Research Article

\(\mathcal{N}\)-Structures Applied to Closed Ideals in BCH-Algebras

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Received 28 December 2009; Revised 2 February 2010; Accepted 9 February 2010

Academic Editor: Ilya M. Spitkovsky

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The notions of \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals in BCH-algebras are introduced, and the relation between \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals is considered. Characterizations of \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals are provided. Using special subsets, \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals are constructed. A condition for an \(\mathcal{N}\)-subalgebra to be an \(\mathcal{N}\)-closed ideal is discussed. Given an \(\mathcal{N}\)-structure, the greatest \(\mathcal{N}\)-closed ideal which is contained in the \(\mathcal{N}\)-structure is established.

1. Introduction

In [1, 2], Hu and Li introduced the notion of BCH-algebras which are a generalization of BCK/BCI-algebras. Ahmad [3] classified BCH-algebras, and decompositions of BCH-algebras are considered by Dudek and Thomys [4]. Jun et al. [5] discussed the notion of \(\mathcal{N}\)-structures and applied it to BCK/BCI-algebras. In [6], Chaudhry et al. studied closed ideals and filters in BCH-algebras. In this paper, we apply the \(\mathcal{N}\)-structures to the closed ideal theory in BCH-algebras. We introduced the notion of \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals in BCH-algebras, and investigate the relation between \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals. We provide characterizations of \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals. Using special subsets, we construct \(\mathcal{N}\)-subalgebras and \(\mathcal{N}\)-closed ideals. We provide a condition for an \(\mathcal{N}\)-subalgebra to be an \(\mathcal{N}\)-closed ideal. Given an \(\mathcal{N}\)-structure \((X, \mu)\), we make the greatest \(\mathcal{N}\)-closed ideal which is contained in \((X, \mu)\).
2. Preliminaries

By a BCH-algebra we mean an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following axioms:

\((H1)\) \(x * x = 0\),
\((H2)\) \(x * y = 0\) and \(y * x = 0\) imply \(x = y\),
\((H3)\) \((x * y) * z = (x * z) * y\)

for all \(x, y, z \in X\).

In a BCH-algebra \(X\), the following conditions are valid (see [1, 4]).

\((a1)\) \(x * 0 = x\),
\((a2)\) \(x * 0 = 0\) implies \(x = 0\),
\((a3)\) \(0 * (x * y) = (0 * x) * (0 * y)\),
\((a4)\) \(0 * (0 * (0 * x)) = 0 * x\).

A nonempty subset \(S\) of a BCH-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A nonempty subset \(A\) of a BCH-algebra \(X\) is called a closed ideal of \(X\) (see [7]) if it satisfies:

\((1)\) (for all \(x \in X\))(\(x \in A \Rightarrow 0 * x \in A\)),
\((2)\) (for all \(y \in X\))(for all \(x \in A\))(\(y * x \in A \Rightarrow y \in A\)).

Note that every closed ideal is a subalgebra, but the converse is not true (see [7]). Since every closed ideal is a subalgebra, we know that any closed ideal contains the element \(0\). Denote by \(S(X)\) and \(\mathcal{O}(X)\) the set of all subalgebras and closed ideals of \(X\), respectively.

For any family \(\{a_i \mid i \in \Lambda\}\) of real numbers, we define

\[ \vee \{a_i \mid i \in \Lambda\} := \begin{cases} \max \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup \{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases} \tag{2.1} \]

\[ \wedge \{a_i \mid i \in \Lambda\} := \begin{cases} \min \{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf \{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases} \tag{2.2} \]

3. \(\mathcal{A}\)-Closed Ideals of BCH-Algebras

Denote by \(\mathcal{F}(X, [-1,0])\) the collection of functions from a set \(X\) to \([-1,0]\). We say that an element of \(\mathcal{F}(X, [-1,0])\) is a negative-valued function from \(X\) to \([-1,0]\) (briefly, \(\mathcal{A}\)-function on \(X\)). By an \(\mathcal{A}\)-structure we mean an ordered pair \((X, \mu)\) of \(X\) and an \(\mathcal{A}\)-function \(\mu\) on \(X\). In what follows, let \(X\) denote a BCH-algebra and \(\mu\) an \(\mathcal{A}\)-function on \(X\) unless otherwise specified.

