Research Article

On Reverses of Some Inequalities in $n$-Inner Product Spaces

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We present some new reverses of Cauchy-Bunyakovsky-Schwarz inequality, and Triangle and Boas-Bellman Type inequalities in $n$-inner product spaces. The results obtained generalize the results of Dragomir [2003–2005] in $n$-inner product spaces. Also we provide some applications for determinantal integral inequalities.

1. Introduction


2. Preliminaries

Definition 2.1 (see [4]). Assume that $n$ is a positive integer and $X$ is a vector space over the field $K = \mathbb{R}$ of real numbers or the field $K = \mathbb{C}$ of complex numbers, such that $\dim X \geq n$ and $\langle \cdot , \cdot | \cdot , \cdot \ldots , \cdot \rangle$ is a $K$ valued function defined on $X \times X \times \cdots \times X$ such that:
(n1) \( \langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0 \), for any \( x_1, x_2, \ldots, x_n \in X \) and \( \langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent vectors;

(n2) \( \langle a, b | x_1, \ldots, x_{n-1} \rangle = \langle a, b | \pi(x_1), \ldots, \pi(x_{n-1}) \rangle \), for any \( a, b, x_1, x_2, \ldots, x_{n-1} \in X \) and for any bijections \( \pi : \{x_1, x_2, \ldots, x_{n-1}\} \rightarrow \{x_1, x_2, \ldots, x_{n-1}\} \);

(n3) If \( n > 1 \), then \( \langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_2, x_2 | x_1, x_3, \ldots, x_n \rangle \), for any \( x_1, x_2, \ldots, x_n \in X \);

(n4) \( \langle a, b | x_1, \ldots, x_{n-1} \rangle = \langle b, a | x_1, \ldots, x_{n-1} \rangle \)

(n5) \( \langle aa, b | x_1, \ldots, x_{n-1} \rangle = a\langle a, b | x_1, \ldots, x_{n-1} \rangle \), for any \( a, b, x_1, \ldots, x_{n-1} \in X \) and any scalar \( a \in R \);

(n6) \( \langle a + a_1, b | x_1, \ldots, x_{n-1} \rangle = \langle a, b | x_1, \ldots, x_{n-1} \rangle + \langle a_1, b | x_1, \ldots, x_{n-1} \rangle \), for any \( a, b, a_1, x_1, \ldots, x_{n-1} \in X \).

Then \( \langle \cdot, \cdot \rangle | \cdot, \ldots, \cdot \rangle \) is called the \( n \)-inner product and \( (X, \langle \cdot, \cdot \rangle | \cdot, \ldots, \cdot \rangle) \) is called the \( n \)-prehilbert space. If \( n = 1 \), then Definition 2.1 reduces to the ordinary inner product. In any given \( n \)-inner product space \( (X, \langle \cdot, \cdot \rangle | \cdot, \ldots, \cdot \rangle) \), we can define a function \( ||\cdot, \ldots, \cdot|| \) on \( X^n = \times_{i=1}^n X \) as

\[
\|x_1, x_2, \ldots, x_n\| = \sqrt{\langle x_1, x_1 | x_2, \ldots, x_n \rangle}
\]

for any \( x_1, x_2, \ldots, x_n \in X \). It is easy to see that this function satisfies the following conditions:

(n1) \( ||x_1, x_2, \ldots, x_n|| \geq 0 \) and \( ||x_1, x_2, \ldots, x_n|| = 0 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent,

(n2) \( ||x_1, x_2, \ldots, x_n|| \) is invariant under any permutation,

(n3) \( ||x_1, x_2, \ldots, ax_n|| = |a||x_1, x_2, \ldots, x_n|| \), for any \( a \in K \),

(n4) \( ||x_1, x_2, \ldots, x_n - 1, y + z|| \leq ||x_1, x_2, \ldots, x_{n-1}, y|| + ||x_1, x_2, \ldots, x_{n-1}, z|| \).

A function \( ||\cdot, \ldots, \cdot|| \) defined on \( X^n \) and satisfying the above conditions is called an \( n \)-norm on \( X \) and the pair \( (X, ||\cdot, \ldots, \cdot||) \) is called \( n \)-normed linear space. Whenever an \( n \)-inner product space \( (X, \langle \cdot, \cdot \rangle | \cdot, \ldots, \cdot \rangle) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space. Whenever an \( n \)-normed linear space \( (X, ||\cdot, \cdot||) \) is given, we consider it as an \( n \)-normed linear space.

A function \( ||\cdot, \ldots, \cdot|| \) defined on \( X^n \) and satisfying the above conditions is called an \( n \)-norm on \( X \) and the pair \( (X, ||\cdot, \ldots, \cdot||) \) is called \( n \)-normed linear space. Whenever an \( n \)-inner product space \( (X, \langle \cdot, \cdot \rangle | \cdot, \ldots, \cdot \rangle) \) is given, we consider it as an \( n \)-normed linear space \( (X, ||\cdot, \ldots, \cdot||) \) with the \( n \)-norm defined by (2.1).

Let \( (H; \langle \cdot, \cdot \rangle) \) be an inner product space over the real or complex number field \( K \). The following inequality is known as Cauchy-Schwarz’s inequality:

\[
|\langle x, y \rangle|^2 \leq \|x\|^2 \|y\|^2, \quad x, y \in H,
\]

where \( \|z\|^2 = \langle z, z \rangle, \quad z \in H \). The equality occurs in (2.2) if and only if \( x \) and \( y \) are linearly dependent.

In [8], Dragomir obtained the following reverse of Cauchy-Schwarz’s inequality:

\[
0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \leq \frac{1}{4} |A - a|^2 \|y\|^4,
\]

(2.3)
provided $x, y \in H$ and $a, A \in K$ are so that either
\[ \text{Re}(Ay - x, x - ay) \geq 0, \] (2.4)
or, equivalently
\[ \left\| x - \frac{a + A}{2}, y \right\| \leq \frac{1}{2} |A - a| \| y \|, \] (2.5)
holds. The constant $1/4$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

If $x, y, A, a$ satisfy either (2.4) or (2.5), then the following reverse of Cauchy-Schwarz’s inequality also holds:
\[ \| x \| \| y \| \leq \frac{1}{2} \cdot \frac{\text{Re}[A(x, y) + \bar{a}(x, y)]}{|\text{Re}(\bar{a}A)|^{1/2}} \leq \frac{1}{2} \cdot \frac{|A| + |a|}{|\text{Re}(\bar{a}A)|^{1/2}} |(x, y)| \] (2.6)
provided that, the complex numbers $a$ and $A$ satisfy the condition $\text{Re}(\bar{a}A) > 0$. In both inequalities in (2.6), the constant $1/2$ is best possible.

