Research Article

A Note on the Range of the Operator
\( X \mapsto TX - XT \) Defined on \( C_2(H) \)

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We show how a proof of J. Stampfli can be extended to prove that the operator \( X \mapsto TX - XT \) defined on the Hilbert-Schmidt class, when \( T \) is an \( M \)-hyponormal, \( p \)-hyponormal, or log-hyponormal operator, has a closed range if and only if \( \sigma(T) \) is finite.

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1. Introduction

Let \( \mathcal{H} \) be a complex, separable, infinite dimensional Hilbert space, and let \( \mathcal{L} (\mathcal{H}) \) denote the algebra of all linear bounded operators on \( \mathcal{H} \). The Hilbert-Schmidt class, denoted by \( C_2(\mathcal{H}) \), is a Hilbert space with the \( \| \cdot \|_2 \)-norm that arises from the inner product \( \langle X, Y \rangle = \text{tr}(XY^*) \), where \( \text{tr} \) is the scalar-valued trace. For \( T \in \mathcal{L} (\mathcal{H}) \), define \( \Delta_T : \mathcal{L} (\mathcal{H}) \to \mathcal{L} (\mathcal{H}) \) by \( \Delta_T(X) = TX - XT \), and let \( \sigma(T) \) denote the spectrum of \( T \). Let the range of a linear operator \( S \) be denoted by \( \mathcal{R}(S) \). For a normal operator \( N \in \mathcal{L} (\mathcal{H}) \), Anderson and Foiaš [1] proved that \( \mathcal{R}(\Delta_N) \) is norm closed if and only if \( \sigma(N) \) is a finite set. In [2], Stampfli extended this result to the class of hyponormal operators.

Theorem A ([2]). Let \( T \in \mathcal{L} (\mathcal{H}) \) be a hyponormal operator. Then \( \mathcal{R}(\Delta_T) \) is norm closed if and only if \( \sigma(T) \) is finite.

In fact, Stampfli provided a proof of the “only if” implication which extends to a larger class of operators than the class of hyponormal operators (see Proposition 2.2). For an operator \( T \in \mathcal{L} (\mathcal{H}) \), let \( \sigma_{\text{nap}} (T) \) denote its normal approximate point spectrum, that is, the set of those \( \lambda \in \mathbb{C} \) for which there exists an orthonormal sequence \( \{ \phi_n \}_n \) in \( \mathcal{H} \) such that

\[ \| (T - \lambda) \phi_n \| + \| (T - \lambda)^* \phi_n \| \to 0. \] (1.1)
Define the class $\mathcal{G}(\mathcal{M})$ as follows:

$$\mathcal{G}(\mathcal{M}) := \{ T \in \mathcal{L}(\mathcal{M}) \mid \sigma_{\text{nap}}(T) \text{ is an infinite set} \}. \quad (1.2)$$

Some classes of hyponormal related operators, such as $M$-hyponormal operators, that is,

$$m \cdot \|(T - \lambda)^* \phi\| \leq \|(T - \lambda) \phi\|, \quad (\forall) \phi \in \mathcal{M}, \quad (\forall) \lambda \in \mathbb{C}, \text{ for some } m > 0, \quad (1.3)$$

$p$-hyponormal operators, that is, $(T^* T)^p \geq (T T^*)^p$ for some $p > 0$, or log-hyponormal operators, that is, invertible operators such that $\log(T^* T) \geq \log(T T^*)$, have spectrum that is finite or they belong to $\mathcal{G}(\mathcal{M})$. Particularly, the hyponormal operators (i.e., 1-hyponormal) have this property.

In [3] Stampfli proved the following lemma which will be used in Section 2.

**Lemma B.** Let $T \in \mathcal{G}(\mathcal{M})$ and let $\{ \lambda_n \}_{n=1}^{\infty}$ be a sequence of distinct points of $\sigma_{\text{nap}}(T)$. Then for any sequence $\{ \varepsilon_n \}_{n=1}^{\infty}$ of positive numbers converging to zero, there exists an orthonormal sequence $\{ \phi_n \}_{n=1}^{\infty}$ of vectors in $\mathcal{M}$ such that

$$\|(T - \lambda_n) \phi_n\| + \|(T - \lambda_n)^* \phi_n\| < \varepsilon_n \quad \text{for } n = 1, 2, \ldots, \quad (1.4)$$

$$\langle \phi_n, T \phi_k \rangle = 0 \quad \text{for } k = 1, \ldots, n - 1. \quad (1.5)$$

### 2. The Closedness of the Range of $\Delta_2^{(2)}$

The operator $\Delta_2$ defined on the Hilbert-Schmidt class will be denoted in the remainder of this note by $\Delta_2^{(2)}$, that is, $\Delta_2^{(2)} : C_2(\mathcal{M}) \rightarrow C_2(\mathcal{M})$, $\Delta_2^{(2)}(X) = TX - XT$. Let $H^M(\mathcal{M})$ denote the set of $M$-hyponormal operators.