For any \(\mathcal{A}\)-structure \((X, \mu)\) and \(t \in [-1,0]\), the set

\[ C(\mu; t) := \{ x \in X \mid \mu(x) \leq t \} \tag{3.1} \]

is called a closed \((\mu, t)\)-cut of \((X, \mu)\).
Using the similar method to the transfer principle in fuzzy theory (see [8, 9]), we can consider transfer principle in \(\mathcal{N}\)-structures. Let \(A\) be a subset of \(X\) and satisfy the following property \(\mathcal{P}\) expressed by a first-order formula:

\[
\mathcal{P}: \quad t_1(x, \ldots, y) \in A, \ldots, t_n(x, \ldots, y) \in A \quad \text{and} \quad t(x, \ldots, y) \in A,
\]

where \(t_1(x, \ldots, y), \ldots, t_n(x, \ldots, y)\) and \(t(x, \ldots, y)\) are terms of \(X\) constructed by variables \(x, \ldots, y\). We note that the subset \(A\) satisfies the property \(\mathcal{P}\) if, for all elements \(a, \ldots, b \in X\), \(t(a, \ldots, b) \in \mathcal{A}\) whenever \(t_1(a, \ldots, b), \ldots, t_n(a, \ldots, b) \in \mathcal{A}\). For the subset \(A\) we define an \(\mathcal{N}\)-structure \((X, \mu_A)\) which satisfies the following property

\[
\mathcal{P}_\mu: \quad \mu_A(t(x, \ldots, y)) \leq \lor \{\mu_A(t_1(x, \ldots, y)), \ldots, \mu_A(t_n(x, \ldots, y))\}.
\]

We establish a statement without proof, and we call it \(\mathcal{N}\)-transfer principle in \(\mathcal{N}\)-structures.

**Theorem 3.1.** (\(\mathcal{N}\)-transfer principle) An \(\mathcal{N}\)-structure \((X, \mu)\) satisfies the property \(\mathcal{P}\) if and only if for all \(\alpha \in [-1, 0]\),

\[
C(\mu; \alpha) \neq \emptyset \iff C(\mu; \alpha) \text{ satisfies the property } \mathcal{P}.
\]

**Definition 3.2.** By an \(\mathcal{N}\)-subalgebra of \(X\) we mean an \(\mathcal{N}\)-structure \((X, \mu)\) in which \(\mu\) satisfies:

\[
(\forall x, y \in X) \quad (\mu(x * y) \leq \lor \{\mu(x), \mu(y)\}).
\]

**Theorem 3.3.** For an \(\mathcal{N}\)-structure \((X, \mu)\), the following are equivalent:

1. \((X, \mu)\) is an \(\mathcal{N}\)-subalgebra of \(X\);
2. \((\forall t \in [-1, 0])(C(\mu; t) \in S(X) \cup \{\emptyset\})\).

**Proof.** It follows from the \(\mathcal{N}\)-transfer principle. \(\square\)

**Definition 3.4.** By an \(\mathcal{N}\)-closed ideal of \(X\) we mean an \(\mathcal{N}\)-structure \((X, \mu)\) in which \(\mu\) satisfies:

\[
(\forall x, y \in X) \quad (\mu(0 * x) \leq \mu(x) \leq \lor \{\mu(x * y), \mu(y)\}).
\]

It is clear that if \((X, \mu)\) is an \(\mathcal{N}\)-closed ideal or an \(\mathcal{N}\)-subalgebra, then \(\mu(0) \leq \mu(x)\) for all \(x \in X\).
Table 1: Cayley table.

<table>
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<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>1</td>
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<td>2</td>
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<td>3</td>
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<td>0</td>
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<td>1</td>
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</tr>
</tbody>
</table>

Theorem 3.5. Every $\mathcal{N}$-closed ideal is an $\mathcal{N}$-subalgebra.

Proof. Let $(X, \mu)$ be an $\mathcal{N}$-closed ideal of $X$. For any $x, y \in X$, we have

\[
\mu(x \ast y) \leq \vee \{ \mu((x \ast y) \ast x), \mu(x) \}
= \vee \{ \mu((x \ast x) \ast y), \mu(x) \}
= \vee \{ \mu(x \ast x), \mu(x) \}
\leq \vee \{ \mu(x), \mu(y) \}.
\]

(3.7)

Hence $(X, \mu)$ is an $\mathcal{N}$-subalgebra of $X$. \hfill $\Box$

The converse of Theorem 3.5 may not be true as seen in the following example.