The following reverse of the triangle inequality in inner product space was also obtained by Dragomir [9].

Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over $K$ and $x, y \in H$, $M \geq m > 0$ such that either $\text{Re}(Ay - x, x - ay) \geq 0$ or, equivalently $\| x - (a + A)/2, y \| \leq 1/2|A - a| \| y \|$, holds.

\[ 0 \leq \| x \| + \| y \| - \| x + y \| \leq \frac{\sqrt{M} - \sqrt{m}}{\sqrt{mM}} \sqrt{\text{Re}(x, y)} \] (2.7)
holds.

Gunawan [6] generalized the Cauchy-Bunyakovsky-Schwarz inequality (shortly, the CBS inequality) for inner product space to $n$-inner product space and obtained the following:

\[ |\langle x, y | z_2, \ldots, z_n \rangle|^2 \leq \langle x, x | z_2, \ldots, z_n \rangle \langle y, y | z_2, \ldots, z_n \rangle. \] (2.8)

Moreover, the equality holds if and only if $x, y, z_2, \ldots, z_n$ are linearly dependent. In terms of the $n$-norms, the (CBS)-inequality (2.8) can be written as
\[ |\langle x, y | z_2, \ldots, z_n \rangle|^2 \leq \| x, z_2, \ldots, z_n \|^2 \| y, z_2, \ldots, z_n \|^2. \] (2.9)
The equality holds if and only if $x, y, z_2, \ldots, z_n$ are linearly dependent.

### 3. Main Results

The aim of the present paper is to generalize the above mentioned results of Dragomir [8, 9], that is, reverses of CBS inequality, Triangle inequality and Boas-bellman type inequalities in inner product space to $n$-inner product spaces.
3.1. Reverses of the CBS Inequality

Theorem 3.1. Let $A, a \in K (K = C, R)$ and $x, y, z_2, \ldots, z_n \in X$, where $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is $n$-inner product space over $K$. If

$$\text{Re}(\langle A y - x, x - a y | z_2, \ldots, z_n \rangle) \geq 0.$$  \hspace{1cm} (3.1)

or, equivalently,

$$\| x - a + \frac{A}{2} y, z_2, \ldots, z_n \| \leq \frac{1}{2} |A - a| \| y, z_2, \ldots, z_n \|. \hspace{1cm} (3.2)$$

holds, then one has

$$0 \leq \| x, z_2, \ldots, z_n \|^2 \| y, z_2, \ldots, z_n \|^2 - \| \langle x, y | z_2, \ldots, z_n \rangle \|^2 \leq \frac{1}{4} |A - a|^2 \| y, z_2, \ldots, z_n \|^4. \hspace{1cm} (3.3)$$

The constant $1/4$ in (3.3) cannot be replaced by a smaller constant.

Proof. Using $(nI_2)-(nI_6)$, we get

$$\langle z_2, z_2 | x \pm y, \ldots, z_n \rangle = \langle x \pm y, x \pm y | z_2, \ldots, z_n \rangle$$

$$= \langle x, x | z_2, \ldots, z_n \rangle + \langle y, y | z_2, \ldots, z_n \rangle \pm 2 \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle),$$

$$\text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)$$

$$= \frac{1}{4} \left[ \langle z_2, z_2 | x + y, \ldots, z_n \rangle - \langle z_2, z_2 | x - y, \ldots, z_n \rangle \right]. \hspace{1cm} (3.5)$$

Considering the vectors $x, u, U, z_2, \ldots, z_n \in X$ and using (3.5) and $(nN1-nN6)$, we have

$$\text{Re}(U - x, x - u | z_2, \ldots, z_n)$$

$$= \frac{1}{4} \left[ \langle z_2, z_2 | U - u, \ldots, z_n \rangle - \langle z_2, z_2 | -2x + U + u, \ldots, z_n \rangle \right]$$

$$= \frac{1}{4} \left[ \langle U - u, U - u | z_2, \ldots, z_n \rangle - 4 \langle x - \frac{u + U}{2}, x - \frac{u + U}{2} | z_2, \ldots, z_n \rangle \right]$$

$$= \frac{1}{4} \| U - u, z_2, \ldots, z_n \|^2 - \left\| x - \frac{u + U}{2}, z_2, \ldots, z_n \right\|^2. \hspace{1cm} (3.6)$$

Therefore, $\text{Re}(U - x, x - u | z_2, \ldots, z_n) \geq 0$ if and only if

$$\left\| x - \frac{u + U}{2}, z_2, \ldots, z_n \right\| \leq \frac{1}{2} \| U - u, z_2, \ldots, z_n \|. \hspace{1cm} (3.7)$$
Applying this to the vectors $U = Ay$ and $u = ay$, we obtain that the (3.1) and (3.2) are equivalent. If we consider the real numbers

$$I_1 = \text{Re} \left[ \left( A \|y, z_2, \ldots, z_n\|^2 - \langle x, y \mid z_2, \ldots, z_n \rangle \right) \times \left( \langle x, y \mid z_2, \ldots, z_n \rangle - \alpha \|y, z_2, \ldots, z_n\|^2 \right) \right],$$

$$I_2 = \|y, z_2, \ldots, z_n\|^2 \text{Re} \langle Ay - x - ay \mid z_2, \ldots, z_n \rangle.$$

