1.1)
\]

which are analytic in the open unit disk \( U = \{ z : z \in \mathbb{C}, |z| < 1 \} \). Also let \( A_0 \) denote the class of all analytic functions \( p(z) \) with \( p(0) = 1 \) which are defined on \( U \). If \( f \) and \( g \) are analytic in \( U \) with \( f(0) = g(0) \), then we say that \( f \) is said to be subordinate to \( g \) in \( U \), written \( f \prec g \) or \( f(z) \prec g(z) \), if there exists the Schwarz function \( \omega \), analytic in \( U \) such that \( \omega(0) = 0, |\omega(z)| < 1 \quad (z \in U) \), and \( f(z) = g(\omega(z)) \quad (z \in U) \). In particular, if the function \( g \) is univalent, then the above subordination is equivalent to \( f(0) = g(0) \) and \( f(U) \subset g(U) \).

Let \( a, b, \) and \( c \) be complex numbers with \( c \neq 0, -1, -2, \ldots \). Then the Gaussian
The hypergeometric function \( _2F_1(a, b; c; z) \) is defined by
\[
_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},
\]
(1.2)

where \( (\eta)_k \) is the Pochhammer symbol defined, in terms of the Gamma function, by
\[
(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 1 & (k = 0), \\ \eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}). \end{cases}
\]
(1.3)

The hypergeometric function \( _2F_1(a, b; c; z) \) is analytic in \( U \), and if \( a \) or \( b \) is a negative integer, then it reduces to a polynomial.

There are a number of definitions for fractional calculus operators in the literature (cf., e.g., [1, 2]). We use here the Saigo-type fractional derivative operator defined as follows (see [3]; see also [4]).

**Definition 1.1.** Let \( 0 \leq \lambda < 1 \) and \( \mu, \nu \in \mathbb{R} \). Then the generalized fractional derivative operator \( \mathcal{D}_{0, z}^{\lambda, \mu, \nu} \) of a function \( f(z) \) is defined by
\[
\mathcal{D}_{0, z}^{\lambda, \mu, \nu} f(z) = \frac{d}{dz} \left( \frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)} \int_0^z (z - \zeta)^{\lambda-1} _2F_1 \left( \mu - \lambda, 1 - \nu; 1 - \lambda; 1 - \frac{\zeta}{z} \right) f(\zeta) d\zeta \right).
\]
(1.4)

The function \( f(z) \) is an analytic function in a simply-connected region of the \( z \)-plane containing the origin, with the order
\[
f(z) = O(|z|^{\epsilon}) \quad (z \to 0)
\]
(1.5)

for \( \epsilon > \max\{|0, \mu - \nu| - 1\} \), and the multiplicity of \( (z - \zeta)^{-1} \) is removed by requiring that \( \log(z - \zeta) \) be real when \( z - \zeta > 0 \).

**Definition 1.2.** Under the hypotheses of Definition 1.1, the fractional derivative operator \( \mathcal{D}_{0, z}^{1+m, \mu+m, \nu+m} \) of a function \( f(z) \) is defined by
\[
\mathcal{D}_{0, z}^{1+m, \mu+m, \nu+m} f(z) = \frac{d^m}{dz^m} \mathcal{D}_{0, z}^{\lambda, \mu, \nu} f(z) \quad (z \in U; m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}).
\]
(1.6)

With the aid of the above definitions, we define a modification of the fractional derivative operator \( \Delta_{z, p}^{\lambda, \mu, \nu} \) by
\[
\Delta_{z, p}^{\lambda, \mu, \nu} f(z) = \frac{\Gamma(p + 1 - \mu)\Gamma(p + 1 - \lambda + \nu)}{\Gamma(p + 1)\Gamma(p + 1 - \mu + \nu)} \left( \frac{p}{p + \lambda - 1} \right)^{\mu - \lambda} \mathcal{D}_{0, z}^{\lambda, \mu, \nu} f(z),
\]
(1.7)
for $f(z) \in \mathcal{A}(p)$ and $\mu - \nu - p < 1$. Then it is observed that $\Delta_{z,p}^{\lambda,\mu,\nu}$ also maps $\mathcal{A}(p)$ onto itself as follows:

$$
\Delta_{z,p}^{\lambda,\mu,\nu} f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+1)_k}{(p+1-\mu)_k} a_{k+p} z^{k+p}
$$

(1.8)

It is easily verified from (1.8) that

$$
z \left( \Delta_{z,p}^{\lambda,\mu,\nu} f(z) \right)' = (p-\mu) \Delta_{z,p}^{\lambda,\mu+1,\nu+1} f(z) + \mu \Delta_{z,p}^{\lambda,\mu,\nu} f(z).
$$

(1.9)

Note that $\Delta_{z,p}^{0,0,0} f = f$, $\Delta_{z,p}^{1,1,0} f = z f'/p$, and $\Delta_{z,p}^{1,1,1} f = \Omega_z^{(1,p)} f$, where $\Omega_z^{(1,p)}$ is the fractional derivative operator defined by Srivastava and Aouf [5, 6].

