Research Article

Generalized Alpha-Close-to-Convex Functions

K. Inayat Noor, Halit Orhan, and Saima Mustafa

1 Department of Mathematics, COMSATS Institute of Information Technology, Islamabad 44000, Pakistan
2 Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey

Correspondence should be addressed to K. Inayat Noor, khalidanoor@hotmail.com

Received 5 January 2009; Revised 26 June 2009; Accepted 13 September 2009

Recommended by Teodor Bulboacă

We define the classes $G_{\alpha,k,\gamma}$ as follows: $f \in G_{\alpha,k,\gamma}$ if and only if, for $z \in E = \{z \in \mathbb{C} : |z| < 1\}$, $|\arg\{(1 - \alpha^2z^2)f(z)/e^{-\beta\phi(z)}\}| \leq \gamma\pi/2$, $0 < \gamma \leq 1$; $\alpha \in [0,1]$; $\beta \in (-\pi/2,\pi/2)$, where $\phi$ is a function of bounded boundary rotation. Coefficient estimates, an inclusion result, arclength problem, and some other properties of these classes are studied.

Copyright © 2009 K. Inayat Noor et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let $A$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. By $S$, $K$, $S^*$, and $C$ denote the subclasses of $A$ which are univalent, close-to-convex, starlike, and convex in $E$, respectively.

Let $V_k$ be the class of functions of bounded boundary rotation. Paate [1] showed that a function $f$, defined by (1.1) and $f''(z) \neq 0$, is in $V_k$ if and only if, for $z = re^{i\theta}$,

$$\int_{0}^{2\pi} \left| \text{Re} \left( \frac{zf'(z)}{f'(z)} \right) \right| d\theta \leq k\pi.$$ \hspace{1cm} (1.2)

It is geometrically obvious that $k \geq 2$ and $V_2 \equiv C$.

A class $T_k$ of analytic functions related with the class $V_k$ was introduced and studied in [2]. A function $f \in A$ is in $T_k$, $k \geq 2$, if and only if there exists a function $g \in V_k$ such that, for $z \in E$, $\text{Re}\{f'(z)/g'(z)\} > 0$. It is clear that $T_2 \equiv K$. 
Let $P$ denote the class of analytic functions $p$ defined by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

(1.3)

with $\Re p(z) > 0$ for $z \in E$.

We denote $K(\gamma)$ as the class of strongly close-to-convex functions of order $\gamma$ in the sense of Pommerenke [3]. A function $f \in A$ belongs to $K(\gamma)$ if and only if there exists $g \in S^*$ such that $|\text{Arg}(zf'(z)/g(z))| \leq \pi\gamma/2$, for $z \in E$ and $\gamma \geq 0$.

Clearly $K(0) = C$, $K(1) = K$, and when $0 \leq \gamma < 1$, $K(\gamma)$ is a subset of $K$ and hence contains only univalent functions. For $\gamma > 1$, $f \in K(\gamma)$ can be of infinite valence; see [4].

We now define the following.

**Definition 1.1.** A function $f \in A$ is said to belong to $G_\beta(\alpha,k,\gamma)$, where $\beta$ is a real number, $\alpha \in \mathbb{C}: |\alpha| \leq 1$, $k \geq 2$, and $\gamma \in (0,1]$ is called generalized alpha-close-to-convex with argument $\beta$ if and only if there exists $\phi \in V_k$ such that

$$\left| \text{Arg}\left\{ \frac{(1 - \alpha^2 z^2)f'(z)}{e^{-i\beta}\phi'(z)} \right\} \right| \leq \frac{\gamma \pi}{2}, \quad z \in E.$$  

(1.4)

In (1.4), we choose this branch of argument which equals $\beta$, $|\beta| < \pi\gamma/2$, $\gamma \in (0,1]$, when $z = 0$. We note that the condition $|\alpha| \leq 1$ implies that $G_\beta(\alpha,k,\gamma)$ is nonempty. From the normalization conditions $f'(0) = \phi'(0) = 1$, it follows from Definition 1.1 that $\Re e^{-i\beta} > 0$ and therefore $|\beta| < \gamma \pi/2$. Also, it follows from (1.4) that if $f \in G_\beta(\alpha,k,\gamma)$, then $f'(z) \neq 0$ for $z \in E$. Condition (1.4) is equivalent to the following $f \in G_\beta(\alpha,k,\gamma)$ if and only if there exists $p \in P$ such that

$$\frac{(1 - \alpha^2 z^2)f'(z)}{e^{-i\beta}\phi'(z)} = \left( p(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^\gamma, \quad \phi \in V_k.$$  