Then we obtain, by using properties of $n$-inner product space,

$$I_1 = \|y, z_2, \ldots, z_n\|^2 \text{Re} \left[ A \langle x, y \mid z_2, \ldots, z_n \rangle + \alpha \langle x, y \mid z_2, \ldots, z_n \rangle \right] - \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 \text{Re}(A \alpha).$$

$$I_2 = \|y, z_2, \ldots, z_n\|^2 \text{Re} \left[ A \langle x, y \mid z_2, \ldots, z_n \rangle + \alpha \langle x, y \mid z_2, \ldots, z_n \rangle \right] - \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 \text{Re}(A \alpha)$$

which gives

$$I_1 - I_2 = \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 - \|\langle x, y \mid z_2, \ldots, z_n \rangle\|^2$$

for any $x, y, z_2, \ldots, z_n \in X$ and $a, A \in K$. If (3.1) holds, then $I_2 \geq 0$ and thus $I_1 - I_2 \leq I_1$

$$\Rightarrow \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 - \|\langle x, y \mid z_2, \ldots, z_n \rangle\|^2 \leq \text{Re} \left[ \left( A \|y, z_2, \ldots, z_n\|^2 - \langle x, y \mid z_2, \ldots, z_n \rangle \right) \times \left( \langle x, y \mid z_2, \ldots, z_n \rangle - \alpha \|y, z_2, \ldots, z_n\|^2 \right) \right].$$

Now using the elementary inequality $\text{Re}(\alpha \beta) \leq 1/4|\alpha + \beta|^2$ for any $\alpha, \beta \in K \ (K = R, C)$, one yields

$$\text{Re} \left[ \left( A \|y, z_2, \ldots, z_n\|^2 - \langle x, y \mid z_2, \ldots, z_n \rangle \right) \times \left( \langle x, y \mid z_2, \ldots, z_n \rangle - \alpha \|y, z_2, \ldots, z_n\|^2 \right) \right] \leq \frac{1}{4} |A - \alpha|^2 \|y, z_2, \ldots, z_n\|^4.$$

If we combine (3.11) and (3.12), we get the required inequality.
To prove the sharpness of the constant $1/4$, assume that (3.3) holds with a constant $C > 0$, that is,

$$
\|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 - |\langle x, y \mid z_2, \ldots, z_n \rangle|^2 \\
\leq C|A - a|^2 \|y, z_2, \ldots, z_n\|^4,
$$

(3.13)

where $x, y, z_2, \ldots, z_n, A, a$ satisfy the hypothesis of the theorem.

Consider $y \in X$ with $\|y, z_2, \ldots, z_n\| = 1$, $a \neq A$, $m \in X$ with $\|m, z_2, \ldots, z_n\| = 1$ and $\langle y, m \mid z_2, \ldots, z_n \rangle = 0$ and define

$$
x = \frac{A + a}{2} y + \frac{A - a}{2} m
$$

(3.14)

then we have

$$
\text{Re}(Ay - x, x - ay \mid z_2, \ldots, z_n) = \frac{|A - a|^2}{4} \text{Re}(y - m, y + m \mid z_2, \ldots, z_n) = 0
$$

(3.15)

and then the condition (3.1) is fulfilled. From (3.13) we deduce

$$
\left\| \frac{A + a}{2} y + \frac{A - a}{2} m, z_2, \ldots, z_n \right\|^2 - \left| \left\langle \frac{A + a}{2} y + \frac{A - a}{2} m, y \mid z_2, \ldots, z_n \right\rangle \right|^2 \\
\leq C|A - a|^2
$$

(3.16)

and, since

$$
\left\| \frac{A + a}{2} y + \frac{A - a}{2} m, z_2, \ldots, z_n \right\|^2 = \left| \frac{A + a}{2} \right|^2 + \left| \frac{A - a}{2} \right|^2,
$$

(3.17)

$$
\left| \left\langle \frac{A + a}{2} y + \frac{A - a}{2} m, y \mid z_2, \ldots, z_n \right\rangle \right|^2 = \left| \frac{A + a}{2} \right|^2.
$$

By (3.16), we have

$$
\left| \frac{A - a}{2} \right|^2 \leq C|A - a|^2,
$$

(3.18)

for any $A, a \in K$ with $a \neq A$, which implies $C \geq 1/4$. This completes the proof. \qed
Theorem 3.2. Assume that \( x, y, z_2, \ldots, z_n, a \) and \( A \) are the same as in above Theorem 3.1. If \( \text{Re}(\bar{a}A) > 0 \), then one has

\[
\|x, z_2, \ldots, z_n\| \cdot \|y, z_2, \ldots, z_n\| \leq \frac{1}{2} \frac{\text{Re}\left(\left(\bar{A} + \bar{a}\right)\langle x, y | z_2, \ldots, z_n\rangle\right)}{[\text{Re}(\bar{a}A)]^{1/2}} \leq \frac{1}{2} \frac{|A + a|}{[\text{Re}(\bar{a}A)]^{1/2}} |\langle x, y | z_2, \ldots, z_n\rangle|.
\]

(3.19)

The constant \( 1/2 \) is best possible in both inequalities in the sense that it cannot be replaced by a smaller constant.

Proof. Define

\[
I = \text{Re}\left(Ay - x, x - ay | z_2, \ldots, z_n\right) = \text{Re}\left(A\langle x, y | z_2, \ldots, z_n\rangle + \bar{a}\langle x, y | z_2, \ldots, z_n\rangle\right) - \|x, z_2, \ldots, z_n\|^2 - \|y, z_2, \ldots, z_n\|^2 \text{Re}(A\bar{a}).
\]

(3.20)

We know that, for a complex number \( a \in \mathbb{C} \), \( \text{Re}(a) = \text{Re}(\bar{a}) \) and thus

\[
\text{Re}\left(A\langle x, y | z_2, \ldots, z_n\rangle\right) = \text{Re}\left(\bar{A} \langle x, y | z_2, \ldots, z_n\rangle\right)
\]

(3.21)

which implies

\[
I = \text{Re}\left(\left(\bar{A} + \bar{a}\right)\langle x, y | z_2, \ldots, z_n\rangle\right) - \|x, z_2, \ldots, z_n\|^2 - \|y, z_2, \ldots, z_n\|^2 \text{Re}(A\bar{a}).
\]

(3.22)

Since \( x, y, z_2, \ldots, z_n, a \) and \( A \) are assumed to satisfy the condition (3.1), by (3.22), we deduce

\[
\|x, z_2, \ldots, z_n\|^2 + \|y, z_2, \ldots, z_n\|^2 \text{Re}(A\bar{a}) \leq \text{Re}\left(\left(\bar{A} + \bar{a}\right)\langle x, y | z_2, \ldots, z_n\rangle\right),
\]

