Research Article

A Study on Degree of Approximation by \((E, 1)\) Summability Means of the Fourier-Laguerre Expansion

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1. Introduction

Let \(\sum_{n=0}^{\infty} u_n\) be an infinite series with the sequence of its \(n\)th partial sums \(\{s_n\}\).

If

\[
E^1_n = \frac{1}{2^n} \sum_{k=0}^{n} C_k s_{n-k} \to s \quad \text{as} \quad n \to \infty,
\]

then we say that \(\{s_n\}\) is summable by \((E, 1)\) means (see the study by Hardy [1]), and it is written as \(s_n \to s (E, 1)\), where \(\{s_n\}\) is the sequence of \(n\)th partial sums of the series \(\sum_{n=0}^{\infty} u_n\).

The Fourier-Laguerre expansion of a function \(f(x) \in L(0, \infty)\) is given by

\[
f(x) \sim \sum_{n=0}^{\infty} a_n L_{n}^{(\alpha)}(x),
\]
where
\[ a_n = \left\{ \Gamma(n + 1) \left( \frac{n + \alpha}{n} \right) \right\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(a)}(y) \, dy \]  
(1.3)

and \( L_n^{(a)}(x) \) denotes the \( n \)th Laguerre polynomial of order \( \alpha > -1 \), defined by generating function
\[ \sum_{n=0}^{\infty} L_n^{(a)}(x) \omega^n = (1 - \omega)^{-\alpha - 1} \exp\left( \frac{-x\omega}{1 - \omega} \right), \]  
(1.4)

and existence of integral (1.3) is presumed.

We write
\[ \phi(y) = \{ \Gamma(\alpha + 1) \}^{-1} e^{-y} y^\alpha \left[ f(y) - f(0) \right]. \]  
(1.5)

Gupta [2] estimated the order of the function by Cesàro means of series (1.2) at the point \( x = 0 \) after replacing the continuity condition in Szegő's theorem [3] by a much lighter condition. He established the following theorem.

**Theorem 1.1.** If
\[ F(t) = \int_0^t \frac{|f(y)|}{y} \, dy = o\left( \log\left( \frac{1}{t} \right) \right)^{1+p}, \quad t \to 0, \quad -1 < p < \infty, \]  
(1.6)

\[ \int_1^\infty e^{-y/2} y^{(3\alpha - 3k - 1)/3} |f(y)| \, dy < \infty, \]

then
\[ \sigma_k^b(0) = o(\log n)^{p+1} \]  
(1.7)

provided that \( k > \alpha + 1/2, \alpha > -1 \), with \( \sigma_k^b(0) \) being the \( n \)th Cesàro mean of order \( k \).

Denoting the harmonic means by \( \{ t_n \} \), Singh [4] estimated the order of function by harmonic means of series (1.2) at point \( x = 0 \) by weaker conditions than those of Theorem 1.1. He proved the following theorem.

**Theorem 1.2.** For \(-5/6 < \alpha < -1/2\),
\[ t_n(0) - f(0) = o(\log n)^{p+1}, \]  
(1.8)

provided that
\[ \int_1^t \frac{|\phi(y)|}{y^{n+1}} \, dy = o\left( \log\left( \frac{1}{t} \right) \right)^{1+p}, \quad t \to 0, \quad -1 < p < \infty, \]  
(1.9)
\[\delta \text{ is a fixed positive constant,}\]
\[
\int_0^n e^{y^2/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left\{ n^{-(2\alpha+1)/4} (\log n)^{1+p} \right\}, \]
\[
\int_n^\infty e^{y^2/2} y^{-1/3} |\phi(y)| dy = o\left\{ (\log n)^{p+1} \right\}, \quad n \to \infty.
\] (1.10)

2. Main Theorem

The objects of present paper are as follows:

(1) We prove our theorem for \((E, 1)\) means which is entirely different from \((C, k)\) and harmonic means.

(2) We employ a condition which is weaker than condition (1.9) of Theorem 1.2.

(3) In our theorem the range of \(\alpha\) is increased to \(-1 < \alpha < -1/2\), which is more useful for application.

In fact, we establish the following theorem.