**Proposition 2.1.** Let $T \in H^M(\mathcal{M})$. If $\sigma(T)$ is finite, then $\mathcal{R}(\Delta_2^{(2)})$ is closed.

**Proof.** It is well known that an operator $T \in H^M(\mathcal{M})$ with finite spectrum is normal. Indeed, for such an operator, the restriction to an invariant subspace $\mathcal{M}$ belongs to $H^M(\mathcal{M})$. On the other hand, if $T \in H^M(\mathcal{M})$ with $\sigma(T) = \{ \lambda \}$, then $T = \lambda I$, (cf. [4]). Thus, we can write $T = \sum_{i=1}^{n_0} \lambda_i E_i$, where $E_i$’s are the spectral projections.

Let $X_n$ and $C$ be in $C_2(\mathcal{M})$ such that $\|\Delta_2^{(2)}(X_n) - C\|_2 \rightarrow 0$. Therefore $\Delta_2(X_n) - C \rightarrow 0$ in the $\mathcal{L}(\mathcal{M})$ norm, and according to Theorem A, there exists $X^0 \in \mathcal{L}(\mathcal{M})$ such that $C = TX^0 - X^0 T$.

For an arbitrary $X \in \mathcal{L}(\mathcal{M})$, let $[X_{ij}]$ be the block-matrix representation of $X$ relative to the decomposition $\mathcal{M} = \sum_{i=1}^{n_0} \oplus E_i \mathcal{M}$. Thus

$$C_{ij} = (\lambda_i - \lambda_j) X^0_{ij}, \quad (2.1)$$

for all $i, j = 1, \ldots, n_0$. This implies that each $X^0_{ij} = 1/(\lambda_i - \lambda_j) C_{ij}$ is a Hilbert-Schmidt operator. Moreover $X^0_{ii}$ can be chosen 0, and thus $X^0 \in C_2(\mathcal{M})$. \qed
Proposition 2.2. Let \( T \in \mathcal{G}(\mathcal{K}) \). Then \( \mathcal{R}(\Delta_T^{(2)}) \) is not closed.

Proof. We will use same notation and circle of ideas as in [2]. Let \( \{\lambda_n\}_{n \geq 1} \) be sequence of distinct points of \( \sigma_{\text{nap}}(T) \) so that \( \lambda_n \to \lambda_0 \). Let

\[
\eta_n = \max \{ |\lambda_{j+1} - \lambda_j|^{-1/2} | j = 1, \ldots, n \},
\]

and choose a nonincreasing sequence \( \{\epsilon_n\}_{n \geq 1} \) so that \( 0 < \epsilon_n \leq |\lambda_{n+1} - \lambda_n|^2, n \geq 1, \) and \( \sum_{n \geq 1} \epsilon_n^2 \eta_n^2 < \infty \). According to Lemma B, there exists an orthonormal sequence \( \{\phi_n\}_{n \geq 1} \) that satisfies (1.4) and (1.5). Let \( \mathcal{K}_1 = \sqrt{\{\phi_n | n \geq 1\}}, \mathcal{K}_2 = \mathcal{K}_1^\perp \), and let \( \delta_n \) such that

\[
T \phi_n = \mu_n \phi_n + \delta_n, \quad \delta_n \perp \phi_n, \quad n \geq 1.
\]

It results that

\[
|\mu_n - \lambda_n| < \epsilon_n, \quad ||\delta_n|| < 2\epsilon_n, \quad n \geq 1.
\]

Define \( V : \mathcal{K} \to \mathcal{K} \) by \( V \phi_n = |\lambda_{j+1} - \lambda_j|^{-1/2} \phi_{j+1}, n \geq 1, \) and \( V g = 0, g \in \mathcal{K}_2 \). Let \( \mathcal{M}_n = \sqrt{\{\phi_j | j = 1, \ldots, n\}} \) and let \( P_n \) be the orthogonal projection onto \( \mathcal{M}_n \), and define \( V_n = VP_n \). A tedious calculation shows that

\[
\Delta_T(V_n)\phi_j = \begin{cases} 
\nu_j (\mu_{j+1} - \mu_j) \phi_{j+1} + \nu_j \delta_{j+1} - V_n \delta_j, & j \leq n, \\
- V_n \delta_j, & j > n,
\end{cases}
\]

where \( \nu_j = |\lambda_{j+1} - \lambda_j|^{-1/2} \). Denoting \( \Delta_T(V_n) - \Delta_T(V_m) \) by \( \Delta_T^{n,m} \), then for \( n < m \),

\[
\Delta_T^{n,m} \phi_j = \begin{cases} 
0, & j \leq n, \\
- \nu_j (\mu_{j+1} - \mu_j) \phi_{j+1} + \nu_j \delta_{j+1} + (V_m - V_n) \delta_j, & n < j \leq m, \\
(V_m - V_n) \delta_j, & j > m.
\end{cases}
\]