In this manuscript, we will use the method of differential subordination to derive certain properties of multivalent functions defined by fractional derivative operator $\Delta_{z,p}^{\lambda,\mu,\nu}$.

### 2. Main Results

In order to establish our results, we require the following lemma due to Miller and Mocanu [7].

**Lemma 2.1.** Let $q(z)$ be univalent in $U$ and let $\theta(w)$ and $\phi(w)$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = z q'(z) \phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that

1. $Q(z)$ is starlike (univalent) in $U$, and
2. $\text{Re} \{ z h'(z)/Q(z) \} = \text{Re} \{ \theta'(q(z))/\phi(q(z)) + z Q'(z)/Q(z) \} > 0$ ($z \in U$).

If $p(z)$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subset D$, and

$$
\theta(p(z)) + z p'(z) \phi(p(z)) < \theta(q(z)) + z q'(z) \phi(q(z)) = h(z),
$$

(2.1)

then $p(z) < q(z)$ and $q(z)$ is the best dominant.

We begin by proving the following

**Theorem 2.2.** Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq -1$, $\mu - \nu - p < 1$, and $\gamma(p - \mu - 1)/\beta < 2$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $U$ and satisfies

$$
\text{Re} \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \begin{cases} 
\frac{\gamma(p - \mu - 1) - \beta}{\beta} if \frac{\gamma(p - \mu - 1)}{\beta} \geq 1, \\
0 if \frac{\gamma(p - \mu - 1)}{\beta} \leq 1.
\end{cases}
$$

(2.2)
If \( f(z) \in \mathcal{A}(p) \) and

\[
\frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+\mu+1,\nu+1} f(z)} \left\{ \alpha \frac{\Delta_{z,p}^{1+\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,\mu+1,v+1} f(z)} + \beta \frac{\Delta_{z,p}^{1+\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} + \gamma \right\} < \frac{1}{p-\mu-1} \left\{ (p-\mu)(\alpha + \beta) - \alpha + [\gamma(p-\mu-1) - \beta]q(z) - \beta zq'(z) \right\},
\]

then

\[
\frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+\mu+1,\nu+1} f(z)} < q(z)
\]

and \( q(z) \) is the best dominant.

**Proof.** Define the function \( p(z) \) by

\[
p(z) = \frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} \quad (z \in \mathbb{U}).
\]

Then \( p(z) \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \). A simple computation using (2.5) gives

\[
\frac{zp'(z)}{p(z)} = \frac{z\left( \frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} \right)'}{\frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)}} - \frac{z\left( \frac{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} \right)'}{\frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)}}.
\]

By applying the identity (1.9) in (2.6), we obtain

\[
\frac{\Delta_{z,p}^{1+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} = \frac{1}{p-\mu-1} \left\{ \frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right\}.
\]

Making use of (2.5) and (2.7), we have

\[
\left\{ \begin{array}{c}
\alpha \frac{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)}{\Delta_{z,p}^{1,\mu+1,v+1} f(z)} + \beta \frac{\Delta_{z,p}^{1+2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} + \gamma \frac{\Delta_{z,p}^{\lambda,\mu,v} f(z)}{\Delta_{z,p}^{1+1,\mu+1,v+1} f(z)} \\
\end{array} \right\} = \left\{ \begin{array}{c}
\alpha \\
\beta \\
\gamma \\
\end{array} \right\} \frac{p(z)}{p-\mu-1} \left\{ \frac{p-\mu}{p(z)} - 1 - \frac{zp'(z)}{p(z)} \right\} + \gamma \right\} \frac{p(z)}{p-\mu-1}.
\]

Finally,

\[
\frac{1}{p-\mu-1} \left\{ (p-\mu)(\alpha + \beta) - \alpha + [\gamma(p-\mu-1) - \beta]p(z) - \beta zp'(z) \right\}.
\]
In view of (2.8), the subordination (2.3) becomes

\[ [\gamma(p - \mu - 1) - \beta]p(z) - \beta z p'(z) < [\gamma(p - \mu - 1) - \beta]q(z) - \beta z q'(z) \]  