(1.5)

We define $G(\alpha,k,\gamma)$ the class of generalized $\alpha$-close-to-convex functions as

$$G(\alpha,k,\gamma) = \bigcup_{|\beta| < \pi/2} G_\beta(\alpha,k,\gamma).$$  

(1.6)

If $\alpha = 0$ in (1.6), then the class $G(0,k,1)$ is identical with the class $T_k$ and $G(\alpha,2,1)$ is the class $K$ of close-to-convex functions. Also $G_\beta(\alpha,2,1)$ in the class of close-to-convex function with argument $\beta$ was defined by Goodman and Saff [5]. For details of special cases of $G_\beta(\alpha,2,1)$ with $\phi(z) = z$ in (1.4), we refer to [6]. The special case with $\gamma = 1 = \alpha$, $k = 2$, and $\phi(z) = z$ in (1.4) leads to the class of functions convex in the direction of the imaginary axis having special normalization; see [7].
2. Main Results

We now prove the main results as follows.

**Theorem 2.1.** Let $\alpha \in [0, 1]$. Then $G(\alpha, k, \gamma) \subset G(0, k, \gamma_1)$, where

\[
\gamma_1(\gamma, \alpha) = \gamma + \frac{2}{\pi} \arcsin(\alpha^2).
\]

The constant $\gamma_1(\gamma, \alpha)$ cannot be smaller.

**Proof.** We will use an extended version of the method given in [8] to prove this result.

For $\alpha = 0$, the result is obvious. Let $f \in G(\alpha, k, \gamma)$. By (1.4), (1.5), and (1.6), then there exists a function $\phi \in V_k$ and a function $p \in P$, $|\beta| < \pi/2$ such that

\[
\frac{f'(z)}{e^{-i\beta}\phi'(z)} = \frac{(p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma))^T}{1 - \alpha^2 z^2}, \quad z \in E.
\]

Let $q(z) = (p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma))^T$, $z \in E$. Then we have

\[
\left| \text{Arg} \frac{f'(z)}{e^{-i\beta}\phi'(z)} \right| = \left| \text{Arg} q(z) - \text{Arg} \left(1 - \alpha^2 z^2\right) \right| < \frac{\pi}{2} \left| \text{Arg} (1 - \alpha^2 z^2) \right|.
\]

We choose in (2.3) this branch of argument which is equal $-\beta$ when $z = 0$.

Since $|\text{Arg} (1 - \alpha^2 z^2)| < \arcsin(\alpha^2)$, $z \in E$, we have from (2.3) $f \in G(0, k, \gamma_1)$, where $\gamma_1$ is given by (2.1). The constant $\gamma_1(\gamma, \alpha)$ cannot be smaller. Let $\alpha \in (0, 1)$ be fixed. Let us consider the point $z_0 \in \mathbb{C}$ with $|z_0| = 1$ and $\text{Arg}(1 - \alpha^2 z_0^2) = -\arcsin(\alpha^2)$. Let $\phi_0 \in V_k$ be such that $\phi_0(z_0)$ is finite. Then, let

\[
f_0'(z) = \frac{e^{-i\beta}\phi_0'(z)}{1 - \alpha^2 z^2} \left[ (p(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma})^T \right], \quad z \in E, \quad |\beta| < \frac{\pi}{2},
\]

where

\[
P_0(z) = \frac{1 + \varepsilon z}{1 - \varepsilon z}, \quad \varepsilon \in \mathbb{C}, \quad |\varepsilon| < 1,
\]

\[
\phi_0'(z) = \frac{(1 + \delta_1 z)^{k/2 - 1}}{(1 + \delta_2 z)^{k/2 + 1}}, \quad |\delta_1| = |\delta_2| = 1.
\]