Example 3.6. Consider a BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1 (see [7]). Let $(X, \mu)$ be an $\mathcal{N}$-structure in which $\mu$ is given by

\[
\mu = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
-0.8 & -0.3 & -0.3 & -0.3 & -0.8
\end{pmatrix}.
\]

(3.8)

It is easy to check that $(X, \mu)$ is an $\mathcal{N}$-subalgebra of $X$ but it is not an $\mathcal{N}$-closed ideal of $X$ since $\mu(3) = -0.3 > -0.8 = \vee \{ \mu(3 \ast 4), \mu(4) \}$.

In order to discuss the converse of Theorem 3.5 we need to strengthen some conditions. We first consider the following lemma.

Lemma 3.7. Every $\mathcal{N}$-subalgebra $(X, \mu)$ of $X$ satisfies the following inequality:

\[
(\forall x \in X) \quad (\mu(x) \geq \mu(0 \ast x)).
\]

(3.9)

Proof. For any $x \in X$, we get

\[
\mu(0 \ast x) \leq \vee \{ \mu(0), \mu(x) \} = \vee \{ \mu(x \ast x), \mu(x) \}
= \vee \{ \vee \{ \mu(x), \mu(x) \}, \mu(x) \} = \mu(x),
\]

(3.10)

which is the desired result. \hfill $\Box$
Theorem 3.8. If an $\mathcal{A}$-subalgebra $(X, \mu)$ satisfies

\[(\forall x, y \in X) \quad (\mu(y) \leq \vee \{\mu(y \ast x), \mu(x)\}), \tag{3.11}\]

then $(X, \mu)$ is an $\mathcal{A}$-closed ideal of $X$.

Proof. It is straightforward by Lemma 3.7. \hfill \Box

Proposition 3.9. Let $(X, \mu)$ be an $\mathcal{A}$-closed ideal of $X$ that satisfies the following inequality

\[(\forall x \in X) \quad (\mu(x) \leq \mu(0 \ast x)). \tag{3.12}\]

Then $(X, \mu)$ satisfies the inequality

\[(\forall x, y \in X) \quad (\mu(y \ast x) \leq \mu(x \ast y)). \tag{3.13}\]

Proof. Using (3.12), (3.6), (a3), (H1), and (H3), we have

\[
\begin{align*}
\mu(y \ast x) & \leq \mu(0 \ast (y \ast x)) \\
& \leq \vee \{\mu((0 \ast (y \ast x)) \ast (x \ast y)), \mu(x \ast y)\} \\
& = \vee \{\mu(((0 \ast y) \ast (0 \ast x)) \ast (x \ast y)), \mu(x \ast y)\} \\
& = \vee \{\mu(((0 \ast y) \ast (x \ast y)) \ast (0 \ast x)), \mu(x \ast y)\} \\
& = \vee \{\mu(((0 \ast x) \ast (0 \ast y)) \ast (0 \ast x) \ast y), \mu(x \ast y)\} \\
& = \vee \{\mu(0 \ast (0 \ast y) \ast y), \mu(x \ast y)\} \\
& = \vee \{\mu(0), \mu(x \ast y)\} = \mu(x \ast y)
\end{align*}
\]

for all $x, y \in X$. \hfill \Box

Using the $\mathcal{A}$-transfer principle, we have a characterization of an $\mathcal{A}$-closed ideal.

Theorem 3.10. For an $\mathcal{A}$-structure $(X, \mu)$, the following are equivalent:

1. $(X, \mu)$ is an $\mathcal{A}$-closed ideal of $X$.
2. (for all $t \in [-1, 0]) (\mathcal{C}(\mu; t) \in \mathcal{D}(X) \cup \{\emptyset\})$.