(2.9)

and this can be written as (2.1), where

\[ \theta(w) = [\gamma(p - \mu - 1) - \beta]w, \quad \phi(w) = -\beta. \]  

(2.10)

Since \( \beta \neq 0 \), we find from (2.10) that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) with \( \phi(w) \neq 0 \). Let the functions \( Q(z) \) and \( h(z) \) be defined by

\[ Q(z) = z q'(z) \phi(q(z)) = -\beta z q'(z), \]
\[ h(z) = \theta(q(z)) + Q(z) = [\gamma(p - \mu - 1) - \beta]q(z) - \beta z q'(z). \]  

(2.11)

Then, by virtue of (2.2), we see that \( Q(z) \) is starlike and

\[ \text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} = \text{Re} \left\{ \frac{\beta - \gamma(p - \mu - 1)}{\beta} + \left( 1 + \frac{z q''(z)}{q'(z)} \right) \right\} > 0. \]  

(2.12)

Hence, by using Lemma 2.1, we conclude that \( p(z) < q(z) \), which completes the proof of Theorem 2.2. \( \square \)

**Remark 2.3.** If we put \( \lambda = \mu \) in Theorem 2.2, then we get new subordination result for the fractional derivative operator \( \Omega_z^{(1,p)} \) due to Srivastava and Aouf [5, 6].

**Theorem 2.4.** Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) and \( \alpha, \beta \neq 0 \), and let \( 0 \leq \lambda < 1, \mu, \nu \in \mathbb{R}, \mu \neq p, \mu - \nu - p < 1, \) and \( 1 + \delta(p - \mu)(\alpha + \gamma) / \alpha > 0 \). Suppose that \( q(z) \in \mathcal{A}_0 \) is univalent in \( \mathbb{U} \) and satisfies

\[ \text{Re} \left( 1 + \frac{z q''(z)}{q'(z)} \right) > \begin{cases} \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \geq 0. \end{cases} \]  

(2.13)

If \( f(z) \in \mathcal{A}(p) \) and

\[ \left\{ \begin{array}{c} \frac{\Delta_z^{1+1/p+1,\nu+1}}{\Delta_z^{1/p}} f(z) + \beta \left( \frac{z^p}{\Delta_z^{1/p}} f(z) \right)^\delta \\ \frac{\Delta_z^{1/p}}{\Delta_z^{1/p}} f(z) + \gamma \left( \frac{\Delta_z^{1/p}}{z^p} f(z) \right)^\delta \end{array} \right\} < \frac{\alpha}{\delta(p - \mu)} z q'(z) + (\alpha + \gamma) q(z) + \beta. \]  

(2.14)
then

$$
\left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^{\delta} < q(z)
$$

(2.15)

and \(q(z)\) is the best dominant.

**Proof.** Define the function \(p(z)\) by

$$
p(z) = \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^{\delta} (z \in U).
$$

(2.16)

Then \(p(z)\) is analytic in \(U\) with \(p(0) = 1\). By a simple computation, we find from (2.16) that

$$
\frac{zp'(z)}{p(z)} = \frac{\delta z \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} - p\delta.
$$

(2.17)

By using the identity (1.9) in (2.17), we obtain

$$
\frac{\Delta_{z,p}^{1+\lambda,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} = \frac{1}{\delta(p - \mu)} \frac{zp'(z)}{p(z)} + 1.
$$

(2.18)

Applying (2.16) and (2.18), we have

\[
\left\{ \begin{array}{l}
\alpha \frac{\Delta_{z,p}^{1+\lambda,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} + \beta \left( \frac{zp'}{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)} \right)^{\delta} + \gamma \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^{\delta} \\
\end{array} \right.
\]

\[
= \left\{ \begin{array}{l}
\alpha \left( \frac{1}{\delta(p - \mu)} \frac{zp'}{p(z)} + 1 \right) + \frac{\beta}{p(z)} + \gamma \left( \frac{\Delta_{z,p}^{\lambda,\mu,\nu} f(z)}{z^p} \right)^{\delta} \\
\end{array} \right.
\]

\[
= \frac{\alpha}{\delta(p - \mu)} zp'(z) + (\alpha + \gamma) p(z) + \beta.
\]

(2.19)

In view of (2.19), the subordination (2.14) becomes

$$
\delta(p - \mu) (\alpha + \gamma) p(z) + \alpha z p'(z) < \delta(p - \mu) (\alpha + \gamma) q(z) + \alpha z q'(z)
$$