Furthermore, from (2.3) it results that

\[
\delta_j \perp \phi_j, \phi_{j+1}, \phi_{j+2}, \ldots
\]

and from (2.4)

\[
\|V_n \delta_j\| \leq 2\eta_j \epsilon_j, \quad \forall j, n \geq 1.
\]

We will show next that \( \|\Delta_T^{n,m}\|_2 \to 0 \) when \( m, n \to \infty \), thus there exists \( C \in C_2(\mathcal{K}) \) such that \( \|\Delta_T(V_n) - C\|_2 \to 0 \), that is, \( C \in \mathcal{R}(\Delta_T^{(2)}). \)
First, we will show that \( \| \Delta T^{n,m} | \omega_t \|^2 \to 0 \), when \( m, n \to \infty \). Indeed,

\[
\| \Delta T^{n,m} | \omega_t \|^2 = \sum_{j=1}^{\infty} \| \Delta T^{n,m}_j \|^2 = \sum_{j=n+1}^{m} \| -\nu_j (\mu_{j+1} - \mu_j) \phi_{j+1} + \nu_j \delta_{j+1} + (V_m - V_n) \delta_j \|^2 + \sum_{j=n+1}^{\infty} \| (V_m - V_n) \delta_j \|^2. \tag{2.9}
\]

The first sum of the right-hand side of the above can be majorized by

\[
2 \cdot \sum_{j=n+1}^{m} \| -\nu_j (\mu_{j+1} - \mu_j) \phi_{j+1} + \nu_j \delta_{j+1} \|^2 + 2 \cdot \sum_{j=n+1}^{m} \| (V_m - V_n) \delta_j \|^2. \tag{2.10}
\]

Since \( \phi_{j+1} \perp \delta_{j+1} \), we have

\[
\| \Delta T^{n,m} | \omega_t \|^2 \leq 2 \left[ \sum_{j=n+1}^{m} \left( \nu_j^2 |\mu_{j+1} - \mu_j|^2 + \nu_j^2 \| \delta_{j+1} \|^2 \right) + \sum_{j=n+1}^{\infty} \| (V_m - V_n) \delta_j \|^2 \right]. \tag{2.11}
\]

According to (2.8),

\[
\| (V_m - V_n) \delta_j \|^2 \leq 16 \eta_j^2 \varepsilon_j^2, \tag{2.12}
\]

and according to (2.4),

\[
\nu_j^2 \| \delta_{j+1} \|^2 \leq 4 \eta_j^2 \varepsilon_{j+1}^2 \leq 4 \eta_j^2 \varepsilon_j^2, \quad |\mu_{j+1} - \mu_j|^2 \leq (2 \varepsilon_j + |\lambda_{j+1} - \lambda_j|^2) \leq 8 \varepsilon_j^2 + 2 |\lambda_{j+1} - \lambda_j|^2, \tag{2.13}
\]

which implies

\[
\nu_j^2 |\mu_{j+1} - \mu_j|^2 \leq 8 \eta_j^2 \varepsilon_j^2 + 2 |\lambda_{j+1} - \lambda_j|. \tag{2.14}
\]

Therefore

\[
\| \Delta T^{n,m} | \omega_t \|^2 \leq c_1 \sum_{j=n+1}^{\infty} \eta_j^2 \varepsilon_j^2 + c_2 \sum_{j=n+1}^{m} |\lambda_{j+1} - \lambda_j|, \tag{2.15}
\]

where \( c_1 \) and \( c_2 \) are some constants. After a careful review of the proof, one can see that the sequence \( \{ \lambda_n \} \) can be assumed to converge fast enough (otherwise choose a subsequence of it), more precisely \( \sum_{j=n+1}^{m} |\lambda_{j+1} - \lambda_j| \to 0 \) when \( n, m \to \infty \).
We show next that \( \| \Delta_{T}^{n,m} H_{2} \|_{2}^{2} \to 0 \), when \( m, n \to \infty \). Indeed, we can write

\[
T^{*} \phi_n = \overline{\mu}_n \phi_n + \gamma_n \quad \text{with} \quad \langle \gamma_n, \phi_n \rangle = 0, \quad \| \gamma_n \| \leq 2 \epsilon_n, \quad n \geq 1.
\]

(2.16)