(3.23)

which gives

\[
\frac{1}{[\text{Re}(\bar{a}A)]^{1/2}} \|x, z_2, \ldots, z_n\|^2 + [\text{Re}(\bar{a}A)]^{1/2} \|y, z_2, \ldots, z_n\|^2 \leq \frac{\text{Re}\left(\left(\bar{A} + \bar{a}\right)\langle x, y | z_2, \ldots, z_n\rangle\right)}{[\text{Re}(\bar{a}A)]^{1/2}}
\]

(3.24)

since \( \text{Re}(A\bar{a}) > 0 \).
On the other hand, by the elementary inequality

$$ap^2 + \frac{1}{a}q^2 \geq 2pq$$

for $p, q \geq 0$ and $a > 0$, we have

$$2\|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\| \leq \frac{1}{[\text{Re}(\overline{a}A)]^{1/2}} \|x, z_2, \ldots, z_n\|^2 + [\text{Re}(\overline{a}A)]^{1/2} \|y, z_2, \ldots, z_n\|^2.$$ 

(3.26)

Using (3.24) and (3.26), we deduce the first inequality in (3.22). The last part is obvious by the fact, for $z \in \mathbb{C}$, $|\text{Re}(z)| \leq |z|$.

To prove the sharpness of the constant $1/2$ in the first inequality in (3.22), we assume that (3.22) holds with a constant $c > 0$, that is,

$$\|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\| \leq c \frac{\text{Re}[\overline{(A + \overline{a})} \langle x, y \mid z_2, \ldots, z_n \rangle]}{[\text{Re}(\overline{a}A)]^{1/2}}.$$ 

(3.27)

provided $x, y, z_2, \ldots, z_n, a$ and $A$ satisfy (2.1). If we take $a = A = 1, y = x \neq 0$, then obviously (2.1) holds and from (3.27), we obtain

$$\|x, z_2, \ldots, z_n\|^2 \leq 2c\|x, z_2, \ldots, z_n\|^2$$

(3.28)

for any linearly independent vectors $x, z_2, \ldots, z_n \in X$, which implies $c \geq 1/2$. This completes the proof.

When the constants involved are assumed to be positive, then we may state the following result.

**Corollary 3.3.** Let $M \geq m > 0$ and assume that, for $x, y, z_2, \ldots, z_n \in X$, one has

$$\text{Re}\langle My - x, x - my \mid z_2, \ldots, z_n \rangle \geq 0$$

(3.29)

or, equivalently,

$$\|x - \frac{M + m}{2}, z_2, \ldots, z_n\| \leq \frac{1}{2} (M - m) \|y, z_2, \ldots, z_n\|.$$ 

(3.30)
Then one has the following reverse of CBS inequality:

\[
\|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\| \leq \frac{1}{2} \frac{M + m}{\sqrt{mM}} \text{Re}(x, y | z_2, \ldots, z_n) \]

\[
\leq \frac{1}{2} \frac{M + m}{\sqrt{mM}} |\langle x, y | z_2, \ldots, z_n \rangle|.
\]

The constant 1/2 is sharp in (3.31).

**Corollary 3.4.** With the assumptions of the Theorem 3.2, one has

\[
0 < \|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\|^2 - |\langle x, y | z_2, \ldots, z_n \rangle|^2 \]

\[
\leq \frac{1}{4} \frac{|A - a|^2}{\text{Re}(\bar{a}A)} |\langle x, y | z_2, \ldots, z_n \rangle|^2.
\]

The constant 1/4 is best possible in (3.32).

**Corollary 3.5.** With the assumption of Corollary 3.3, one has

\[
0 \leq \|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\|-|\langle x, y | z_2, \ldots, z_n \rangle| \]

\[
\leq \|x, z_2, \ldots, z_n\| \|y, z_2, \ldots, z_n\| - \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle) \]

\[
\leq \frac{1}{2} \left( \sqrt{M} - \sqrt{m} \right)^2 \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle) \leq \frac{1}{2} \left( \sqrt{M} - \sqrt{m} \right)^2 |\langle x, y | z_2, \ldots, z_n \rangle|,
\]

0 \leq \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|-|\langle x, y | z_2, \ldots, z_n \rangle|^2 \]

\[
\leq \|x, z_2, \ldots, z_n\|^2 \|y, z_2, \ldots, z_n\|^2 - |\text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)|^2 \]

\[
\leq \frac{1}{4} \frac{(M - m)^2}{mM} \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)^2 \leq \frac{1}{4} \frac{(M - m)^2}{mM} |\langle x, y | z_2, \ldots, z_n \rangle|^2.
\]

The constant 1/2 in (3.33) and the constant 1/4 in (3.34) are best possible.

### 3.2. Reverse of the Triangle Inequality

**Corollary 3.6.** Assume \(x, y, z_2, \ldots, z_n, m, M\) are the same as in Corollary 3.3 and \(\text{Re}(x, y | z_2, \ldots, z_n) \geq 0\). Then one has the following reverse of the triangle inequality:

\[
0 \leq \|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\| - \|x + y, z_2, \ldots, z_n\| \]

\[
\leq \frac{\sqrt{M} - \sqrt{m}}{(mM)^{1/4}} \sqrt{\text{Re}(x, y | z_2, \ldots, z_n)}.
\]

(3.35)
Proof. It is easy to see that

\[
0 \leq \left( \|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\| \right) - \|x + y, z_2, \ldots, z_n\|^2
\]

for any \(x, y, z_2, \ldots, z_n \in X\). If the assumption of Corollary 3.3 holds, then (3.33) is valid and, by (3.36), we deduce

\[
0 \leq \left( \|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\| \right)^2 - \|x + y, z_2, \ldots, z_n\|^2
\]

\[
\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mm}} \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)
\]

which gives

\[
\left( \|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\| \right)^2 - \|x + y, z_2, \ldots, z_n\|^2
\]

\[
\leq \|x + y, z_2, \ldots, z_n\|^2 + \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mm}} \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle).
\]

