Research Article

Derivations of MV-Algebras

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Received 26 August 2010; Revised 8 November 2010; Accepted 16 December 2010

Academic Editor: Howard Bell

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We introduce the notion of derivation for an MV-algebra and discuss some related properties. Using the notion of an isotone derivation, we give some characterizations of a derivation of an MV-algebra. Moreover, we define an additive derivation of an MV-algebra and investigate some of its properties. Also, we prove that an additive derivation of a linearly ordered MV-algebra is an isotone.

1. Introduction

In his classical paper [1], Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of infinite valued Lukasiewicz propositional calculus. Recently, the algebraic theory of MV-algebras is intensively studied, see [2–5].

The notion of derivation, introduced from the analytic theory, is helpful to the research of structure and property in algebraic system. Several authors [6–9] studied derivations in rings and near rings. Jun and Xin [10] applied the notion of derivation in ring and near-ring theory to BCI-algebras. In [11], Szász introduced the concept of derivation for lattices and investigated some of its properties, for more details, the reader is referred to [9, 12–19].

In this paper, we apply the notion of derivation in ring and near-ring theory to MV-algebras and investigate some of its properties. Using the notion of an isotone derivation, we characterize a derivation of MV-algebra. We introduce a new concept, called an additive derivation of MV-algebras, and then we investigate several properties. Finally, we prove that an additive derivation of a linearly ordered MV-algebra is an isotone.

2. Preliminaries

Definition 2.1 (see [5]). An MV-algebra is a structure \((M, \oplus, *, 0)\) where \(\oplus\) is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for
any \(a, b \in M\):

(MV1) \((M, \oplus, 0)\) is a commutative monoid,
(MV2) \((a^*)^* = a\),
(MV3) \(0^* \oplus a = 0^*\),
(MV4) \((a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a\).

If we define the constant \(1 = 0^*\) and the auxiliary operations \(\ominus, \lor, \land\) by

\[
\begin{align*}
    a \ominus b & = (a^* \oplus b^*)^*, \\
    a \lor b & = a \oplus (b \ominus a^*), \\
    a \land b & = a \ominus (b \oplus a^*),
\end{align*}
\]

(2.1)
then \((M, \circ, 1)\) is a commutative monoid and the structure \((M, \lor, \land, 0, 1)\) is a bounded distributive lattice. Also, we define the binary operation \(\ominus\) by \(x \ominus y = x \oplus y^*\). A subset \(X\) of an MV-algebra \(M\) is called subalgebra of \(M\) if and only if \(X\) is closed under the MV-operations defined in \(M\). In any MV-algebras, one can define a partial order \(\leq\) by putting \(x \leq y\) if and only if \(x \land y = x\) for each \(x, y \in M\). If the order relation \(\leq\), defined over \(M\), is total, then we say that \(M\) is linearly ordered. For an MV-algebra \(M\), if we define \(B(M) = \{x \in M : x \ominus x = x\}\) = \(\{x \in M : x \ominus x = x\}\). Then, \((B(M), \oplus, *, 0)\) is both a largest subalgebra of \(M\) and a Boolean algebra.

An MV-algebra \(M\) has the following properties for all \(x, y, z \in M\)

1. \(x \ominus 1 = 1\),
2. \(x \ominus x^* = 1\),
3. \(x \ominus x^* = 0\),
4. If \(x \ominus y = 0\), then \(x = y = 0\),
5. If \(x \ominus y = 1\), then \(x = y = 1\),
6. If \(x \leq y\), then \(x \lor z \leq y \lor z\) and \(x \land z \leq y \land z\),
7. If \(x \leq y\), then \(x \ominus z \leq y \ominus z\) and \(x \ominus z \leq y \ominus z\),
8. \(x \leq y\) if and only if \(y^* \leq x^*\),
9. \(x \ominus y = y\) if and only if \(x \ominus y = x\).

**Theorem 2.2** (see [1]). The following conditions are equivalent for all \(x, y \in M\)

(i) \(x \leq y\),
(ii) \(y \ominus x^* = 1\),
(iii) \(x \ominus y^* = 0\).

**Definition 2.3** (see [1]). Let \(M\) be an MV-algebra and \(I\) be a nonempty subset of \(M\). Then, we say that \(I\) is an ideal if the following conditions are satisfied:

(i) \(0 \in I\),
(ii) \(x, y \in I\) imply \(x \ominus y \in I\),
(iii) \(x \in I\) and \(y \leq x\) imply \(y \in I\).

**Proposition 2.4** (see [1]). Let \(M\) be a linearly ordered MV-algebra, then \(x \ominus y = x \ominus z\) and \(x \ominus z \not= 1\) implies that \(y = z\).
3. Derivations of MV-Algebras

Definition 3.1. Let $M$ be an MV-algebra, and let $d : M \to M$ be a function. We call $d$ a derivation of $M$, if it satisfies the following condition for all $x, y \in M$

$$d(x \odot y) = (dx \odot y) \oplus (x \odot dy).$$  \hfill (3.1)

We often abbreviate $d(x)$ to $dx$.

