Research Article

Brandt Extensions and Primitive Topological Inverse Semigroups

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We study countably compact and absolutely $H$-closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

In this paper all spaces are Hausdorff. A semigroup is a nonempty set with a binary associative operation. A semigroup $S$ is called inverse if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element $y$ in $S$ is called inverse to $x$ and denoted by $x^{-1}$. The map defined on an inverse semigroup $S$ which maps to any element $x$ of $S$ its inverse $x^{-1}$ is called the inversion.

A topological semigroup is a Hausdorff topological space with a jointly continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an inverse topological semigroup. A topological inverse semigroup is an inverse topological semigroup with continuous inversion. A topological group is a topological space with a continuous group operation and an inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [1, Proposition II.1]). A Hausdorff topology $\tau$ on a (inverse) semigroup $S$ is called (inverse) semigroup if $(S, \tau)$ is a topological (inverse) semigroup.

Further we shall follow the terminology of [2-8]. If $S$ is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of $S$, and by $S^1 [S^0]$ we denote the semigroup $S$ with the adjoined unit [zero] (see [7, page 2]). Also if a semigroup $S$ has zero $0_S$, then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$. If $Y$ is a subspace of a topological space $X$ and $A \subseteq Y$, then
by $\text{cl}_Y(A)$ we denote the topological closure of $A$ in $Y$. The set of positive integers is denoted by $\mathbb{N}$.

If $E$ is a semilattice, then the semilattice operation on $E$ determines the partial order $\preceq$ on $E$:

$$e \preceq f \quad \text{iff} \quad ef = fe = e.$$  \hspace{1cm} (1)

This order is called \emph{natural}. An element $e$ of a partially ordered set $X$ is called \emph{minimal} if $f \preceq e$ implies $f = e$ for $f \in X$. An idempotent $e$ of a semigroup $S$ without zero (with zero) is called \emph{primitive} if $e$ is a minimal element in $E(S)$ (in $(E(S))^*$).

Let $S$ be a semigroup with zero and let $I_1$ be a set of cardinality $\lambda \geq 1$. On the set $B_3(S) = (I_1 \times S \times I_1) \cup \{0\}$ we define the semigroup operation as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} 
(\alpha, ab, \delta), & \text{if } \beta = \gamma, \\
0, & \text{if } \beta \neq \gamma,
\end{cases}$$  \hspace{1cm} (2)

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_1$ and $a, b \in S$. If $S = S^1$, then the semigroup $B_3(S)$ is called the \emph{Brandt $\lambda$-extension of the semigroup $S$} [9]. Obviously, $\mathcal{J} = \{0\} \cup \{(a, \emptyset, \beta) \mid \emptyset$ is the zero of $S\}$ is an ideal of $B_3(S)$. We put $B_3^0(S) = B_3(S)/\mathcal{J}$ and we shall call $B_3^0(S)$ the \emph{Brandt $\lambda$-extension of the semigroup $S$ with zero} [10]. Further, if $A \subseteq S$, then we shall denote $A_{\alpha, \beta} = \{(a, s, \beta) \mid s \in A\}$ if $A$ does not contain zero, and $A_{\alpha, \beta} = \{(a, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_1$. If $\mathcal{J}$ is a trivial semigroup (i.e., $\mathcal{J}$ contains only one element), then by $\mathcal{J}^0$ we denote the semigroup $\mathcal{J}$ with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt $\lambda^0$-extension of the semigroup $\mathcal{J}^0$ is isomorphic to the semigroup of $I_1 \times I_1$-matrix units and any Brandt $\lambda^0$-extension of a semigroup with zero contains the semigroup of $I_1 \times I_1$-matrix units. Further by $B_3$ we shall denote the semigroup of $I_1 \times I_1$-matrix units and by $B_3^0(1)$ the subsemigroup of $I_1 \times I_1$-matrix units of the Brandt $\lambda^0$-extension of a monoid $S$ with zero. A completely 0-simple inverse semigroup is called a \emph{Brandt semigroup} [8]. A semigroup $S$ is a Brandt semigroup if and only if $S$ is isomorphic to a Brandt $\lambda$-extension $B_3(G)$ of some group $G$ [8, Theorem II.3.5].

A nontrivial inverse semigroup is called a \emph{primitive inverse semigroup} if all its nonzero idempotents are primitive [8]. A semigroup $S$ is a primitive inverse semigroup if and only if $S$ is an orthogonal sum of Brandt semigroups [8, Theorem II.4.3].

Green’s relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ on a semigroup $S$ are defined by

(i) $a\mathcal{L}b$ if and only if $a \cup Sa = b \cup Sb$;

(ii) $a\mathcal{R}b$ if and only if $a \cup aS = b \cup bS$;

(iii) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$

for $a, b \in S$. For details about Green’s relations, see [4, Section 2.1] or [11]. We observe that two nonzero elements $(\alpha_1, s, \beta_1)$ and $(\alpha_2, t, \beta_2)$ of a Brandt semigroup $B_3(G)$, $s, t \in G$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_1$, are $\mathcal{H}$-equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ (see [8, page 93]).

By $\mathcal{S}$ we denote some class of topological semigroups.