Taking square root in (3.38), we have

\[
\|x, z_2, \ldots, z_n\| + \|y, z_2, \ldots, z_n\|
\]

\[
\leq \sqrt{\|x + y, z_2, \ldots, z_n\|^2 + \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{\sqrt{mm}} \text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)}
\]

\[
\leq \|x + y, z_2, \ldots, z_n\| + \frac{\sqrt{M} - \sqrt{m}}{(mm)^{1/4}} \sqrt{\text{Re}(\langle x, y | z_2, \ldots, z_n \rangle)}.
\]

From where we deduce the desire inequality (3.35). This completes the proof. \(\square\)

Let \((X, \langle \cdot, \cdot \rangle_1, \ldots, \langle \cdot, \cdot \rangle_m)\) be a \(n\)-inner product space over real or complex number field \(K\). If \((e_i)_{1 \leq i \leq m}\) are linearly independent vector in the \(n\)-inner product space \(X\), and, for given \(z_2, \ldots, z_n \in X\), \(\langle e_i, e_j | z_2, \ldots, z_n \rangle = \delta_{ij}\) for all \(i,j \in \{1,2,\ldots,m\}\) where \(\delta_{ij}\) is the Kronecker
delta, then the following inequality is the Bessel’s inequality for $z_2, \ldots, z_n$ orthonormal family $(e_i)_{1 \leq i \leq m}$ in $n$-inner product space $(X, \langle \cdot, \cdot | \cdot, \cdot \rangle)$:

$$
\sum_{i=1}^{m} |\langle x, e_i | z_2, \ldots, z_n \rangle|^2 \leq \|x, z_2, \ldots, z_n\|^2 \quad \text{for any } x \in X.
$$

(3.40)

Theorem 3.7. Let $x_1, x_2, \ldots, x_m, z_2, \ldots, z_n \in X$ and $\alpha_1, \ldots, \alpha_m \in K$, then one has

$$
\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \ldots, z_n \right\|^2 \leq \left\{ \begin{array}{l}
\max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^{m} \|x_i, z_2, \ldots, z_n\|^2, \\
\left( \sum_{i=1}^{m} |\alpha_i|^2 \right)^{1/a} \left( \sum_{i=1}^{m} \|x_i, z_2, \ldots, z_n\|^2 \right)^{1/b}, \quad \text{where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\
\sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \ldots, z_n\|^2
\end{array} \right.
$$

(3.41)

Proof. We observe that

$$
\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \ldots, z_n \right\|^2 = \left\| \sum_{i=1}^{m} \alpha_i x_i \sum_{j=1}^{m} \alpha_j x_j | z_2, \ldots, z_n \right\|
$$

$$
= \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \langle x_i, x_j | z_2, \ldots, z_n \rangle
$$

$$
= \left| \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j \langle x_i, x_j | z_2, \ldots, z_n \rangle \right|
$$

$$
\leq \sum_{i=1}^{m} \sum_{j=1}^{m} |\alpha_i| |\alpha_j| \langle x_i, x_j | z_2, \ldots, z_n \rangle
$$

$$
= \sum_{i=1}^{m} |\alpha_i|^2 \|x_i, z_2, \ldots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\alpha_i| |\alpha_j| \langle x_i, x_j | z_2, \ldots, z_n \rangle.
$$

(3.42)
Using Holder’s inequality, we may write that

\[
\sum_{i=1}^{m} |\alpha_i|^2 \|x_i, z_2, \ldots, z_n\|^2
\]

\[
\leq \begin{cases} 
\max_{1 \leq i \leq m} |\alpha_i|^2 \sum_{i=1}^{m} \|x_i, z_2, \ldots, z_n\|^2, \\
\left( \sum_{i=1}^{m} |\alpha_i|^{2a} \right)^{1/a} \left( \sum_{i=1}^{m} \|x_i, z_2, \ldots, z_n\|^{2b} \right)^{1/b}, & \text{where } a > 1, \frac{1}{a} + \frac{1}{b} = 1, \\
m \sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \ldots, z_n\|^2. \end{cases}
\]

(3.43)

By Holder’s inequality for double sum, we also have

\[
\sum_{1 \leq i \neq j \leq m} |\alpha_i||\alpha_j| |\langle x_i, x_j | z_2, \ldots, z_n \rangle|
\]

\[
\leq \begin{cases} 
\max_{1 \leq i \neq j \leq m} |\alpha_i\alpha_j| \sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \ldots, z_n \rangle|, \\
\left( \sum_{1 \leq i \neq j \leq m} |\alpha_i|^c |\alpha_j|^c \right)^{1/c} \\
\times \left( \sum_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \ldots, z_n \rangle|^d \right)^{1/d}, & \text{where } c > 1, \frac{1}{c} + \frac{1}{d} = 1, \\
m \sum_{1 \leq i \neq j \leq m} |\alpha_i||\alpha_j| \max_{1 \leq i \neq j \leq m} |\langle x_i, x_j | z_2, \ldots, z_n \rangle| \end{cases}
\]

(3.44)

Using (3.43) and (3.44) in (3.42), we may deduce the desired result.
Corollary 3.8. With the assumption in above Theorem 3.7, one has

\[
\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \ldots, z_n \right\|^2 \leq \sum_{i=1}^{n} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq n} \|x_i, z_2, \ldots, z_n\|^2 \right. \\
\left. + \left( \sum_{i=1}^{n} |\alpha_i|^2 \right)^2 - \left( \sum_{i=1}^{n} |\alpha_i|^4 \right) \right\}^{1/2} \sum_{i=1}^{n} |\alpha_i|^2 \right\}^{1/2} \right\} \right. \\
\left. \leq \sum_{i=1}^{m} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|x_i, z_2, \ldots, z_n\|^2 \right. \\
\left. + \left( \sum_{1 \leq i \neq j \leq m} |(x_i, x_j | z_2, \ldots, z_n)|^2 \right) \right\}^{1/2} \right\}.
\] (3.45)

The first inequality follows by taking the third branch in the first curly bracket with the second branch in the second curly bracket for \(c = d = 2\). The second inequality in (3.45) follows by the fact that

\[
\left[ \left( \sum_{i=1}^{m} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{m} |\alpha_i|^4 \right]^{1/2} \leq \sum_{i=1}^{m} |\alpha_i|^2.
\] (3.46)