Now, for $z \in E$,

\[
\left| \text{Arg} \frac{f_0'(z)}{e^{-i\beta}\phi_0'(z)} \right| = \left| \text{Arg} \left( p_0(z) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right)^T - \text{Arg} \left(1 - \alpha^2 z^2\right) \right|,
\]

97x642
and $\text{Arg} e^{-i\beta} = -\beta$. Since $p_0$ maps the unit circle $|z| = 1$ onto imaginary axis, we may choose $\varepsilon_0$, $|\varepsilon_0| = 1$ such that $\varepsilon_0 \neq 1/z_0$, $p_0(z_0) = (1 + \varepsilon_0 z_0)/(1 - \varepsilon_0 z_0) \neq i \tan \beta$, $p_0(z_0) = ai, a > 0$. This means that $p_0(z_0)$ is finite and $\text{Arg} p_0(z_0) = \pi/2$. Hence

$$\text{Arg} \left[ \left( p_0(z) \cos \frac{\beta}{\gamma} i \sin \frac{\beta}{\gamma} \right)^T \right] = \frac{\gamma \pi}{2}. \quad (2.7)$$

Thus, from (2.4) and (2.6), we have

$$\left| \text{Arg} \frac{f_1'(z)}{e^{-i\beta} \phi'(z)} \right| = \frac{\pi}{2} \left[ \gamma + \frac{2}{\pi} \arcsin(a^2) \right] = \frac{\gamma \pi}{2}. \quad (2.8)$$

Therefore $\gamma_1$ cannot be smaller.

For $a = 1$, consider the sequence $\{z_n\}$, $z_n = e^{i\theta_n}$, $\theta_n \in (0, \pi/4)$, $n \in \mathbb{N} = 1$ such that $\lim_{n \to \infty} z_n = 1$. So

$$\lim_{n \to \infty} \text{Arg}(1 - z_n^2) = -\frac{\pi}{2}. \quad (2.9)$$

Let $\phi \in V_k$ with $\phi(e^{i\theta})$ finite and $\theta \in (0, \pi/2)$. The function $f_1$ defined as

$$f_1(z) = \frac{e^{-i\beta} \phi'(z)}{1 - z^2} \left[ \left( \frac{1 + z}{1 - z} \right) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right], \quad z \in E, \ |\beta| < \frac{\pi}{2} \quad (2.10)$$

belongs to $G(1, k, \gamma)$. Thus, from (2.9), it follows that

$$\lim_{n \to \infty} \left| \text{Arg} \frac{f_1'(z)}{e^{-i\beta} \phi'(z)} \right| = \lim_{n \to \infty} \left| \text{Arg} \left\{ \left( \frac{1 + z_n}{1 - z_n} \right) \cos \frac{\beta}{\gamma} - i \sin \frac{\beta}{\gamma} \right\} - \text{Arg}(1 - z_n^2) \right| = (1 + \gamma) \frac{\pi}{2}. \quad (2.11)$$

This means that $\gamma_1(1, \gamma) = 1 + \gamma$ is best possible.

We note that, for $\gamma = 1, k \geq 2$, we obtain a result proved in [8].

Theorem 2.2. Let $f \in G(\alpha, k, \gamma), \alpha \in [0, 1]$. Then, for every $\gamma \in (0, 1)$ and $\theta_1, \theta_2$ with $0 \leq \theta_2 - \theta_1 \leq 2\pi$, one has

$$\int_{\theta_1}^{\theta_2} \text{Re} \left\{ 1 + r e^{i\theta} f''(r e^{i\theta}) \right\} d\theta > -\left( \gamma + \frac{k}{2} - 1 - R \right) \pi, \quad (2.12)$$

where

$$R = \frac{1}{\pi} \left\{ \varphi(r, \theta_2) - \varphi(r, \theta_1) \right\},$$

$$\varphi(r, \theta) = -\text{Arg} \left( 1 - \alpha^2 \gamma^2 e^{2i\theta} \right) = \arctan \frac{\alpha^2 r^2 \sin 2\theta}{1 - \alpha^2 r^2 \cos 2\theta}. \quad (2.13)$$
Proof. To prove this result, we shall essentially use the similar method given by Kaplan [9].