\textbf{Definition 1} (see [9, 12]). A semigroup $S \in \mathcal{S}$ is called \emph{H-closed in $\mathcal{S}$}, if $S$ is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains $S$ as a subsemigroup.
If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup $S$ is called $H$-closed.

**Definition 2** (see [13, 14]). A topological semigroup $S \in \mathcal{S}$ is called **absolutely $H$-closed in the class $\mathcal{S}$** if any continuous homomorphic image of $S$ into $T \in \mathcal{S}$ is $H$-closed in $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup $S$ is called **absolutely $H$-closed**.

A semigroup $S$ is called **algebraically closed in $\mathcal{S}$** if $S$ with any semigroup topology $\tau$ is $H$-closed in $\mathcal{S}$ and $(S, \tau) \in \mathcal{S}$ [9]. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup $S$ is called **algebraically closed**. A semigroup $S$ is called **algebraically $h$-closed in $\mathcal{S}$** if $S$ with the discrete topology $\delta$ is absolutely $H$-closed in $\mathcal{S}$ and $(S, \delta) \in \mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then the semigroup $S$ is called **algebraically $h$-closed**.

Absolutely $H$-closed semigroups and algebraically $h$-closed semigroups were introduced by Stepp in [14]. There, they were called **absolutely maximal** and **algebraic maximal**, respectively.

**Definition 3** (see [9]). Let $\lambda$ be a cardinal $\geq 1$ and $(S, \tau) \in \mathcal{S}$. Let $\tau_B$ be a topology on $B_1(S)$ such that

(i) $(B_1(S), \tau_B) \in \mathcal{S}$;
(ii) $\tau_B|_{(S,\alpha)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_1(S), \tau_B)$ is called a **topological Brandt $\lambda$-extension of $(S, \tau)$ in $\mathcal{S}$**. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_1(S), \tau_B)$ is called a **topological Brandt $\lambda$-extension of $(S, \tau)$**.

**Definition 4** (see [10]). Let $\mathcal{S}_0$ be some class of topological semigroups with zero. Let $\lambda$ be a cardinal $\geq 1$ and $(S, \tau) \in \mathcal{S}_0$. Let $\tau_B$ be a topology on $B_1^0(S)$ such that

(a) $(B_1^0(S), \tau_B) \in \mathcal{S}_0$;
(b) $\tau_B|_{(S,\alpha,\delta)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_1^0(S), \tau_B)$ is called a **topological Brandt $\lambda^0$-extension of $(S, \tau)$ in $\mathcal{S}_0$**. If $\mathcal{S}_0$ coincides with the class of all topological semigroups, then $(B_1^0(S), \tau_B)$ is called a **topological Brandt $\lambda^0$-extension of $(S, \tau)$**.

Gutik and Pavlyk in [9] proved that the following conditions for a topological semigroup $S$ are equivalent:

(i) $S$ is an $H$-closed semigroup in the class of topological inverse semigroups;
(ii) there exists a cardinal $\lambda \geq 1$ such that any topological Brandt $\lambda$-extension of $S$ is $H$-closed in the class of topological inverse semigroups;
(iii) for any cardinal $\lambda \geq 1$ every topological Brandt $\lambda$-extension of $S$ is $H$-closed in the class of topological inverse semigroups.

In [13] they showed that the similar statement holds for absolutely $H$-closed topological semigroups in the class of topological inverse semigroups.

In [10], Gutik and Pavlyk proved the following.
Theorem 5. Let $S$ be a topological inverse monoid with zero. Then the following conditions are equivalent:

(i) $S$ is an (absolutely) $H$-closed semigroup in the class of topological inverse semigroups;
(ii) there exists a cardinal $\lambda \geq 1$ such that any topological Brandt $\lambda^0$-extension $B_\lambda^0(S)$ of the semigroup $S$ is (absolutely) $H$-closed in the class of topological inverse semigroups;
(iii) for each cardinal $\lambda \geq 1$, every topological Brandt $\lambda^0$-extension $B_\lambda^0(S)$ of the semigroup $S$ is (absolutely) $H$-closed in the class of topological inverse semigroups.

Also, an example of an absolutely $H$-closed topological semilattice $\mathcal{H}$ with zero and a topological Brandt $\lambda^0$-extension $B_\lambda^0(\mathcal{H})$ of $\mathcal{H}$ with the following properties was constructed in [10]:

(i) $B_\lambda^0(\mathcal{H})$ is an absolutely $H$-closed semigroup for any infinite cardinal $\lambda$;
(ii) $B_\lambda^0(\mathcal{H})$ is a $\sigma$-compact inverse topological semigroup for any countable cardinal $\lambda$;
(iii) $B_\lambda^0(\mathcal{H})$ contains an absolutely $H$-closed ideal $J$ such that the Rees quotient semigroup $B_\lambda^0(\mathcal{H})/J$ is not a topological semigroup.

We observe that for any topological Brandt $\lambda$-extension $B_\lambda(S)$ of a topological semigroup $S$ there exist a topological monoid $T$ with zero and a topological Brandt $\lambda^0$-extension $B_\lambda^0(T)$ of $T$, such that the semigroups $B_\lambda(S)$ and $B_\lambda^0(T)$ are topologically isomorphic. Algebraic properties of Brandt $\lambda^0$-extensions of monoids with zero and nontrivial homomorphisms between Brandt $\lambda^0$-extensions of monoids with zero and a category whose objects are ingredients of the construction of Brandt $\lambda^0$-extensions of monoids with zeros were described in [15]. Also, in [15, 16] was described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt $\lambda^0$-extensions of topological monoids with zeros.