Consider two subsets of $X$ as follows:

\[
D_1 := \{x \in X \mid 0 \ast x = 0\}, \quad D_2 := \{x \in X \mid 0 \ast (0 \ast x) = x\}. \tag{3.15}
\]

Since $D_1$ and $D_2$ are a closed ideal and a subalgebra, respectively, the following theorems are direct results of the $\mathcal{A}$-transfer principle.
Theorem 3.11. Let \((X, \mu)\) be an \(\mathcal{A}\)-structure in which \(\mu\) is given by

\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in D_1, \\
\beta & \text{otherwise}
\end{cases}
\]  
(3.16)

for all \(x \in X\) where \(\alpha < \beta\). Then \((X, \mu)\) is an \(\mathcal{A}\)-closed ideal of \(X\).

Theorem 3.12. Let \((X, \mu)\) be an \(\mathcal{A}\)-structure in which \(\mu\) is given by

\[
\mu(x) = \begin{cases} 
\alpha & \text{if } x \in D_2, \\
\beta & \text{otherwise}
\end{cases}
\]  
(3.17)

for all \(x \in X\) where \(\alpha < \beta\). Then \((X, \mu)\) is an \(\mathcal{A}\)-subalgebra of \(X\).

We provide a condition for an \(\mathcal{A}\)-subalgebra to be an \(\mathcal{A}\)-closed ideal.

Theorem 3.13. Let \((X, \mu)\) be an \(\mathcal{A}\)-subalgebra of \(X\) in which \(\mu\) satisfies

\[
(\forall x, y \in X) \quad (\mu(y \ast x) \geq \mu(x \ast y)).
\]  
(3.18)

Then \((X, \mu)\) is an \(\mathcal{A}\)-closed ideal of \(X\).

Proof. Taking \(x = 0\) in (3.18) induces \(\mu(0 \ast y) \leq \mu(y \ast 0) = \mu(y)\) for all \(y \in X\). Using (a1), (3.18), (H1), (H3), and (3.5), we have

\[
\begin{align*}
\mu(y) &= \mu(y \ast 0) \leq \mu(0 \ast y) \\
&= \mu((x \ast x) \ast y) = \mu((x \ast y) \ast x) \\
&\leq \vee \{\mu(x \ast y), \mu(x)\} \leq \vee \{\mu(y \ast x), \mu(x)\}
\end{align*}
\]  
(3.19)

for all \(x, y \in X\). Therefore \((X, \mu)\) is an \(\mathcal{A}\)-closed ideal of \(X\).

For any \(\mathcal{A}\)-structure \((X, \mu)\) and any element \(w \in X\), we consider the set

\[
X_w := \{x \in X \mid \mu(x) \leq \mu(w)\}.
\]  
(3.20)

Then \(X_w\) is nonempty subset of \(X\).
Theorem 3.14. If an $\mathcal{A}$-structure $(X, \mu)$ is an $\mathcal{A}$-closed ideal of $X$, then $X_w$ is a closed ideal of $X$ for all $w \in X$.

Proof. If $x \in X_w$, then $\mu(x) \leq \mu(w)$ which implies from (3.6) that $\mu(0 \ast x) \leq \mu(x) \leq \mu(w)$. Thus $0 \ast x \in X_w$. Let $x, y \in X$ be such that $y \in X_w$ and $x \ast y \in X_w$. Then $\mu(y) \leq \mu(w)$ and $\mu(x \ast y) \leq \mu(w)$. Using (3.6), we have

$$
\mu(x) \leq \mu(\{x \ast y, \mu(y)\} \leq \mu(w), \ i.e., \ x \in X_w.
$$

Therefore $X_w$ is a closed ideal of $X$.

Proposition 3.15. Let $(X, \mu)$ be an $\mathcal{A}$-structure such that $X_w$ is a closed ideal of $X$ for all $w \in X$. Then $(X, \mu)$ satisfies the following assertion:

$$
\mu(x) \geq \mu(y \ast z), \mu(z) \implies \mu(x) \geq \mu(y)
$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $\mu(x) \geq \mu(y \ast z), \mu(z)$. Then $y \ast z \in X_x$ and $z \in X_x$. Since $X_x$ is a closed ideal of $X$, it follows that $y \in X_x$ so that $\mu(y) \leq \mu(x)$. This completes the proof.

Theorem 3.16. If an $\mathcal{A}$-structure $(X, \mu)$ satisfies (3.22) and $\mu(0 \ast x) \leq \mu(x)$ for all $x \in X$, then $X_w$ is a closed ideal of $X$ for all $w \in X$.