Let $f \in G(\alpha, k, \gamma)$ for fixed $\alpha \in [0, 1]$. Then $f$ satisfies the inequality (1.4) for some $\beta$, $|\beta| < \pi/2$ and $\phi \in V_k$. Let $\phi_1(z) = \phi(z)e^{i\beta}$, $z \in E$. Since $f'(z) \neq 0$, $\phi_1'(z) \neq 0$ for $z \in E$, we can define, for $z = re^{i\theta}$, $r \in (0, 1)$, $\theta$ is a real number, the following:

\[
\varphi(r, \theta) = \operatorname{Arg}\left\{ 1 - \alpha^2 r^2 e^{2i\theta} \right\} f'(re^{i\theta}),
\]

(2.14)

\[
V(r, \theta) = \operatorname{Arg} \phi_1'(re^{i\theta}),
\]

(2.15)

\[
\varphi(r, \theta) = \operatorname{Arg}\left\{ 1 - \alpha^2 r^2 e^{-2i\theta} \right\} re^{i\theta} f'(re^{i\theta}) = \varphi(r, \theta) + \theta,
\]

(2.16)

\[
V(r, \theta) = \operatorname{Arg}\left\{ re^{i\theta} \phi_1'(re^{i\theta}) \right\} = \tau(r, \theta) + \theta.
\]

(2.17)

The functions $\varphi$, $\tau$, $\varphi$, and $V$ are continuous and periodic with period $2\pi$. From (1.4), we can choose the branches of argument of $\varphi(z)$ and $\tau(z)$ as

\[
|\varphi(r, \theta) - \tau(r, \theta)| < \frac{\gamma \pi}{2}, \quad \gamma \in [0, 1].
\]

(2.18)

Now, for $\phi_1 \in V_k$, it is known [10] that, for $\theta_1 < \theta_2$, $z = re^{i\theta}$,

\[
\int_{\theta_1}^{\theta_2} \frac{Re\left\{ (z\phi_1'(z)')^r \frac{\phi_1(z)}{\phi_1'(z)} \right\} d\theta}{Re\left\{ (z\phi_1'(z)')^r \frac{\phi_1(z)}{\phi_1'(z)} \right\} d\theta} > -\left( \frac{k}{2} - 1 \right) \pi.
\]

(2.19)

From (2.16), (2.17), and (2.19), we have

\[
\varphi(r, \theta_2) - \varphi(r, \theta_1) = \varphi(r, \theta_2) + \theta_2 - \varphi(r, \theta_1) - \theta_1
\]

\[
= [\varphi(r, \theta_2) - \tau(r, \theta_2)] + [\tau(r, \theta_2) + \theta_2 - \tau(r, \theta_1) - \theta_1] - [\varphi(r, \theta_1) - \tau(r, \theta_1)]
\]

\[
> \gamma \pi - \left( \frac{k}{2} - 1 \right) \pi = -\left( r + \frac{k}{2} - 1 \right) \pi.
\]

(2.20)

Moreover, by (2.16), we have

\[
\frac{d}{d\theta}\varphi(r, \theta) = \frac{d}{d\theta}\operatorname{Arg}\left( 1 - \alpha^2 r^2 e^{2i\theta} \right) + \operatorname{Re}\left\{ 1 + re^{i\theta} \phi_1''(re^{i\theta}) \right\} f'(re^{i\theta}).
\]

(2.21)
and therefore, from (2.20)

$$
\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta = \int_{\theta_1}^{\theta_2} \frac{d}{d\theta} \psi(r, \theta) d\theta - \int_{\theta_1}^{\theta_2} \text{Arg} \left( 1 + a^2 r^2 e^{2i\theta} \right) d\theta
$$

$$
> - \left( \gamma + \frac{k}{2} - 1 \right) \pi - [\psi(r, \theta_1) - \psi(r, \theta_2)]
$$

(2.22)

$$
= - \left( \gamma + \frac{k}{2} - 1 - \Re \right) \pi,
$$

where \( \psi(r, \theta) \) and \( \Re \) are defined by (2.13). This completes the proof. \( \square \)

We note that, for \( \gamma = 1, k = 2, a = 0 \), we obtain the necessary condition for \( f \) to be close-to-convex in \( E \), proved in [9].