By applying the following Cauchy-Bunyakovsky-Schwarz inequality:

\[
\left( \sum_{i=1}^{m} a_i \right)^2 \leq m \sum_{i=1}^{m} a_i^2, \quad a_i \in \mathbb{R}_+, \quad 1 \leq i \leq m.
\] (3.47)

one may write that

\[
\left( \sum_{i=1}^{m} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{m} |\alpha_i|^{2c} \leq (m - 1) \sum_{i=1}^{m} |\alpha_i|^{2c}, \quad (m \geq 1),
\] (3.48)

\[
\left( \sum_{i=1}^{m} |\alpha_i|^2 \right)^2 - \sum_{i=1}^{m} |\alpha_i|^2 \leq (m - 1) \sum_{i=1}^{m} |\alpha_i|^2, \quad (m \geq 1).
\] (3.49)

It is obvious that

\[
\max_{1 \leq i \neq j \leq m} \{|\alpha_i \alpha_j|\} \leq \max_{1 \leq i \leq m} |\alpha_i|^2.
\] (3.50)
Corollary 3.9. With the assumption in above Theorem 3.7, one has

\[
\left\| \sum_{i=1}^{m} \alpha_i x_i, z_2, \ldots, z_n \right\|^2 \\
\leq \left\{ \left( \sum_{i=1}^{m} |\alpha_i|^{2a} \right)^{1/a} \left( \sum_{i=1}^{m} \|x_i, z_2, \ldots, z_n\|^2 \right)^{1/b} \right\}, \quad \text{where } a > 1, \quad \frac{1}{a} + \frac{1}{b} = 1,
\]

\[
= \left\{ \sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|x_i, z_2, \ldots, z_n\|^2 \right\}
\]

The proof is obvious by Theorem 3.7 on applying (3.48)–(3.50).

Theorem 3.10. Let \( x, y_1, \ldots, y_m, z_2, \ldots, z_n \) be vectors of an \( n \)-inner product space \((X, \langle \cdot, \cdot | \cdot, \cdot \rangle)\) and \( \alpha_1, \alpha_2, \ldots, \alpha_n \in K (K = \mathbb{R}, \mathbb{C}) \). Then

\[
\left\| \sum_{i=1}^{m} \alpha_i \langle x, y_i | z_2, \ldots, z_n \rangle \right\|^2 \\
\leq \|x, z_2, \ldots, z_n\|^2 \left\{ \sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 \right\}
\]

\[
= \left\{ \left( \sum_{i=1}^{m} |\alpha_i|^{2a} \right)^{1/a} \left( \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^2 \right)^{1/b} \right\}, \quad \text{where } a > 1, \quad \frac{1}{a} + \frac{1}{b} = 1,
\]

\[
= \left\{ \sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 \right\}
\]

\[
\left( \sum_{1 \leq i \neq j \leq m} |\alpha_i\alpha_j| \right) \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j | z_2, \ldots, z_n \rangle|,\]

\[
+ \|x, z_2, \ldots, z_n\|^2 \left\{ \left[ \left( \sum_{i=1}^{m} |\alpha_i| \right)^2 - \left( \sum_{i=1}^{m} |\alpha_i|^2 \right) \right]^{1/c} \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j | z_2, \ldots, z_n \rangle|^d \right) \right\},
\]

\[
= \left\{ \sum_{i=1}^{m} |\alpha_i|^2 \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 \right\}
\]

(3.52)
Proof. Since

\[ \sum_{i=1}^{m} \alpha_i (x, y_i | z_2, \ldots, z_n) = \langle x, \sum_{i=1}^{m} \alpha_i y_i | z_2, \ldots, z_n \rangle \] (3.53)

and by Schwarz's inequality in n-inner product spaces,

\[ \left| \sum_{i=1}^{m} \alpha_i (x, y_i | z_2, \ldots, z_n) \right|^2 \leq \|x, z_2, \ldots, z_n\|^2 \left\| \sum_{i=1}^{m} \alpha_i y_i, z_2, \ldots, z_n \right\| ^2. \] (3.54)

Using \( \alpha_i = \bar{\alpha}_i \), \( z_i = y_i \) (\( i = 1, 2, \ldots, n \)) in the Theorem 3.7, we get the desired inequality (3.52).

**Corollary 3.11.** With the assumptions in Theorem 3.10, the following holds:

\[
\left| \sum_{i=1}^{m} \alpha_i (x, y_i | z_2, \ldots, z_n) \right|^2 \\
\leq \|x, z_2, \ldots, z_n\|^2 \\
\leq \left\{ \sum_{i=1}^{m} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 + \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j | z_2, \ldots, z_n \rangle| \right)^2 \right\}^{1/2} \right\}, \\
\times \left\{ \sum_{i=1}^{m} |\alpha_i|^{2p} \right\}^{1/p} \\
\times \left\{ \left( \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^2 q \right)^{1/q} + (m - 1) \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j | z_2, \ldots, z_n \rangle|^q \right)^{1/q} \right\}, \\
\left\{ \sum_{i=1}^{m} |\alpha_i|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 + (m - 1) \max_{1 \leq i \neq j \leq m} |\langle y_i, y_j | z_2, \ldots, z_n \rangle| \right\} \right\}. \] (3.55)
3.3. Some Boas-Bellman Type Inequalities in $n$-Inner Product Spaces

If we put $\alpha_i = \langle x, y_i \mid z_2, \ldots, z_n \rangle$ ($i = 1, 2, \ldots, m$), in the first inequality of (3.55)

$$\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^2 \leq \|x, z_2, \ldots, z_n\|^2 \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2$$

(3.56)

$$\times \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 + \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^2 \right)^{1/2} \right\}$$

which is equivalent to the following Boass-Bellman type inequality for $n$-inner products:

$$\sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \leq \|x, z_2, \ldots, z_n\|^2 \left\{ \max_{1 \leq i \leq m} \|y_i, z_2, \ldots, z_n\|^2 + \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^2 \right)^{1/2} \right\}. \tag{3.57}$$

Now, if we take $\alpha_i = \langle x, y_i \mid z_2, \ldots, z_n \rangle$ ($i = 1, 2, \ldots, m$), in second inequality of (3.55), we have

$$\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^2 \leq \|x, z_2, \ldots, z_n\|^2 \max_{1 \leq i \leq m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2$$