Proof. For each $w \in X$, let $x, y \in X$ be such that $x \ast y \in X_w$ and $y \in X_w$. Then $\mu(x \ast y) \leq \mu(w)$ and $\mu(y) \leq \mu(w)$, which imply that $\mu(x \ast y) \leq \mu(w)$. It follows from (3.22) that $\mu(x) \leq \mu(w)$ so that $x \in X_w$. If $x \in X_w$, then $\mu(0 \ast x) \leq \mu(x) \leq \mu(w)$ by assumption. Hence $0 \ast x \in X_w$. Therefore $X_w$ is a closed ideal of $X$.

Theorem 3.17. Given an $\mathcal{A}$-structure $(X, \mu)$, let $(X, \mu^*)$ be an $\mathcal{A}$-structure in which $\mu^*$ is defined by

$$
\mu^*(x) = \land \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \}
$$

for all $x \in X$. Then $(X, \mu^*)$ is the greatest $\mathcal{A}$-closed ideal of $X$ such that $(X, \mu^*) \subseteq (X, \mu)$, where $\langle C(\mu; t) \rangle$ is a closed ideal of $X$ generated by $C(\mu; t)$.

Proof. For any $s \in \text{Im}(\mu^*)$, let $s_n = s + (1/n)$ for any $n \in \mathbb{N}$. Let $x \in C(\mu^*; s)$. Then $\mu^*(x) \leq s$, and so

$$
\land \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \} \leq s < s + \frac{1}{n} = s_n
$$

for all $n \in \mathbb{N}$. Hence there exists $t^* \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle \}$ such that $t^* < s_n$. Thus $C(\mu; t^*) \subseteq C(\mu; s_n)$, and so $x \in \langle C(\mu; t^*) \rangle \subseteq \langle C(\mu; s_n) \rangle$ for all $n \in \mathbb{N}$. Consequently
so that $x \in \bigcap_{n \in \mathbb{N}} (C(\mu; s_n))$. On the other hand, if $x \in \bigcap_{n \in \mathbb{N}} (C(\mu; s_n))$, then $s_n \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}$ for any $n \in \mathbb{N}$. Therefore

$$
\mu^*(x) = \land \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\} \leq s_n = s + \frac{1}{n}
$$

(3.25)

for all $n \in \mathbb{N}$. Since $n$ is arbitrary, it follows that $\mu^*(x) \leq s$ so that $x \in C(\mu^*; s)$. Thus $C(\mu^*; s) = \bigcap_{n \in \mathbb{N}} (C(\mu; s_n))$, which is a closed ideal of $X$. Using Theorem 3.10, we conclude that $(X, \mu^*)$ is an $\mathcal{A}$-closed ideal of $X$. For any $x \in X$, let

$$
s \in \{t \in [-1, 0] \mid x \in C(\mu; t)\}.
$$

(3.26)

Then $x \in C(\mu; s)$ and thus $x \in \langle C(\mu; s) \rangle$. It follows that

$$
s \in \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}
$$

(3.27)

so that $\{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\} \subseteq \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}$. Hence

$$
\mu(x) = \land \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}
$$

$$
\geq \land \{t \in [-1, 0] \mid x \in \langle C(\mu; t) \rangle\}
$$

$$
= \mu^*(x),
$$

(3.28)

and so $(X, \mu^*) \subseteq (X, \mu)$. Finally, let $(X, \nu)$ be an $\mathcal{A}$-closed ideal of $X$ such that $(X, \nu) \subseteq (X, \mu)$. Let $x \in X$. If $\mu^*(x) = 0$, then clearly $\nu(x) \leq \mu^*(x)$. Assume that $\mu^*(x) > 0$. Then $x \in C(\mu^*; s) = \bigcap_{n \in \mathbb{N}} (C(\mu; s_n))$, and so $x \in \langle C(\mu; s_n) \rangle$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \leq \mu(x) \leq s_n = s + (1/n)$ for all $n \in \mathbb{N}$ so that $\nu(x) \leq s = \mu^*(x)$ since $n$ is arbitrary. This shows that $(X, \nu) \subseteq (X, \mu^*)$. This completes the proof.

\section*{Acknowledgments}

The authors wish to thank the anonymous reviewers for their valuable suggestions. The first author was supported by the fund of sabbatical year program (2009), Gyeongsang National University.

\section*{References}