(2.20)

and this can be written as (2.1), where

$$
\theta(w) = \delta(p - \mu) (\alpha + \gamma) w, \quad \phi(w) = \alpha.
$$

(2.21)
Since \( \alpha \neq 0 \), it follows from (2.21) that \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \) with \( \phi(w) \neq 0 \). Let the functions \( Q(z) \) and \( h(z) \) be defined by

\[
Q(z) = zq'(z)\phi(q(z)) = azq'(z),
\]
\[
h(z) = \theta(q(z)) + Q(z) = \delta(p - \mu)(\alpha + \gamma)q(z) + azq'(z).
\]

Then, by virtue of (2.13), we see that \( Q(z) \) is starlike and

\[
\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{\frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0.
\]

Hence, by using Lemma 2.1, we conclude that \( p(z) < q(z) \), which proves Theorem 2.4. \( \square \)

If we put \( \lambda = \mu = 0 \) in Theorem 2.4, then we have the following.

**Corollary 2.5.** Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) and \( \alpha, \delta \neq 0 \), and let \( 1 + p\delta(\alpha + \gamma)/\alpha > 0 \). Suppose that \( q(z) \in \mathcal{A}_0 \) is univalent in \( \mathbb{U} \) and satisfies

\[
\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} -p\delta(\alpha + \gamma)/\alpha & \text{if } \frac{\delta(\alpha + \gamma)}{\alpha} \leq 0, \\ 0 & \text{if } \frac{\delta(\alpha + \gamma)}{\alpha} \geq 0. \end{cases}
\]

If \( f(z) \in \mathcal{A}(p) \) and

\[
\left\{\alpha z f'(z)/f(z) + \beta \left(\frac{z^p}{f(z)}\right)^\delta + \gamma\right\}\left(\frac{f(z)}{z^p}\right)^\delta < \frac{\alpha}{p\delta} zq'(z) + (\alpha + \gamma)q(z) + \beta,
\]

then \( (f(z)/z^p)^\delta < q(z) \) and \( q(z) \) is the best dominant.

By putting \( \delta = \alpha \) in Corollary 2.5, we obtain the following.

**Corollary 2.6.** Let \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( \alpha \neq 0 \), and let \( 1 + p(\alpha + \gamma) > 0 \). Suppose that \( q(z) \in \mathcal{A}_0 \) is univalent in \( \mathbb{U} \) and satisfies

\[
\text{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} -p(\alpha + \gamma) & \text{if } \alpha + \gamma \leq 0, \\ 0 & \text{if } \alpha + \gamma \geq 0. \end{cases}
\]

If \( f(z) \in \mathcal{A}(p) \) and

\[
\left\{\alpha z f'(z)/f(z) + \beta \left(\frac{z^p}{f(z)}\right)^a + \gamma\right\}\left(\frac{f(z)}{z^p}\right)^a < \frac{zq'(z)}{p} + (\alpha + \gamma)q(z) + \beta,
\]

then \( (f(z)/z^p)^a < q(z) \) and \( q(z) \) is the best dominant.
By using Lemma 2.1, we obtain the following.

**Theorem 2.7.** Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta \neq 0$, and let $0 \leq \lambda < 1$, $\mu, \nu \in \mathbb{R}$, $\mu \neq 0$, $\mu - \nu - p < 1$, and $1 + \gamma / \beta > 0$. Suppose that $q(z) \in \mathcal{A}_0$ is univalent in $U$ and satisfies

\[
\Re \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} 
-\frac{\gamma}{\beta} & \text{if } \frac{\gamma}{\beta} \leq 0, \\
0 & \text{if } \frac{\gamma}{\beta} \geq 0.
\end{cases}
\]  

(2.28)

If $f(z) \in \mathcal{A}(p)$ and

\[
\left\{ \alpha \beta \left[ (p - \mu - 1) \frac{\Delta_{z,p}^{1,2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} + 1 \right] + Y \right\}
\]

\[
\cdot \left( \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} \right)^{\alpha} < \beta z q'(z) + \gamma q(z),
\]

then

\[
\left( \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} \right)^{\alpha} < q(z)
\]  

(2.30)

and $q(z)$ is the best dominant.