Obviously, we can write \( T^{*} \phi_n = \theta_n \phi_n + \gamma_n \) with \( \langle \gamma_n, \phi_n \rangle = 0 \), which implies

\[
\theta_n = \langle \theta_n \phi_n + \gamma_n, \phi_n \rangle = \langle T^{*} \phi_n, \phi_n \rangle = \langle \phi_n, \mu_n \phi_n + \delta_n \rangle = \overline{\mu}_n^{(n)} \cdot \lambda_n^{(n)},
\]

\[
\| \gamma_n \| = \| (T^{*} - \overline{\mu}_n) \phi_n \| \leq \| (T - \lambda_n) \phi_n \| + \| \lambda_n - \overline{\mu}_n \| \leq 2 \epsilon_n.
\]

(2.17)

For an orthonormal basis \( \{ \psi_i \}_{i \geq 1} \) of \( H_2 \), we will show that

\[
\sum_{i=1}^{\infty} \| \Delta_{T}^{n,m} \psi_i \|^2 \to 0 \quad \text{when} \quad n, m \to \infty.
\]

(2.18)

For each \( i \), write \( T \psi_i = \sum_{k=1}^{\infty} a_{k}^{(i)} \phi_k + w_i \) with \( w_i \in H_2 \). Thus

\[
V_m T \psi_i = \sum_{k=1}^{m} a_{k}^{(i)} V_m \phi_k + V_m w_i = \sum_{k=1}^{m} a_{k}^{(i)} v_k \phi_{k+1}.
\]

(2.19)

Since \( V_m \psi_i = 0 \), we have \( \Delta_{T}^{m} (V_m) \psi_i = -V_m \psi_i \), and consequently, for \( n < m \),

\[
\Delta_{T}^{n,m} \psi_i = \sum_{k=n+1}^{m} a_{k}^{(i)} v_k \phi_{k+1}.
\]

(2.20)

Since the sequence \( \{ \phi_k \} \) is orthonormal, we have

\[
\| \Delta_{T}^{n,m} \psi_i \|^2 = \sum_{k=n+1}^{m} | a_{k}^{(i)} |^2 \cdot v_k^2.
\]

(2.21)

Therefore

\[
\sum_{i=1}^{\infty} \| \Delta_{T}^{n,m} \psi_i \|^2 = \sum_{i=1}^{\infty} \sum_{k=n+1}^{m} | a_{k}^{(i)} |^2 \cdot v_k^2 = \sum_{k=n+1}^{m} v_k^2 \left( \sum_{i=1}^{\infty} | a_{k}^{(i)} |^2 \right).
\]

(2.22)
For a fixed \( k \),
\[
\sum_{i=1}^{\infty} |a_k^{(i)}|^2 = \sum_{i=1}^{\infty} |\langle T \psi_i, \phi_k \rangle|^2 = \sum_{i=1}^{\infty} |\langle \psi_i, T^* \phi_k \rangle|^2
\]

\[\stackrel{(2.16)}{=} \sum_{i=1}^{\infty} |\langle \psi_i, \mu_k \phi_k + \gamma_k \rangle|^2 = \sum_{i=1}^{\infty} |\langle \psi_i, \gamma_k \rangle|^2 \leq \|\gamma_k\|^2 \tag{2.23}\]

Consequently,
\[
\sum_{i=1}^{\infty} \|\Delta_T^{n,m} \psi_i\|^2 \leq 4 \sum_{k=n+1}^{m} \nu_k^2 \cdot \epsilon_k^2 \rightarrow 0 \text{ for } n, m \rightarrow \infty.
\]

The operator \( C \) is not in \( \mathcal{R}(\Delta_T^{(2)}) \) since, according to the proof of Theorem A in [2], \( C \notin \mathcal{R}(\Delta_T) \). \( \square \)

**Theorem 2.3.** Let \( T \in H^M(\mathcal{A}) \). Then \( \mathcal{R}(\Delta_T^{(2)}) \) is closed if and only if \( \sigma(T) \) is finite.

**Proof.** If \( T \in H^M(\mathcal{A}) \) and \( \sigma(T) \) are finite, then according to Proposition 2.1, \( \mathcal{R}(\Delta_T^{(2)}) \) is closed. Conversely, if \( T \in H^M(\mathcal{A}) \) has an infinite spectrum, then there are infinitely many distinct points \( \{\lambda_n\}_n \) that are either isolated points of the spectrum, in which case they are eigenvalues, or accumulation points of the spectrum, in which case they are in \( \sigma_{ap}(T) \). Since \( T \in H^M(\mathcal{A}) \), we have \( \sigma_p(T) \subseteq \sigma_{ap}(T) \). Thus \( T \in \mathcal{G}(\mathcal{A}) \) and according to Proposition 2.2, \( \mathcal{R}(\Delta_T^{(2)}) \) is not closed. \( \square \)

**References**