Example 3.2. Let $M = \{0, a, b, 1\}$. Consider Tables 1 and 2.

Then $(M, \oplus, \ast, 0)$ is an MV-algebra. Define a map $d : M \to M$ by

$$dx = \begin{cases} 0 & \text{if } x = 0, a, 1, \\ a & \text{if } x = b. \end{cases} \hfill (3.2)$$

Since $d(a \odot b) = 0$ and $(da \odot b) \oplus (a \odot db) = (0 \odot b) \oplus (a \odot a) = 0 \odot a = a$, $d$ is not derivation.

Example 3.3. Let $M = \{0, x_1, x_2, x_3, x_4, 1\}$. Consider Tables 3 and 4.

Then, $(M, \oplus, \ast, 0)$ is an MV-algebra. Define a map $d : M \to M$ by

$$dx = \begin{cases} 0 & \text{if } x = 0, x_1, x_3, \\ x_2 & \text{if } x = x_2, x_4, 1. \end{cases} \hfill (3.3)$$

Then, it is easily checked that $d$ is a derivation of $M$.

Proposition 3.4. Let $M$ be an MV-algebra, and let $d$ be a derivation on $M$. Then, the following hold for every $x \in M$:

(i) $d0 = 0$,
(ii) $dx \odot x^* = x \odot dx^* = 0$,
(iii) $dx = dx \oplus (x \odot d1)$,
(iv) $dx \leq x$,
(v) If $I$ is an ideal of an MV-algebra $M$, then $d(I) \subseteq I$. 

\newpage

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\begin{tabular}{c|ccc}
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$\oplus$ & 0 & $a$ & $b$ \\
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0 & 0 & $a$ & $b$ \\
$a$ & $a$ & $a$ & 1 \\
b & $b$ & 1 & $b$ \\
1 & 1 & 1 & 1 \\
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Table 1

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$\ast$ & 0 & $a$ & $b$ \\
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1 & $b$ & $a$ & 0 \\
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Proof. (i) $d0 = d(x \odot 0) = (dx \odot 0) \oplus (x \odot d0) = x \odot d0$. Putting $x = 0$, we get $d0 = 0$.

(ii) Let $x \in M$, then

$$0 = d0 = d(x \odot x^*) = (dx \odot x^*) \oplus (x \odot dx^*),$$

(3.4)

and so (ii) follows from (4).

(iii) It is clear.

(iv) Let $x \in M$, from (ii), we have

$$1 = 0^* = (dx \odot x^*)^* = (dx)^* \oplus x,$$

(3.5)

from Theorem 2.2 we get $dx \leq x$.

(v) Let $y \in d(I)$, then $y = d(x)$ for some $x \in I$. Since $y = d(x) \leq x \in I$, thus $y \in I$ and so $d(I) \subseteq I$.

Proposition 3.5. Let $d$ be a derivation of an MV-algebra $M$, and let $x, y \in M$. If $x \leq y$. Then, the following hold:

(i) $d(x \odot y^*) = 0$,

(ii) $dy^* \leq x^*$,

(iii) $dx \odot dy^* = 0$.

Proof. (i) Let $x \leq y$, then Theorem 2.2 implies that $x \odot y^* = 0$, and so $d(x \odot y^*) = d0 = 0$.

(ii) From (i), we get

$$0 = d(x \odot y^*) = (dx \odot y^*) \oplus (x \odot dy^*),$$

(3.6)

and by (4), we have $x \odot dy^* = 0$. Therefore, $dy^* \leq x^*$. 

Example 3.8. Let $M$ be an MV-algebra, and let $d$ be a derivation on $M$. Then, the following hold:

(i) $dx \odot dx^* = 0$,
(ii) $dx^* = (dx)^*$ if and only if $d$ is the identity on $M$.

Proof. (i) It follows directly from Proposition 3.5(iii).

(ii) It is sufficient to show that if $dx^* = (dx)^*$, then $d$ is the identity on $M$.

Assume that $dx^* = (dx)^*$, from Proposition 3.4(ii), we have $x \odot (dx)^* = 0$, which implies that $x \leq dx$. Therefore, $dx = x$. □

Definition 3.7. Let $M$ be an MV-algebra and $d$ be a derivation on $M$. If $x \leq y$ implies $dx \leq dy$ for all $x, y \in M$, $d$ is called an isotone derivation.

Example 3.8. Let $M$ be an MV-algebra as in Example 3.3. It is easily checked that $d$ is an isotone derivation of $M$.