**Remark 2.3.** From Theorem 2.2, we can interpret some geometrical meaning for the functions in \( G(\alpha, k, \gamma) \). For simplicity, let us suppose that the image domain is bounded by an analytic curve \( \Gamma \). At a point on \( \Gamma \), the outward drawn normal turns back at most \( = - (\gamma + k/2 - 1 - \Re) \pi \), where \( A \) is given by (2.13). This is a necessary condition for a function \( f \) to belong to \( G(\alpha, k, \gamma) \). Goodman [4] showed that if \( f \in K(\sigma) \), \( \sigma \geq 0 \), then, for \( z = re^{i\theta} \), \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \),

\[
\int_{\theta_1}^{\theta_2} \Re \left( (zf'(z))' / f'(z) \right) d\theta > -\sigma \pi.
\]

We note that \( f \in G(\alpha, k, \gamma) \) is univalent for \( k + 2(\gamma - \Re) \leq 4 \), since

\[
G(\alpha, k, \gamma) \subset K \left( \gamma + \frac{k}{2} - 1 - \Re \right).
\]

The functions in \( K(\gamma + k/2 - 1 - \Re) \) need not even be finitely valent in \( E \) for \( k + 2(\gamma - \Re) > 4 \).

**Remark 2.4.** From Theorem 2.2 and [11, Lemma 1.3] by Pommerenke, it follows that \( G(\alpha, k, \gamma) \) is a linearly invariant family of order \( (\gamma + k/2 - \Re) \). Therefore, the image of \( E \) under functions in \( G(\alpha, k, \gamma) \) contains the schlicht disk \( |z| < 1/(k + 2(\gamma - \Re)) \).

**Theorem 2.5.** Let \( f \in G_{\beta}(\alpha, k, \gamma), \gamma \in (0, 1), |\beta| < \pi/2 \), be of the form (1.1). Then \( |a_2| \leq k/2 + ((1 + \gamma)/2)|\cos(\beta/\gamma)| \). This estimate is best possible, extremal function being \( f_0(z) \) defined by (2.4).

**Proof.** Let \( \hat{\phi}(z) = z + \sum_{n=2}^{\infty} b_n z^n \), \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \) in (1.5).

Now, it is known that, for functions \( p \) of positive real part with \( \gamma \in (0, 1) \), \( p^r \) is subordinate to \( ((1 + z)/(1 - z))^r \). Also \( |b_2| \leq k/2 \), see [1, 12]. Therefore, from (1.5), we have \( 2a_2 = 2b_2 + (e^{i\beta} \cos(\beta/\gamma))(1 + \gamma) \), and this gives us the required result. \( \square \)

**Remark 2.6.** Let \( f \in G(\alpha, k, \gamma) \), for \( 2 \leq k \leq 4 - 2(\gamma - \Re) \), and be given by (1.1). Then \( f \) is univalent in \( E \) by Remark 2.3 and \( w_0 f(z)/(w_0 - f(z)) \) is univalent in \( E \) for \( w_0 \neq 0, w_0 \neq f(z) \). Now

\[
\frac{w_0 f(z)}{w_0 - f(z)} = z + \left( a_2 + \frac{1}{w_0} \right) z^2 + \cdots,
\]

(2.24)
and therefore $|a_2 + 1/w_0| \leq 2$ and so $|1/w_0| \geq 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$, on using Theorem 2.5. Hence it follows that the image of $E$ under $f \in G(\alpha, k, \gamma)$ with $2 \leq k + 2(\gamma - \Re) \leq 4$ contains the schlicht disc $|z| < 2/(4+k+(1+\gamma)\cos(\beta/\gamma))$.

From Remark 2.3, and the results proved for the class $K(\sigma)$, $\sigma \geq 0$ in [4], we at once have the following.