**Theorem 2.1.** If

\[
E_n^1 = \frac{1}{2^n} \sum_{k=0}^n n! C_k \xi_k \to \infty \quad \text{as} \quad n \to \infty,
\] (2.1)

then the degree of approximation of Fourier-Laguerre expansion (1.2) at the point \(x = 0\) by \((E, 1)\) means \(E_n^1\) is given by

\[
E_n^1(0) - f(0) = o\{ \xi(n) \}
\] (2.2)

provided that

\[
\Phi(t) = \int_0^t |\phi(y)| dy = o\left\{ t^{\alpha+1} \xi\left(\frac{1}{t}\right) \right\}, \quad t \to 0,
\] (2.3)

\(\delta\) is a fixed positive constant and \(-1 < \alpha < -1/2,\)

\[
\int_0^n e^{y^2/2} y^{-(2\alpha+3)/4} |\phi(y)| dy = o\left\{ n^{-(2\alpha+1)/4} \xi(n) \right\}, \]
\[
\int_n^\infty e^{y^2/2} y^{-1/3} |\phi(y)| dy = o\{ \xi(n) \}, \quad n \to \infty,
\] (2.4) (2.5)

where \(\xi(t)\) is a positive monotonic increasing function of \(t\) such that \(\xi(n) \to \infty\) as \(n \to \infty.\)
3. Lemmas

Lemma 3.1 (see the study by Szegö, 1959, [3, page 175]). Let $\alpha$ be arbitrary and real, let $c$ and $\epsilon$ be fixed positive constants, and let $n \to \infty$. Then

$$L_n^{(\alpha)}(x) = O\left(x^{-(2\alpha+1)/4}n^{(2\alpha-1)/4}\right) \quad \text{if} \quad \frac{c}{n} \leq x \leq \epsilon,$$

(3.1)

$$L_n^{(\alpha)}(x) = O(n^\alpha) \quad \text{if} \quad 0 \leq x \leq \frac{\epsilon}{n}.$$

(3.2)

4. Proof of the Main Theorem

Since

$$L_n^{(\alpha)}(0) = \left(\begin{array}{c} n + \alpha \\ \alpha \end{array}\right),$$

(4.1)

therefore,

$$s_n(0) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) = \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y) dy$$

$$= \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y) dy.$$  

(4.2)

Now,

$$E_n^1(0) = \frac{1}{2^n} \sum_{k=0}^{n} n^k s_k(0)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} n^k \{\Gamma(\alpha+1)\}^{-1} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y) dy.$$  

(4.3)

Using orthogonal property of Laguerre’s polynomial and (1.5), we have

$$E_n^1(0) - f(0) = \frac{1}{2^n} \sum_{k=0}^{n} n^k \int_0^\infty \phi(y) L_k^{(\alpha+1)}(y) dy$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{n} + \int_n^\infty\right) \frac{1}{2^n} \sum_{k=0}^{n} n^k \phi(y) L_k^{(\alpha+1)}(y) dy$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}.$$  

(4.4)
Using orthogonal property and condition (3.2) (taking $\alpha + 1$ for $\alpha$ and $\delta$ for $\varepsilon$) of Lemma 3.1, we get

$$I_1 = \frac{1}{2\pi} \sum_{k=0}^{n} nC_k O\left\{ n^{\alpha+1} \right\} \int_{0}^{1/n} |\phi(y)| dy$$

$$= \frac{1}{2\pi} \sum_{k=0}^{n} nC_k O\left\{ n^{\alpha+1} \right\} o\left\{ \frac{1}{n^{\alpha+1}} \xi(n) \right\},$$

$$I_1 = o\left( \frac{1}{2\pi} \sum_{k=0}^{n} nC_k \xi(n) \right)$$

$$= o\{\xi(n)\} \quad \text{since} \quad \sum_{k=0}^{n} nC_k = 2^n.$$ 

Further, using orthogonal property and condition (3.1) (taking $\alpha + 1$ for $\alpha$, 1 for $c$, and $\delta$ for $\varepsilon$) of Lemma 3.1, we get

$$I_2 = \frac{1}{2\pi} \sum_{k=0}^{n} nC_k O\left\{ n^{(2\alpha+1)/4} \right\} \int_{1/n}^{\delta} |\phi(y)| y^{-(2\alpha+3)/4} dy.$$ 

Now,

$$\sum_{k=0}^{n} nC_k n^{(2\alpha+1)/4} = \left\{ \sum_{k=0}^{[n/2]} + \sum_{k=[n/2]+1}^{n} \right\} nC_k n^{(2\alpha+1)/4}$$

$$= (n)^{(2\alpha+1)/4} \sum_{k=0}^{[n/2]} nC_k + nC_{[n/2]} n^{(2\alpha+5)/4}$$

$$\leq (n)^{(2\alpha+1)/4} \sum_{k=0}^{n} nC_k + nC_{[n/2]} n^{(2\alpha+5)/4},$$

$$\sum_{k=0}^{n} nC_k n^{(2\alpha+1)/4} = (n)^{(2\alpha+1)/4} 2^n + nC_{[n/2]} n^{(2\alpha+5)/4}$$

since

$$2^n = \sum_{k=0}^{n} nC_k$$

$$= nC_0 + nC_1 + nC_2 + \cdots + nC_{[n/2]} + nC_{[n/2]+1} + \cdots$$

$$\geq nC_{[n/2]} + nC_{[n/2]} + \cdots + nC_n$$

$$\geq nC_{[n/2]} + nC_{[n/2]} + \cdots + nC_{[n/2]} = \left\{ \left\lfloor \frac{n}{2} \right\rfloor + 1 \right\} nC_{[n/2]} \geq \frac{n}{2} nC_{[n/2]}.$$
Therefore,

\[ \frac{n}{2} C_{[n/2]} \leq 2^n. \quad (4.9) \]