In [9, 17] for every infinite cardinal $\lambda$, semigroup topologies on Brandt $\lambda$-extensions which preserve an $H$-closedness and an absolute $H$-closedness were constructed. An example of a non-$H$-closed topological inverse semigroup $S$ in the class of topological inverse semigroups such that for any cardinal $\lambda \geq 1$ there exists an absolute $H$-closed topological Brandt $\lambda$-extension of the semigroup $S$ in the class of topological semigroups was constructed in [17].

In this paper we study (countably) compact and (absolutely) $H$-closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Lemma 6. Let $E$ be a topological semilattice with zero 0 such that every nonzero idempotent of $E$ is primitive. Then every nonzero element of $E$ is an isolated point in $E$.

Proof. Let $x \in E^*$. Since $E$ is a Hausdorff topological semilattice, for every open neighbourhood $U(x)\not=\emptyset$ of the point $x$ there exists an open neighbourhood $V(x)$ of $x$ such that $V(x) \cdot V(x) \subseteq U(x)$. If $x$ is not an isolated point of $E$, then $V(x) \cdot V(x) \not= \emptyset$ which contradicts to the choice of $U(x)$. This implies the assertion of the lemma. \[\square\]

Lemma 7. Let $S$ be a primitive inverse topological semigroup and let $S$ be an orthogonal sum of the family $\{B_\lambda(G_i)\}_{i \in A}$ of topological Brandt semigroups with zeros, that is, $S = \sum_{i \in A} B_\lambda(G_i)$. 
Let \((\alpha_i, g_i, \beta_i) \in B_\lambda(G_i)\) be a nonzero element of \(S\). Then

(i) there exists an open neighbourhood \(U\) of \((\alpha_i, g_i, \beta_i)\) such that \(U \subseteq S_{\alpha_i,\beta_i}^* \subseteq B_\lambda(G_i)\); 

(ii) every nonzero idempotent of \(S\) is an isolated point in \(E(S)\).

Proof. (i) Suppose to the contrary that \(U \notin S_{\alpha_i,\beta_i}^* \subseteq B_\lambda(G_i)\) for any open neighbourhood \(U\) of the point \((\alpha_i, g_i, \beta_i)\). Since \(S\) is a Hausdorff space, there exists an open neighbourhood \(V\) of the point \((\alpha_i, g_i, \beta_i)\) such that \(0 \notin V\). The continuity of the semigroup operation in \(S\) implies that there exists an open neighbourhood \(W\) of the point \((\alpha_i, g_i, \beta_i)\) such that \((\alpha_i, 1, \alpha_i) \cdot W \cdot (\beta_i, 1, \beta_i) \subseteq V\). Since \(W \notin S_{\alpha_i,\beta_i}^*\), we have that \(0 \in V\), a contradiction.

Statement (ii) follows from Lemma 6.

Lemma 7 implies the following.

Corollary 8. Every nonzero \(\mathcal{H}\)-class of a primitive inverse topological semigroup \(S\) is an open subset in \(S\).

Lemma 9. If \(S\) is a primitive topological inverse semigroup, then every nonzero \(\mathcal{H}\)-class of \(S\) is a clopen subset in \(S\).

Proof. Let \(H(e, f)\) be a nonzero \(\mathcal{H}\)-class in \(S\) for \(e, f \in (E(S))^*\), that is,

\[
H(e, f) = \left\{ x \in S \mid x \cdot x^{-1} = e, x^{-1} \cdot x = f \right\}.
\]

(3)

Since \(S\) is a topological inverse semigroup, the maps \(\varphi : S \to E(S)\) and \(\psi : S \to E(S)\) defined by the formulae \(\varphi(x) = x \cdot x^{-1}\) and \(\psi(x) = x^{-1} \cdot x\) are continuous. By Lemma 6, \(e\) and \(f\) are isolated points in \(E(S)\). Then the continuity of the maps \(\varphi\) and \(\psi\) implies the statement of the lemma.

The following example shows that the statement of Lemma 9 does not hold for primitive inverse locally compact \(H\)-closed topological semigroups.

Example 10. Let \(\mathbb{Z}\) be the discrete additive group of integers. We extend the semigroup operation from \(\mathbb{Z}\) onto \(\mathbb{Z}^0 = \mathbb{Z} \cup \{\infty\}\) as follows:

\[
x \cdot \infty = \infty \cdot x = \infty \cdot \infty = \infty, \quad \forall x \in \mathbb{Z}.
\]

(4)

We observe that \(\mathbb{Z}^0\) is the group with adjoined zero \(\infty\). We determine a semigroup topology \(\tau\) on \(\mathbb{Z}^0\) as follows:

(i) every nonzero element of \(\mathbb{Z}^0\) is an isolated point;

(ii) the family \(\mathcal{B}(\infty) = \{U_n = \{\infty\} \cup \{x \in \mathbb{Z} \mid x \geq n\} \mid n\) is a positive integer\} is a base of the topology \(\tau\) at the point \(\infty\).