**Theorem 2.7.** Let $f \in G(\alpha, k, \gamma)$ and be given by (1.1). Let $F_\alpha$ be defined by

$$F_\alpha(z) = \frac{1}{2(\sigma + 1)} \left[ \left( \frac{1 + z}{1 - z} \right)^{\sigma + 1} - 1 \right] = z + \sum_{n=2}^{\infty} A_n(\sigma) z^n, \quad (2.25)$$

where $\sigma = (\gamma + k/2 - 1 - \Re)$, and $\Re$ is given by (2.13). Clearly $F_\alpha \in G(\alpha, k, r)$.

(i) Denote by $L(r, f)$ the length of the image of the circle $|z| = r$ under $f$ and by $A(r, f)$ the area of $f(|z| \leq r)$. Then, for $0 \leq r < 1$

(a) $L(r, f) \leq L(r, F_\alpha)$,

(b) $A(r, f) \leq A(r, F_\alpha)$.

(ii) For $z = re^{i\theta}$, $0 \leq r < 1$, $|f(z)| = (1/2(\sigma + 1))[(1 + z)/(1 - z)]^{\sigma + 1} - 1]$. The function $F_\alpha$, defined by (2.25), shows that this upper bound is sharp.

**Theorem 2.8.** Let $f \in G(\alpha, k, \gamma)$. Then, for $0 < r < 1$, $\alpha, r \in (0, 1)$, $k \geq 2$,

$$L(r, f) \leq c(\alpha, k, r) \left( \frac{1}{1 - r} \right)^{k/2+\gamma}, \quad (2.26)$$

where $c(\alpha, k, r)$ is a constant depending upon $\alpha, k, \gamma$ only.

**Proof.** With $z = re^{i\theta}$,

$$L(r, f) = \int_{0}^{2\pi} |z f'(z)| d\theta = \int_{0}^{2\pi} r \left| e^{-i\phi} \varphi(z) \frac{p(z) \cos(\beta/\gamma) - i \sin(\beta/\gamma)}{1 - \alpha^2 z^2} \right| d\theta, \quad \varphi \in V_k, \; p \in P, \; z \in E. \quad (2.27)$$

For $\varphi \in V_k$, it is known [10] that there exist $s_1, s_2 \in S^*$ such that

$$z \varphi(z) = \frac{(s_1(z))^{k + 1/2}}{(s_2(z))^{k + 1/2}}. \quad (2.28)$$

Also, for $p \in P$ we have for $z = re^{i\theta}$,

$$\frac{1}{2\pi} \int_{0}^{2\pi} |p(z)|^2 d\theta \leq \frac{1 + 3r^2}{1 - r^2} \quad (2.29)$$
(see [13]). Now, from (2.27), (2.28), and (2.29), we have
\[
L(r, f) \leq \frac{c_1(a, k, \gamma)}{r^{(k/4-1/2)}} \left( \frac{1}{2\pi} \int_0^{2\pi} |s_1(z)|^{(k/4+1/2)(2/(2-\gamma))} d\theta \right)^{(2-\gamma)/2} \left( \frac{1}{2\pi} \int_0^{2\pi} |p(z)|^2 \right)^{\gamma/2}
\]
\[
\leq c(a, k, \gamma) \left( \frac{1}{1-r} \right)^{k/2+\gamma},
\]
where we have used distortion theorems, subordination for the starlike functions, and Holder’s inequality, and \( c \) and \( c_1 \) are constants.

**Theorem 2.9.** Let \( f \in G(a, k, \gamma) \) and be given by (1.1). Then, for \( a, \gamma \in [0, 1], k \geq 2, \) one has \( a_n = o(1)n^{k/2+\gamma-1}, \) \( (n \rightarrow \infty) \) where \( o(1) \) is a constant depending only on \( k, a, \) and \( \gamma. \)

**Proof.** With \( z = re^{i\theta}, \) Cauchy’s theorem gives
\[
n a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} z f'(z) e^{-i\theta} d\theta.
\]
Thus
\[
n |a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |z f'(z)| d\theta = (1/2\pi r^n)L(r, f).
\]
Using Theorem 2.8 and putting \( r = 1 - 1/n, \) we prove this result. □

**Acknowledgment**

The authors would like to thank the referee for thoughtful comments and suggestions.

**References**