Proposition 3.9. Let $M$ be an MV-algebra, and let $d$ be a derivation of $M$. If $dx^* = dx$ for all $x \in M$, then the following hold:

(i) $d1 = 0$,
(ii) $dx \odot dx = 0$,
(iii) If $d$ is an isotone derivation of $M$, then $d$ is zero.

Proof. (i) It follows by putting $x = 0$.

(ii) It follows from Proposition 3.6(i).

(iii) Since $d$ is an isotone, hence $dx \leq d1$ for all $x \in M$. By (i), we have $dx \leq 0$, and so $d$ is zero. □

Definition 3.10. Let $M$ be an MV-algebra, and let $d$ be a derivation on $M$. If $d(x \oplus y) = dx \oplus dy$ for all $x, y \in M$, $d$ is called an additive derivation.

Example 3.11. Let $M$ be an MV-algebra as in Example 3.3. It is easily checked that $d$ is an additive derivation of $M$.

Theorem 3.12. Let $M$ be an MV-algebra, and let $d$ be a nonzero additive derivation of $M$. Then, $d(B(M)) \subseteq B(M)$.

Proof. Let $y \in d(B(M))$, thus $y = d(x)$ for some $x \in B(M)$. Then,

$$y \oplus y = dx \oplus dx = d(x \oplus x) = dx = y. \quad (3.7)$$

Therefore $y \in B(M)$, this complete the proof. □

Theorem 3.13. Let $d$ be an additive derivation of a linearly ordered MV-algebra $M$. Then, either $d = 0$ or $d1 = 1$. 
Proof. Let $d$ be an additive derivation of a linearly ordered MV-algebra $M$. Hence,

$$d1 = d(x \oplus x^*) = dx \oplus dx^*, \hspace{1cm} (3.8)$$

also,

$$d1 = d(x \oplus 1) = dx \oplus d1, \hspace{1cm} (3.9)$$

for all $x \in M$. If $d1 \neq 0$, then Proposition 2.4 implies that $dx^* = d1$. Putting $x = 1$, we get that $d1 = 0$. Therefore,

$$0 = d1 = dx \oplus d1 = dx, \hspace{1cm} (3.10)$$

for all $x \in M$, and so $d$ is zero. \[ \square \]

**Proposition 3.14.** Let $M$ be a linearly ordered MV-algebra, and let $d_1, d_2$ additive derivations of $M$. Define $d_1 d_2(x) = d_1(d_2(x))$ for all $x \in M$. If $d_1 d_2 = 0$, then $d_1 = 0$ or $d_2 = 0$. 

**Proof.** Let $d_1 d_2 = 0$, $x \in M$, and suppose that $d_2 \neq 0$. Then,

$$0 = d_1 d_2 x = d_1(d_2 x \oplus (x \oplus d_2 1)) = d_1 (d_2 x \oplus d_1 x) = d_1 x, \hspace{1cm} (3.11)$$

thus $d_1 = 0$. Similarly, we can prove that $d_2 = 0$. \[ \square \]

**Proposition 3.15.** Let $M$ be a linearly ordered MV-algebra, and let $d$ be a nonzero additive derivation of $M$. Then,

$$d(x \odot x) = x \odot x, \hspace{1cm} \forall x \in M. \hspace{1cm} (3.12)$$

**Proof.** From Proposition 3.4(iii) and Theorem 3.13, we get that $dx = dx \odot x$; applying (9), we have $dx \odot x = x$. Thus,

$$d(x \odot x) = (dx \odot x) \oplus (dx \odot x)$$

$$= x \odot x. \hspace{1cm} (3.13)$$

\[ \square \]

**Theorem 3.16.** Every nonzero additive derivation of a linearly ordered MV-algebra $M$ is an isotone derivation.

**Proof.** Assume that $d$ is an additive derivation of $M$, and $x, y \in M$. If $x \leq y$, then $x^* \odot y = 1$, hence

$$1 = d1 = d(x^* \odot y) = dx^* \odot dy, \hspace{1cm} (3.14)$$

and so, $(dy)^* \leq dx^*$, from (8), we have $(dx^*)^* \leq dy$. Otherwise, $dx^* \leq x^*$, again by (8) $x \leq (dx^*)^*$. Since $dx \leq x$, we get $dx \leq dy$. \[ \square \]
Theorem 3.17. Let $M$ be a linearly ordered MV-algebra, and let $d$ be a nonzero additive derivation of $M$. Then, $d^{-1}(0) = \{ x \in M \mid dx = 0 \}$ is an ideal of $M$.

Proof. From Proposition 3.4(i), we get that $0 \in d^{-1}(0)$. Let $x, y \in d^{-1}(0)$; this implies that $d(x \oplus y) = 0$. And so $x \oplus y \in d^{-1}(0)$.

Now, let $x \in d^{-1}(0)$ and $y \leq x$. Using Theorem 3.16, we have that $dy \leq dx$, and so $dy = 0$. \hfill $\square$

References