By (4.7) and (4.9), we have,

\[
\sum_{k=0}^{n} n C_k \left\{ n^{(2\alpha+1)/4} \right\} \leq (n)^{(2\alpha+1)/4} 2^n + 2(n)^{(2\alpha+1)/4} \\
= O\left\{ (n)^{(2\alpha+1)/4} 2^n \right\}. \quad (4.10)
\]

Thus,

\[
I_2 = O\left\{ (n)^{(2\alpha+1)/4} \right\} \int_{1/n}^{\delta} y^{-(2\alpha+3)/4} \Phi(y) dy \\
= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[ \left\{ y^{-(2\alpha+3)/4} \Phi(y) \right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} \frac{(2\alpha + 3)}{4} y^{-(2\alpha+7)/4} \Phi(y) dy \right] \\
= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[ \left\{ y^{-(2\alpha+3)/4} \Phi(y) \right\}_{1/n}^{\delta} + \int_{1/n}^{\delta} y^{-(2\alpha+7)/4} \Phi(y) dy \right] \\
= O\left\{ (n)^{(2\alpha+1)/4} \right\} \left[ O(1) + o\left\{ n^{-(2\alpha+1)/4} \xi(n) \right\} \right] + \xi(n) o\left\{ \int_{1/n}^{\delta} y^{-(2\alpha+3)/4} dy \right\} \\
= o\{\xi(n)\} + o\left\{ (n)^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \int_{1/n}^{\delta} y^{-(2\alpha-3)/4} dy \right\} \\
= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \frac{y^{(2\alpha+1)/4}}{(2\alpha - 3)/4 + 1} \right\}_{1/n}^{\delta} \\
= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ \frac{y^{(2\alpha+1)/4}}{(2\alpha + 1)/4} \right\}_{1/n}^{\delta} \\
= o\{\xi(n)\} + o\left\{ n^{(2\alpha+1)/4} \xi(n) \right\} \left\{ n^{-(2\alpha+1)/4} \right\} \\
= o\{\xi(n)\} + o\{\xi(n)\}, \\
I_2 = o\{\xi(n)\}. \quad (4.11)
\]
Now, we consider

\[
I_3 = \left( \frac{1}{2n} \right) \left\{ \sum_{k=0}^{n} C_k \int_{\delta}^{n} e^{y/2} y^{-(2+3)/4} | \phi(y) | e^{-y/2} y^{(2+3)/4} L_n^{(a+1)}(y) \, dy \right\} \\
= \left( \frac{1}{2n} \right) \left\{ \sum_{k=0}^{n} C_k O\left\{ n^{(2+1)/4} \right\} \int_{\delta}^{n} e^{y/2} y^{-(2+3)/4} | \phi(y) | \, dy \right\},
\]

(4.12)

\[
I_3 = O\left\{ (n)^{(2+1)/4} \right\} o\left\{ (n)^{-2+1/4} \xi(n) \right\}, \text{ using (2.4),}
\]

\[
I_3 = o\{ \xi(n) \}.
\]

Finally,

\[
I_4 = \left( \frac{1}{2n} \right) \left\{ \sum_{k=0}^{n} C_k \int_{\delta}^{n} e^{y/2} y^{-(3+5)/6} | \phi(y) | e^{-y/2} y^{(3+5)/6} L_n^{(a+1)}(y) \, dy \right\} \\
= \left( \frac{1}{2n} \right) \left\{ \sum_{k=0}^{n} C_k O\left\{ k^{(a+1)/2} \right\} \int_{\delta}^{n} e^{y/2} y^{-1/3} | \phi(y) | \, dy \right\},
\]

(4.13)

\[
I_4 = O\left\{ \left( \frac{1}{2n} \right) (2^n) \left\{ n^{(a+1)/2} n^{-2(a+1)/2} \right\} o\{ \xi(n) \right\}, \text{ by (2.5)}
\]

\[
I_4 = o\{ \xi(n) \}.
\]

Combining (4.4), (4.5), (4.11), (4.12), and (4.13), we get

\[
E_n(0) - f(0) = o\{ \xi(n) \}.
\]

(4.14)

This completes the proof of the theorem.

**References**