A simple verification shows that \((\mathbb{Z}^0, \tau)\) is a primitive inverse locally compact topological semigroup.
Proposition 11. \((\mathbb{Z}^0, \tau)\) is an \(H\)-closed topological semigroup.

Proof. Suppose that \(\mathbb{Z}^0\) is embedded into a topological semigroup \(T\). If \(\{n_i\}\) is a net in \(\mathbb{N}\) for which \(\{-n_i\}\) converges in \(T\) to \(t \in T \setminus \mathbb{Z}^0\), then the equation \(-n_i + (n_i + k) = k\) implies that \(t \cdot \infty = k\) for every \(k \in \mathbb{N}\) which is impossible. So \(\mathbb{Z}^0\) is closed in \(T\). \(\Box\)

Proposition 12. Every completely 0-simple topological inverse semigroup \(S\) is topologically isomorphic to a topological Brandt \(\lambda\)-extension \(B_1(G)\) of some topological group \(G\) and cardinal \(\lambda \geq 1\) in the class of topological inverse semigroups. Furthermore one has the following:

(i) any nonzero subgroup of \(S\) is topologically isomorphic to \(G\) and every nonzero \(\mathcal{K}\)-class of \(S\) is homeomorphic to \(G\) and is a clopen subset in \(S\);

(ii) the family \(\mathcal{B}(\alpha, g, \beta) = \{(\alpha, g \cdot U, \beta) \mid U \in \mathcal{B}_G(e)\}\), where \(\mathcal{B}_G(e)\) is a base of the topology at the unity \(e\) of \(G\), is a base of the topology at the nonzero element \((\alpha, g, \beta) \in B_1(G)\).

Proof. Let \(G\) be a nonzero subgroup of \(S\). Then by Theorem 3.9 of \([4, 5]\) the semigroup \(S\) is isomorphic to the Brandt \(\lambda\)-extension of the subgroup \(G\) for some cardinal \(\lambda \geq 1\). Since \(S\) is a topological inverse semigroup, we have that \(G\) is a topological group.

(i) Let \(e\) be the unity of \(G\). We fix arbitrary \(\alpha, \beta, \gamma, \delta \in I_\lambda\) and define the maps

\[
\psi_{\alpha g} : B_1(G) \rightarrow B_1(G) \quad \text{and} \quad \psi_{\alpha g} : B_1(G) \rightarrow B_1(G)
\]

by the formulae \(\psi_{\alpha g}(s) = (\gamma, e, \alpha) \cdot s \cdot (\beta, e, \delta)\) and \(\psi_{\alpha g}(s) = (\alpha, e, \gamma) \cdot s \cdot (\delta, e, \beta), s \in B_1(G)\). We observe that \(\psi_{\alpha g}(\psi_{\alpha g}((\alpha, x, \beta))) = (\alpha, x, \beta)\) and \(\psi_{\alpha g}(\psi_{\alpha g}((\alpha, x, \beta))) = (\alpha, x, \beta)\) for all \(\alpha, \beta, \gamma, \delta \in I_\lambda, x \in G\), and hence the restrictions \(\psi_{\alpha g}|_{(\alpha, G, \beta)}\) and \(\psi_{\alpha g}|_{(\gamma, G, \delta)}\) are mutually invertible. Since the maps \(\psi_{\alpha g}\) and \(\psi_{\alpha g}\) are continuous on \(B_1(G)\), the map \(\psi_{\alpha g}|_{(\alpha, G, \beta)} : (\alpha, G, \beta) \rightarrow (\gamma, G, \delta)\) is a homeomorphism and the map \(\psi_{\alpha g}|_{(\alpha, G, \beta)} : (\gamma, G, \gamma) \rightarrow (\gamma, G, \gamma)\) is a topological isomorphism. We observe that the subset \((\alpha, G, \beta)\) of \(B_1(G)\) is an \(\mathcal{K}\)-class of \(B_1(G)\) and \((\alpha, G, \beta)\) is a subgroup of \(B_1(G)\) for all \(\alpha, \beta \in I_\lambda\). This completes the proof of assertion (i).

(ii) The statement follows from assertion (i) and Theorem 4.3 of \([18]\). \(\Box\)

We observe that Example 10 implies that the statements of Proposition 12 are not true for completely 0-simple inverse topological semigroups. Definition 3 implies that \(S\) is a topological Brandt \(\lambda\)-extension \(B_1(G)\) of the topological group \(G\).

Gutik and Repovš, in \([19]\), studied the structure of 0-simple countably compact topological inverse semigroups. They proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt \(\lambda\)-extension \(B_1(H)\) of a countably compact topological group \(H\) in the class of topological inverse semigroups for some finite cardinal \(\lambda \geq 1\). This implies Pavlyk's Theorem (see \([20]\)) on the structure of 0-simple compact topological inverse semigroups: every 0-simple compact topological inverse semigroup is topologically isomorphic to a topological Brandt \(\lambda\)-extension \(B_1(H)\) of a compact topological group \(H\) in the class of topological inverse semigroups for some finite cardinal \(\lambda \geq 1\).