(3.58)

$$\times \left\{ \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle| \right\}.$$

By taking the square root in

$$\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^2 \leq \|x, z_2, \ldots, z_n\| \max_{1 \leq i \leq m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|$$

(3.59)

$$\times \left\{ \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^2 + \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle| \right\}^{1/2}.$$
for any $x, y_1, \ldots, y_m, z_2, \ldots, z_n$ be vectors of an $n$-inner product space $(X, \langle \cdot, \cdot \mid \cdot, \cdot \rangle)$. If we assume that $(e_i)_{1 \leq i \leq m}$ is an orthonormal family in $X$ with respect to the vector $z_2, \ldots, z_n$, $(e_i, e_j \mid z_2, \ldots, z_n) = \delta_{ij}$ for all $i, j \in \{1, \ldots, m\}$ then by (3.57) we deduce Bessel’s inequality $\sum_{i=1}^{m} |\langle x, e_i \mid z_2, \ldots, z_n \rangle|^2 \leq \|x, z_2, \ldots, z_n\|^2$, and (3.59) implies

$$
\sum_{i=1}^{m} |\langle x, e_i \mid z_2, \ldots, z_n \rangle|^2 \leq \sqrt{m} \|x, z_2, \ldots, z_n\| \max_{1 \leq i \leq m} |\langle x, e_i \mid z_2, \ldots, z_n \rangle|^2.
$$

(3.60)

For the third inequality in (3.55) $a_i = \overline{\langle x, y_i \mid z_2, \ldots, z_n \rangle}$ ($i = 1, 2, \ldots, m$), we have

$$
\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^2 \leq \|x, z_2, \ldots, z_n\|^2 \left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^{2p} \right)^{1/p}
$$

$$
\times \left\{ \left( \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^{2q} \right)^{1/q} + (m-1) \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^q \right)^{1/q} \right\}
$$

(3.61)

for $p > 1$, $(1/p) + (1/q) = 1$. Taking the square root in this inequality, we get

$$
\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^{1/2} \leq \|x, z_2, \ldots, z_n\| \left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^{2p} \right)^{1/2p}
$$

$$
\times \left\{ \left( \sum_{i=1}^{m} \|y_i, z_2, \ldots, z_n\|^{2q} \right)^{1/q} + (m-1) \left( \sum_{1 \leq i \neq j \leq m} |\langle y_i, y_j \mid z_2, \ldots, z_n \rangle|^q \right)^{1/q} \right\}^{1/2}.
$$

(3.62)

For any $x, y_1, \ldots, y_m, z_2, \ldots, z_n \in X$, and $p > 1$, with $(1/p) + (1/q) = 1$, then the above inequality (3.62) becomes, for an orthonormal family with respect to the vector $z_2, \ldots, z_n$,

$$
\left( \sum_{i=1}^{m} |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right)^{1/2p} \leq m^{1/q} \|x, z_2, \ldots, z_n\| \left( \sum_{i=1}^{m} |\langle x, e_i \mid z_2, \ldots, z_n \rangle|^{2p} \right)^{1/2p}.
$$

(3.63)
We take \( \alpha_i = \langle x, y_i \mid z_2, \ldots, z_n \rangle \) \( (i = 1, 2, \ldots, m) \), in the last inequality of (3.55)

\[
\left( \sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right| \right)^2 \\
\leq \| x, z_2, \ldots, z_n \|^2 \sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 \\
\times \left\{ \max_{1 \leq i \leq m} \| y_i, z_2, \ldots, z_n \|^2 + (m - 1) \max_{1 \leq i \neq j \leq m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right| \right\}
\]

which implies

\[
\left( \sum_{i=1}^{m} \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right| \right)^2 \\
\leq \| x, z_2, \ldots, z_n \|^2 \left\{ \max_{1 \leq i \leq m} \| y_i, z_2, \ldots, z_n \|^2 + (m - 1) \max_{1 \leq i \neq j \leq m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right| \right\}
\]

for any \( x, y_1, \ldots, y_m, z_2, \ldots, z_n \in X \).

**4. Applications for Integral Inequalities**

Let \((\Omega, \Sigma, \mu)\) be a measure space consisting of a set \(\Omega\), a \(\sigma\)-algebra \(\Sigma\) of subsets of \(\Omega\) and a countably additive measure \(\mu\) on \(\Sigma\) with values in \(\mathbb{R} \cup \{\infty\}\). Denote by \(L^2(\rho(\Omega))\) the Hilbert space of all real-valued functions \(f\) defined on \(\Omega\) that are \(\rho\)-integrable on \(\Omega\), that is, \(\int_{\Omega} \rho(s)|f(s)|^2 d\mu(s) < \infty\) where \(\rho : \Omega \to [0, \infty)\) is a measurable function on \(\Omega\).

We can introduce the following \(n\)-inner product on \(L^2(\rho(\Omega))\):

\[
\langle f, g \mid h_2, \ldots, h_n \rangle_{\rho} = \frac{1}{n!} \int_{\Omega} \cdots \int_{\Omega} \rho(s_1) \rho(s_2) \cdots \rho(s_n) \\
\times \begin{vmatrix} f(s_1) & f(s_2) & \cdots & f(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & h_2(s_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & h_n(s_n) \end{vmatrix} \\
\times \begin{vmatrix} g(s_1) & g(s_2) & \cdots & g(s_n) \\ h_2(s_1) & h_2(s_2) & \cdots & h_2(s_n) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(s_1) & h_n(s_2) & \cdots & h_n(s_n) \end{vmatrix} d\mu(s_1) \cdots d\mu(s_n).
\]  