**Proof.** Define the function $p(z)$ by

\[
p(z) = \left( \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} \right)^{\alpha} \quad (z \in U).
\]  

(2.31)

Then $p(z)$ is analytic in $U$ with $p(0) = 1$. A simple computation using (1.9) and (2.31) gives

\[
\frac{1}{\alpha} \frac{zp'(z)}{p(z)} = (p - \mu - 1) \frac{\Delta_{z,p}^{1,2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} + 1.
\]  

(2.32)

By using (2.29), (2.31), and (2.32), we get

\[
\left\{ \alpha \beta \left[ (p - \mu - 1) \frac{\Delta_{z,p}^{1,2,\mu+2,\nu+2} f(z)}{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)} - (p - \mu) \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} + 1 \right] + Y \right\}
\]

\[
\cdot \left( \frac{\Delta_{z,p}^{1,1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,1,\mu,\nu} f(z)} \right)^{\alpha} = \beta z p'(z) + \gamma p(z).
\]  

(2.33)
And this can be written as (2.1) when \( \theta(w) = \gamma w \) and \( \phi(w) = \beta \). Note that \( \phi(w) \neq 0 \) and \( \theta(w) \) and \( \phi(w) \) are analytic in \( \mathbb{C} \). Let the functions \( Q(z) \) and \( h(z) \) be defined by

\[
Q(z) = zq(z)\phi(q(z)) = \beta zq(z),
\]

\[
h(z) = \theta(q(z)) + Q(z) = \gamma q(z) + \beta zq(z).
\]

(2.34)

Then, by virtue of (2.28), we see that \( Q(z) \) is starlike and

\[
\Re\left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re\left\{ \frac{1 + \frac{zq''(z)}{q'(z)}}{\beta} + 1 \right\} > 0.
\]

(2.35)

Hence, by applying Lemma 2.1, we observe that \( p(z) < q(z) \), which evidently proves Theorem 2.7.

Finally, we prove

**Theorem 2.8.** Let \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) and \( \alpha, \delta \neq 0 \), and let \( 0 \leq \lambda < 1, \mu, \nu \in \mathbb{R}, \mu \neq p, p + 1 - \mu + \nu > 0 \) and \( 1 - \delta(p - \mu)(\alpha + \gamma)/\alpha > 0 \). Suppose that \( q(z) \in A_{\theta} \) be univalent in \( \mathbb{U} \) and satisfies

\[
\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \begin{cases} \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \geq 0, \\ 0 & \text{if } \frac{\delta(p - \mu)(\alpha + \gamma)}{\alpha} \leq 0. \end{cases}
\]

(2.36)

If \( f(z) \in A(p) \) and

\[
\left\{ \alpha \frac{\Delta_{z,p}^{1+1,\mu+1,\nu+1} f(z)}{\Delta_{z,p}^{1,\mu+1,\nu+1} f(z)} + \beta \left( \frac{\Delta_{z,p}^{1,\mu+1} f(z)}{z^p} \right)^\delta + \gamma \left( \frac{z^p}{\Delta_{z,p}^{1,\mu+1} f(z)} \right)^\delta \right\} < \beta + (\alpha + \gamma)q(z) - \frac{\alpha}{\delta(p - \mu)} zq'(z),
\]

then

\[
\left( \frac{z^p}{\Delta_{z,p}^{1,\mu+1} f(z)} \right)^\delta < q(z)
\]

(2.37)

and \( q(z) \) is the best dominant.

**Proof.** If we define the function \( p(z) \) by

\[
p(z) = \left( \frac{z^p}{\Delta_{z,p}^{1,\mu+1} f(z)} \right)^\delta (z \in \mathbb{U}),
\]

(2.38)
then \( p(z) \) is analytic in \( \mathbb{U} \) with \( p(0) = 1 \). Hence, by using the same techniques as detailed in the proof of Theorem 2.2, we obtain the desired result.

By taking \( \lambda = \mu = 0 \) in Theorem 2.8 and after a suitable change in the parameters, we have the following.

**Corollary 2.9.** Let \( \alpha \in \mathbb{R} \setminus \{0\} \) and \( p\alpha < 1/2 \). Suppose that \( q(z) \in \mathcal{A}_0 \) is univalent in \( \mathbb{U} \) and satisfies

\[
\text{Re} \left( 1 + \frac{zq''(z)}{q'(z)} \right) > \begin{cases} 2p\alpha & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha < 0. \end{cases}
\] (2.40)

If \( f(z) \in \mathcal{A}(p) \) and

\[
\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left( \frac{z''}{f''(z)} \right)^{\alpha} < 2\alpha q(z) - \frac{1}{p} zq'(z),
\] (2.41)

then \( (z''/f(z))^{\alpha} < q(z) \) and \( q(z) \) is the best dominant.

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**References**