The following theorem describes the structure of primitive countably compact topological inverse semigroups.

Theorem 13. Every primitive countably compact topological inverse semigroup \(S\) is topologically isomorphic to an orthogonal sum \(\sum_{i \in \mathcal{I}} B_i(G_i)\) of topological Brandt \(\lambda\)-extensions \(B_i(G_i)\) of
countably compact topological groups $G_i$ in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Moreover the family

$$\mathcal{B}(0) = \left\{ S \setminus \left( B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}) \right)^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N} \right\}$$

(5)

determines a base of the topology at zero 0 of $S$.

**Proof.** By Theorem II.4.3 of [8] the semigroup $S$ is an orthogonal sum of Brandt semigroups and hence $S$ is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of groups $G_i$. We fix any $i_0 \in \mathcal{A}$. Since $S$ is a topological inverse semigroup, Proposition II.2 [1] implies that $B_{\lambda_i}(G_i)$ is a topological inverse semigroup. By Proposition 12, $B_{\lambda_i}(G_i)$ is a closed subsemigroup of $S$ and hence by Theorem 3.10.4 [6], $B_{\lambda_i}(G_i)$ is a countably compact 0-simple topological inverse semigroup. Then, by Theorem 2 of [19], the semigroup $B_{\lambda_i}(G_i)$ is a topological Brandt $\lambda_i$-extension of countably compact topological group $G_i$ in the class of topological inverse semigroups for some finite cardinal $\lambda_i \geq 1$. This completes the proof of the first assertion of the theorem.

Suppose on the contrary that $\mathcal{B}(0)$ is not a base at zero 0 of $S$. Then, there exists an open neighbourhood $U(0)$ of zero 0 such that $U(0) \cup (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \neq S$ for finitely many indexes $i_1, i_2, \ldots, i_n \in \mathcal{A}$. Therefore there exists an infinitely family $\mathcal{F}$ of nonzero disjoint $\mathcal{A}$-classes such that $HgU(0)$ for all $H \in \mathcal{F}$. Let $\mathcal{F}_0$ be an infinite countable subfamily of $\mathcal{F}$. We put $W = \bigcup \{ H \mid H \in \mathcal{F} \setminus \mathcal{F}_0 \}$. Lemma 9 implies that the family $\mathcal{C} = \{ U(0), W \} \cup \mathcal{F}_0$ is an open countable cover of $S$. Simple observation shows that the cover $\mathcal{C}$ does not contain a finite subcover. This contradicts to the countable compactness of $S$. The obtained contradiction implies the last assertion of the theorem.

Since any maximal subgroup of a compact topological semigroup $T$ is a compact subset in $T$ (see [2, Vol. 1, Theorem 1.11]), Theorem 13 implies the following.

**Corollary 14.** Every primitive compact topological inverse semigroup $S$ is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of compact topological groups $G_i$ in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$ and the family

$$\mathcal{B}(0) = \left\{ S \setminus \left( B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \cdots \cup B_{\lambda_{i_n}}(G_{i_n}) \right)^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N} \right\}$$

(6)

determines a base of the topology at zero 0 of $S$.

**Theorem 15.** Every primitive countably compact topological inverse semigroup $S$ is a dense subsemigroup of a primitive compact topological inverse semigroup.

**Proof.** By Theorem 13 the topological semigroup $S$ is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups $G_i$ in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Since any countably compact topological group $G_i$ is pseudocompact, the Comfort-Ross Theorem (see [21, Theorem 4.1]) implies that the Stone-Cech compactification $\beta(G_i)$ is a compact topological group and the inclusion mapping $f_i$ of $G_i$ into $\beta(G_i)$ is
a topological isomorphism for all \( i \in \mathcal{A} \). On the orthogonal sum \( \sum_{i \in \mathcal{A}} B_{i}^{1}(G_{i}) \) of Brandt \( \lambda \)-extensions \( B_{i}^{1}(\beta(G_{i})) \), \( i \in \mathcal{A} \), we determine a topology \( \tau \) as follows:

(a) the family \( \mathcal{B}(a_{i_{1}}, g_{i_{1}}, \beta_{i_{1}}) = \{ (a_{i_{1}}, g_{i_{1}}, U, \beta_{i_{1}}) \mid U \in \mathcal{B}_{\beta(G_{i_{1}})}(e_{i_{1}}) \} \) is a base of the topology at the nonzero element \( (a_{i_{1}}, g_{i_{1}}, \beta_{i_{1}}) \in B_{i_{1}}^{1}(\beta(G_{i_{1}})) \), where \( \mathcal{B}_{\beta(G_{i})}(e_{i}) \) is a base of the topology at the unity \( e_{i} \) of the compact topological group \( \beta(G_{i}) \);

(b) the family

\[
\mathcal{B}(0) = \left\{ S \setminus \left( B_{i_{1}}^{1}(\beta(G_{i_{1}})) \cup B_{i_{2}}^{1}(\beta(G_{i_{2}})) \cup \cdots \cup B_{i_{n}}^{1}(\beta(G_{i_{n}})) \right)^{*} \mid i_{1}, i_{2}, \ldots, i_{n} \in \mathcal{A}, n \in \mathbb{N} \right\}
\] (7)

determines a base of the topology at zero 0 of \( \sum_{i \in \mathcal{A}} B_{i}^{1}(G_{i}) \).