(4.1)
where by
\[
\begin{vmatrix}
  f(s_1) & f(s_2) & \ldots & f(s_n) \\
  h_2(s_1) & h_2(s_2) & \ldots & h_2(s_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  h_n(s_1) & h_n(s_2) & \ldots & h_n(s_n)
\end{vmatrix}
\]
we denote the determinant of matrix
\[
\begin{vmatrix}
  f(s_1) & f(s_2) & \ldots & f(s_n) \\
  h_2(s_1) & h_2(s_2) & \ldots & h_2(s_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  h_n(s_1) & h_n(s_2) & \ldots & h_n(s_n)
\end{vmatrix}
\]
Generating the \( n \)-norm on \( L^2(\rho(\Omega)) \) is expressed by
\[
\| f, h_2, \ldots, h_n \|_p = \left( \frac{1}{n!} \int_{\Omega} \cdots \int_{\Omega} \rho(s_1) \rho(s_2) \cdots \rho(s_n) \right. \\
\left. \times \begin{vmatrix}
  f(s_1) & f(s_2) & \ldots & f(s_n) \\
  h_2(s_1) & h_2(s_2) & \ldots & h_2(s_n) \\
  \vdots & \vdots & \ddots & \vdots \\
  h_n(s_1) & h_n(s_2) & \ldots & h_n(s_n)
\end{vmatrix}^2 \\
\int_{\Omega} \cdots \int_{\Omega} \rho(s_1) \rho(s_2) \cdots \rho(s_n) \right)^{1/2}
\]
A simple calculation with integrals shows that
\[
\langle f, g \mid h_2, \ldots, h_n \rangle_p = \begin{vmatrix}
  \int_{\Omega} \rho f g \, d\mu & \int_{\Omega} \rho f h_2 \, d\mu & \ldots & \int_{\Omega} \rho f h_n \, d\mu \\
  \int_{\Omega} \rho h_2 g \, d\mu & \int_{\Omega} \rho h_2^2 \, d\mu & \ldots & \int_{\Omega} \rho h_2 h_n \, d\mu \\
  \vdots & \vdots & \ddots & \vdots \\
  \int_{\Omega} \rho h_n g \, d\mu & \int_{\Omega} \rho h_n h_2 \, d\mu & \ldots & \int_{\Omega} \rho h_n^2 \, d\mu
\end{vmatrix}
\]
\[
\| f, h_2, \ldots, h_n \|_p = \begin{vmatrix}
  \int_{\Omega} \rho f^2 \, d\mu & \int_{\Omega} \rho f h_2 \, d\mu & \ldots & \int_{\Omega} \rho f h_n \, d\mu \\
  \int_{\Omega} \rho h_2^2 \, d\mu & \int_{\Omega} \rho h_2^2 \, d\mu & \ldots & \int_{\Omega} \rho h_2 h_n \, d\mu \\
  \vdots & \vdots & \ddots & \vdots \\
  \int_{\Omega} \rho h_n^2 \, d\mu & \int_{\Omega} \rho h_n h_2 \, d\mu & \ldots & \int_{\Omega} \rho h_n^2 \, d\mu
\end{vmatrix}
\]
where, for simplicity, instead of \( \int_{\Omega} \rho(s)f(s)g(s)\,d\mu(s) \), we have written \( \int_{\Omega} \rho f g \, d\mu \).
Proposition 4.1. Let \( f, g_1, \ldots, g_m, h_2, \ldots, h_n \in L^2(\rho(\Omega)) \), where \( \rho : \Omega \to [0, \infty) \) a measure function is on \( \Omega \). Then one has

\[
\sum_{i=1}^m \left| \int_{\Omega} \rho f g_i d\mu \int_{\Omega} \rho f h_2 d\mu \cdots \int_{\Omega} \rho f h_n d\mu \right|^2
\leq \max_{1 \leq i \leq m} \left| \int_{\Omega} \rho g_i^2 d\mu \int_{\Omega} \rho g_i h_2 d\mu \cdots \int_{\Omega} \rho g_i h_n d\mu \right|
\]

\[
\times \left\{ \left( \sum_{1 \leq i \neq j \leq m} \left| \int_{\Omega} \rho g_i g_j d\mu \int_{\Omega} \rho g_i h_2 d\mu \cdots \int_{\Omega} \rho g_i h_n d\mu \right|^2 \right)^{1/2} \right. \}
\]

\[
+ \left. \left( \sum_{1 \leq i \neq j \leq m} \left| \int_{\Omega} \rho g_i h_2 d\mu \int_{\Omega} \rho h_2^2 d\mu \cdots \int_{\Omega} \rho h_2 h_n d\mu \right|^2 \right)^{1/2} \right\}.
\]

Proof. Applying the \( n \)-inner product and \( n \)-norm defined in (4.1) and (4.4), and using (4.5) and (4.6) in (3.57)

\[
\sum_{i=1}^m \left| \langle x, y_i \mid z_2, \ldots, z_n \rangle \right|^2 
\leq \|x, z_2, \ldots, z_n\|^2
\]

\[
\leq \|x\|^2 + \sum_{1 \leq i \neq j \leq m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right|^2 \right\}^{1/2},
\]

we get the required proof of the proposition. \( \square \)
Proposition 4.2. Let \( f, g_1, \ldots, g_m, h_2, \ldots, h_n \in L^2\rho(\Omega) \), where \( \rho: \Omega \rightarrow [0, \infty) \) a measure function is on \( \Omega \). Then one has

\[
\sum_{i=1}^{m} \left| \int_{\Omega} \rho f g_i d\mu \int_{\Omega} \rho f h_2 d\mu \cdots \int_{\Omega} \rho f h_n d\mu \right|^2 \\
\leq \max_{1 \leq i \leq m} \left| \int_{\Omega} \rho g_i h_2 d\mu \int_{\Omega} \rho h_2^2 d\mu \cdots \int_{\Omega} \rho h_n^2 d\mu \right| \\
\times \left\{ \max_{1 \leq i \neq j \leq m} \left| \int_{\Omega} \rho g_i h_2 d\mu \int_{\Omega} \rho h_2^2 d\mu \cdots \int_{\Omega} \rho h_n^2 d\mu \right| + (m-1) \max_{1 \leq i \leq m} \left| \langle x_i, y_i \mid z_2, \ldots, z_n \rangle \right| \right\}.
\]

Proof. By (3.65),

\[
\left( \sum |\langle x, y_i \mid z_2, \ldots, z_n \rangle|^2 \right) \\
\leq \|x, z_2, \ldots, z_n\|^2 + (m-1) \max_{1 \leq i \neq j \leq m} \left| \langle y_i, y_j \mid z_2, \ldots, z_n \rangle \right|,
\]

and using (4.1) and (4.4), we get the proof of the proposition.
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References