By Theorem II.4.3 of [8], \( \sum_{i \in \mathcal{A}} B_{i}^{1}(\beta(G_{i})) \) is a primitive inverse semigroup and simple verifications show that \( \sum_{i \in \mathcal{A}} B_{i}^{1}(\beta(G_{i})) \) is a topological inverse semigroup.

We define a map \( f : \sum_{i \in \mathcal{A}} B_{i}^{1}(G_{i}) \rightarrow \sum_{i \in \mathcal{A}} B_{i}^{1}(\beta(G_{i})) \) as follows:

\[
f(0) = 0, \quad f((a_{i_{1}}, g_{i_{1}}, \beta_{i_{1}})) = (a_{i_{1}}, f_{i}(g_{i_{1}}), \beta_{i_{1}}) \in B_{i_{1}}^{1}(\beta(G_{i_{1}})) \quad \text{for } (a_{i_{1}}, g_{i_{1}}, \beta_{i_{1}}) \in B_{i_{1}}^{1}(G_{i_{1}}).
\] (8)

Simple verifications show that \( f \) is a continuous homomorphism. Since \( f_{i} : G_{i} \rightarrow \beta(G_{i}) \) is a topological isomorphism, we have that \( f : \sum_{i \in \mathcal{A}} B_{i}^{1}(G_{i}) \rightarrow \sum_{i \in \mathcal{A}} B_{i}^{1}(\beta(G_{i})) \) is a topological isomorphism too. \( \square \)

Gutik and Repovš in [19] showed that the Stone-Čech compactification \( \beta(T) \) of a 0-simple countably compact topological inverse semigroup \( T \) is a 0-simple compact topological inverse semigroup. In this context the following question arises naturally.

**Question 1.** Is the Stone-Čech compactification \( \beta(T) \) of a primitive countably compact topological inverse semigroup \( T \) a topological semigroup (a primitive topological inverse semigroup)?

**Theorem 16.** Let \( S = \bigcup_{\alpha \in \mathcal{A}} S_{\alpha} \) be a topological inverse semigroup such that

(i) \( S_{\alpha} \) is an \( H \)-closed (resp., absolutely \( H \)-closed) semigroup in the class of topological inverse semigroups for any \( \alpha \in \mathcal{A} \);

(ii) there exists an \( H \)-closed (resp., absolutely \( H \)-closed) subsemigroup \( T \) of \( S \) in the class of topological inverse semigroups such that \( S_{\alpha} \cdot S_{\beta} \subseteq T \) for all \( \alpha \neq \beta, \alpha, \beta \in \mathcal{A} \).

Then \( S \) is an \( H \)-closed (resp., absolutely \( H \)-closed) semigroup in the class of topological inverse semigroups.

**Proof.** We consider the case of absolute \( H \)-closedness only.

Suppose on the contrary that there exist a topological inverse semigroup \( G \) and a continuous homomorphism \( h : S \rightarrow G \) such that \( h(S) \) is not closed subsemigroup in \( G \). Without loss of generality we can assume that \( \text{cl}_{c}(h(S)) = G \). Thus, by Proposition II.2 of [1], \( G \) is a topological inverse semigroup.

Then, \( G \setminus h(S) \neq \emptyset \). Let \( x \in G \setminus h(S) \). Since \( S \) and \( G \) are topological inverse semigroups we have that \( h(S) \) is an inverse subsemigroup in \( G \) and hence \( x^{-1} \in G \setminus h(S) \). The semigroup
obtained contradiction implies that there exists an open neighbourhood \( U(x) \) of the point \( x \) in \( T \) such that \( U(x) \cap h(T) = \emptyset \). Since \( G \) is a topological inverse semigroup there exist open neighbourhoods \( V(x) \) and \( V(x^{-1}) \) of the points \( x \) and \( x^{-1} \) in \( G \), respectively, such that \( V(x) \cdot V(x^{-1}) \cdot V(x) \subseteq U(x) \). But \( x, x^{-1} \in cl_c(h(S)) \setminus h(S) \) and since \( \{ S_a \mid a \in \mathcal{A} \} \) is the family of absolutely \( H \)-closed semigroups in the class of topological inverse semigroups, each of the neighbourhoods \( V(x) \) and \( V(x^{-1}) \) intersects infinitely many subsemigroups \( h(S_\beta) \) in \( G, \beta \in \mathcal{A} \). Hence, \( (V(x) \cdot V(x^{-1}) \cdot V(x)) \cap h(T) \neq \emptyset \). This contradicts the assumption that \( U(x) \cap h(T) = \emptyset \). The obtained contradiction implies that \( S \) is an absolutely \( H \)-closed semigroup in the class of topological inverse semigroups.

The proof in the case of \( H \)-closeness is similar to the previous one. \( \square \)

Theorem 16 implies the following.

**Corollary 17.** Let \( S = \bigcup_{a \in \mathcal{A}} S_a \) be an inverse semigroup such that

(i) \( S_a \) is an algebraically closed (resp., algebraically \( H \)-closed) semigroup in the class of topological inverse semigroups for any \( a \in \mathcal{A} \);

(ii) there exists an algebraically closed (resp., algebraically \( H \)-closed) sub-semigroup \( T \) of \( S \) in the class of topological inverse semigroups such that \( S_a \cdot S_\beta \subseteq T \) for all \( a \neq \beta, a, \beta \in \mathcal{A} \).

Then \( S \) is an algebraically closed (resp., algebraically \( H \)-closed) semigroup in the class of topological inverse semigroups.

Theorem 16 implies the following.

**Theorem 18.** Let a topological inverse semigroup \( S \) be an orthogonal sum of the family \( \{ S_a \}_{a \in \mathcal{A}} \) of \( H \)-closed (resp., absolutely \( H \)-closed) topological inverse semigroups with zeros in the class of topological inverse semigroups. Then \( S \) is an \( H \)-closed (resp., absolutely \( H \)-closed) topological inverse semigroup in the class of topological inverse semigroups.

**Corollary 19.** Let an inverse semigroup \( S \) be an orthogonal sum of the family \( \{ S_a \}_{a \in \mathcal{A}} \) of algebraically closed (resp., algebraically \( H \)-closed) inverse semigroups with zeros in the class of topological inverse semigroups. Then \( S \) is an algebraically closed (resp., algebraically \( H \)-closed) inverse semigroup in the class of topological inverse semigroups.

Recall in [22], that a topological group \( G \) is called **absolutely closed** if \( G \) is a closed subgroup of any topological group which contains \( G \) as a subgroup. In our terminology such topological groups are called \( H \)-closed in the class of topological groups. In [23] Raikov proved that a topological group \( G \) is absolutely closed if and only if it is Raikov complete, that is, \( G \) is complete with respect to the two sided uniformity.

A topological group \( G \) is called **\( h \)-complete** if for every continuous homomorphism \( f : G \to H \) into a topological group \( H \) the subgroup \( f(G) \) of \( H \) is closed [24]. The \( h \)-completeness is preserved under taking products and closed central subgroups [24).

Gutik and Pavlyk in [13] showed that a topological group \( G \) is \( H \)-closed (resp., absolutely \( H \)-closed) in the class of topological inverse semigroups if and only if \( G \) is absolutely closed (resp., \( h \)-complete).
Theorem 20. For a primitive topological inverse semigroup $S$ the following assertions are equivalent:

(i) every maximal subgroup of $S$ is absolutely closed;

(ii) the semigroup $S$ with every inverse semigroup topology $\tau$ is $H$-closed in the class of topological inverse semigroups.

Proof. (i)$\Rightarrow$(ii) Suppose that a primitive topological inverse semigroup $S$ is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i} (G_i)$ of topological Brandt $\lambda$-extensions $B_{\lambda_i} (G_i)$ of topological groups $G_i$ in the class of topological inverse semigroups and every topological group $G_i$ is absolutely closed. Then, by Theorem 3 of [9] any topological Brandt $\lambda$-extension $B_{\lambda_i} (G_i)$ of topological group $G_i$ is $H$-closed in the class of topological inverse semigroups. Theorem 18 implies that $S$ is an $H$-closed topological inverse semigroup in the class of topological inverse semigroups.

(ii)$\Rightarrow$(i) Let $G$ be any maximal nonzero subgroup of $S$. Since $S$ is a primitive topological inverse semigroup, we have that $S$ is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i} (G_i)$ of Brandt $\lambda$-extensions $B_{\lambda_i} (G_i)$ of topological groups $G_i$ and hence there exists a topological Brandt $\lambda_{i_0}$-extension $B_{\lambda_{i_0}} (G_{i_0})$, $i \in \mathcal{A}$, such that $B_{\lambda_{i_0}} (G_{i_0})$ contains the maximal subgroup $G$ and $B_{\lambda_{i_0}} (G_{i_0})$ is a subgroup of $S$.

Suppose on the contrary that the topological group $G = G_{i_0}$ is not absolutely closed. Then there exists a topological group $H$ which contains $G$ as a dense proper subgroup. For every $i \in \mathcal{A}$ we put

$$H_i = \begin{cases} G_{i_0} & \text{if } i \neq i_0, \\ H & \text{if } i = i_0. \end{cases}$$

(9)

On the orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i)$ of Brandt $\lambda$-extensions $B_{\lambda_i} (H_i)$, $i \in \mathcal{A}$, we determine a topology $\tau_0$ as follows:

(a) the family $\mathcal{B}(\alpha_i, g, h)$ = $\{ (\alpha_i, g, h) \cup U_\beta \cup \beta_i, U \in \mathcal{B}_{H_i} (e_i) \}$ is a base of the topology at the nonzero element $\alpha_i, g, h$, $i \in \mathcal{A}$, where $\mathcal{B}_{H_i} (e_i)$ is a base of the topology at the unity $e_i$ of the topological group $H_i$;

(b) the zero 0 is an isolated point in $(\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i), \tau_0)$.

By Theorem II.4.3 of [8], $\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i)$ is a primitive inverse semigroup. Also we observe that the semigroup $\sum_{i \in \mathcal{A}} B_{\lambda_i} (G_i)$ which is induced from $(\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i), \tau_0)$ topology is a topological inverse semigroup which is a dense proper inverse sub-semigroup of $(\sum_{i \in \mathcal{A}} B_{\lambda_i} (H_i), \tau_0)$. The obtained contradiction completes the statement of the theorem.

Theorem 20 implies the following.

Corollary 21. For a primitive inverse semigroup $S$ the following assertions are equivalent:

(i) every maximal subgroup of $S$ is algebraically closed in the class of topological inverse semigroups;

(ii) the semigroup $S$ is algebraically closed in the class of topological inverse semigroups.
Theorem 22. For a primitive topological inverse semigroup $S$ the following assertions are equivalent:

(i) every maximal subgroup of $S$ is $h$-complete;

(ii) the semigroup $S$ with every inverse semigroup topology $\tau$ is absolutely $H$-closed in the class of topological inverse semigroups.

Proof. (i)$\Rightarrow$(ii) Suppose that a primitive topological inverse semigroup $S$ is an orthogonal sum $\sum_{i\in A} B_{\lambda_i}(G_i)$ of topological Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of topological groups $G_i$ in the class of topological inverse semigroups and every topological group $G_i$ is $h$-complete. Then by Theorem 14 of [13] any topological Brandt $\lambda_i$-extension $B_{\lambda_i}(G_i)$ of topological group $G_i$ is absolutely $H$-closed in the class of topological inverse semigroups. Theorem 18 implies that $S$ is an absolutely $H$-closed topological inverse semigroup in the class of topological inverse semigroups.

(ii)$\Rightarrow$(i) Let $G$ be any maximal nonzero subgroup of $S$. Since $S$ is a primitive topological inverse semigroup, $S$ is an orthogonal sum $\sum_{i\in A} B_{\lambda_i}(G_i)$ of Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$ of topological groups $G_i$. Hence there exists a topological Brandt $\lambda_i$-extension $B_{\lambda_i}(G_i)$, $i \in A$, such that $B_{\lambda_i}(G_i)$ contains the maximal subgroup $G$ and $B_{\lambda_i}(G_i)$ is a subsemigroup of $S$.

Suppose on the contrary that the topological group $G = G_{\lambda_i}$ is not $h$-completed. Then there exist a topological group $H$ and continuous homomorphism $h : G \to H$ such that $h(G)$ is a dense proper subgroup of $H$. On the Brandt $\lambda$-extension $B_{\lambda_{i_0}}(H)$, we determine a topology $\tau_H$ as follows:

(a) the family $\mathcal{B}(a_{i_0}, g_{i_0}, \beta_{i_0}) = \{ (a_{i_0}, g_{i_0} \cdot U, \beta_{i_0}) | U \in B_H(e) \}$ is a base of the topology at the nonzero element $(a_{i_0}, g_{i_0}, \beta_{i_0}) \in B_{\lambda_{i_0}}(H)$, where $B_H(e)$ is a base of the topology at the unity $e$ of the topological group $H$;

(b) the zero 0 is an isolated point in $(B_{\lambda_{i_0}}(H), \tau_H)$.

Then $B_{\lambda_{i_0}}(H)$ is an inverse semigroup and simple verifications show that $B_{\lambda_{i_0}}(H)$ with the topology $\tau_H$ is a topological inverse semigroup.

On the orthogonal sum $\sum_{i\in A} B_{\lambda_i}(G_i)$ of Brandt $\lambda_i$-extensions $B_{\lambda_i}(G_i)$, $i \in A$, we determine a topology $\tau_*$ as follows:

(a) the family $\mathcal{B}(a_i, g_i, \beta_i) = \{ (a_i, g_i \cdot U, \beta_i) | U \in B_{G_i}(e_i) \}$ is a base of the topology at the nonzero element $(a_i, g_i, \beta_i) \in B_{\lambda_i}(G_i)$, where $B_{G_i}(e_i)$ is a base of the topology at the unity $e_i$ of the topological group $G_i$;

(b) the zero 0 is an isolated point in $(\sum_{i\in A} B_{\lambda_i}(G_i), \tau_*)$.

By Theorem II.4.3 of [8], $\sum_{i\in A} B_{\lambda_i}(G_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i\in A} B_{\lambda_i}(G_i)$ with the topology $\tau_*$ is a topological inverse semigroup.

We define the map $f : S \to B_{\lambda_{i_0}}(H)$ as follows:

$$f(x) = \begin{cases} h(x), & \text{if } x \in B_{\lambda_{i_0}}(G_{i_0}), \\ 0, & \text{if } x \notin B_{\lambda_{i_0}}(G_{i_0}), \end{cases}$$

where 0 is zero of $S$. Evidently the defined map $f$ is a continuous homomorphism. Then $f(S) = B_{\lambda_{i_0}}(h(G_{i_0}))$ is a dense proper inverse subsemigroup of the topological
inverse semigroup \((B_{\lambda_0}(H), \tau_H)\). The obtained contradiction completes the statement of the theorem. □

Theorem 22 implies the following.

**Corollary 23.** For a primitive inverse semigroup \(S\) the following assertions are equivalent:

(i) every maximal subgroup of \(S\) is algebraically \(h\)-closed in the class of topological inverse semigroups;

(ii) the semigroup \(S\) is algebraically \(h\)-closed in the class of topological inverse semigroups.

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**References**


